ON EQUATIONAL THEORIES OF SEMILATTICES WITH OPERATORS

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0. Introduction

The question of characterizing the lattices of subvarieties of a variety of universal algebras seems to be very difficult. Some years ago a conjecture was that a lattice is isomorphic to the lattice of all extensions of an equational theory (or dually isomorphic to the lattice of subvarieties of a variety) iff it is algebraic and its greatest element is compact. In [2] and [3] W. Lampe proved this to be false and found further conditions that such a lattice must satisfy. The purpose of the present paper is to find a class K of lattices with the following properties:

THEOREM.

    (1) *every lattice from K is isomorphic to the lattice of equational theories extending an equational theory;*

    (2) *K contains all infinitely distributive algebraic lattices with compact 1;*

    (3) *the parallel join of any pair of lattices from K belongs to K.*

(The parallel join of a pair of lattices L_1, L_2 is defined in this way: it is the lattice obtained from the disjoint union $L_1 \cup L_2$, in which $a \leq b$ holds iff $a \leq b$ holds either in L_1 or in L_2, by adding a greatest and a least element.)

The class K is constructed in this paper as the class of congruence lattices of well-behaved 0,1-semilattices with operators. A universal algebra A is said to be well-behaved if its congruence lattice is isomorphic in a canonical way (see Section 1 for a precise definition) to the lattice of equational theories extending the equational theory of the algebra obtained from A by considering every element as a nullary operation.

Let us remark that there are many lattices representable as lattices of extensions of an equational theory and not belonging to K. This follows from the fact that congruence lattices of semilattices are rather special; see the papers [1], [4] and [6].

The methods used in the present paper are related to those employed in D. Pigozzi [5]. Also, it seems probable that they are related to those in an unpublished paper of W. Lampe and J. Sichler, in which it is proved that every finite distributive lattice is isomorphic to the lattice of equational theories extending an equational theory (see a mention in [5]).

1. Universal algebras

Let A be a universal algebra of similarity type τ. Then $\tau + A$ denotes the disjoint union of the two types, where A is conceived as a set of nullary operation symbols. By the nominal expansion of A we shall mean the algebra of type $\tau + A$ whose τ-reduct coincides with A and in which every element serves as its own name.

In the following let A be an algebra and N be its nominal expansion. Denote by $Con(A)$ the congruence lattice of A (we have $Con(A) = Con(N)$) and by $L(N)$ the lattice of the equational theories extending the equational theory $Eq(N)$ of N.

We shall often neglect to specify the similarity type when speaking about terms, equations, equational theories, etc. The convention is that by a term we shall mean a term of the type of N, and similarly for equations, equational theories, etc. By a strictly constant equation we shall mean an equation, both sides of which are elements of A; the set of strictly constant equations is thus equal to $A^2 = N^2$. (Equations are identified with the ordered pairs of terms, and an equation (a, b) will be often denoted by $a \approx b$.)

Let us define a mapping ϱ of $L(N)$ into $Con(A)$ by $\varrho(E) = E \cap A^2$. Furthermore, define two mappings $\varepsilon_1, \varepsilon_2$ of $Con(A)$ into $L(N)$ as follows: for a congruence r of A, let $\varepsilon_1(r)$ be the equational theory generated by $r \cup Eq(N)$ and let $\varepsilon_2(r)$ be the equational theory of N/r.

1.1.PROPOSITION. *Let A be an algebra and N be its nominal expansion. Then ϱ is a complete lattice homomorphism of $L(N)$ onto $Con(A)$. The mappings ε_1 and ε_2 are both injective; ε_1 is a complete join-homomorphism and ε_2 is a complete meet-homomorphism; for a congruence r, $\varepsilon_1(r)$ is the least and $\varepsilon_2(r)$ is the largest element of the interval $\varrho^{-1}(r)$ in $L(N)$.*

Proof. The fact that ϱ is a complete meet-homomorphism follows immediately from the definition of ϱ. In order to prove that it is a complete lattice homomorphism, it remains to prove that if E is the join of a subset S of $L(N)$ then the congruence $\varrho(E)$ is contained in the join of the congruences $\varrho(H)$ for $H \in S$. Let $(a,b) \in \varrho(E) = E \cap A^2$. There exists a finite sequence $u_0, \ldots, u_k$ of terms such that $u_0 = a$, $u_k = b$ and $(u_{i-1}, u_i) \in E_i$ for some $E_i \in S$, for any $i = 1, \ldots, k$. Take an element $c \in A$ and define a substitution f by $f(x) = c$ for all variables x. The sequence $v_i = f(u_i)$ has the properties $v_0 = a$, $v_k = b$ and $(v_{i-1}, v_i) \in E_i \cap A^2 = \varrho(E_i)$ for all i. It follows that (a,b) belongs to the join of the congruences $\varrho(H)$.

It is clear from the definitions that $r = \varepsilon_2(r) \cap A^2$ for any congruence r. From this we get $r = \varrho(\varepsilon_2(r))$ and also $r = \varrho(\varepsilon_1(r))$, since clearly $\varepsilon_1(r) \subseteq \varepsilon_2(r)$. Especially, the mapping ϱ is surjective and both ε_1 and ε_2 are injective.

Evidently, $\varepsilon_1(r)$ is the least equational theory that is mapped onto r by ϱ. In order to prove that $\varepsilon_2(r)$ is the largest one, let E be any equational theory with $\varrho(E) = r$ and take an equation $(u,v) \in E$; we need to show that (u,v) is satisfied in N/r. Let f be a homomorphism of the algebra of terms into N and denote by g the substitution, mapping every variable x onto $f(x)$. It is easy to see that both $(f(u), g(u))$ and $(f(v), g(v))$ are satisfied in N and so belong to E; since $(g(u), g(v)) \in E$, we get $(f(u), f(v)) \in E \cap A^2$ and consequently $(f(u), f(v)) \in r$. Consequently, $h(u) = h(v)$ for any homomorphism h of the algebra of terms into N/r.

It is not difficult to show that if φ is a complete lattice homomorphism of a complete lattice L_1 onto a complete lattice L_2 then the two mappings, assigning to any element $a \in L_2$ the least and the greatest element of the interval $\varphi^{-1}(a)$, are a complete join homomorphism and a complete meet homomorphism, respectively. $\square$

Let an algebra A be given and let N be its nominal expansion. An equation is said to be a consequence of a set of equations S if it belongs to the equational theory generated by $S \cup Eq(N)$. An equation is said to be good (more precisely, A-good) if it is a consequence of the set of its own strictly constant consequences.

1.2.PROPOSITION. *The following are equivalent for an algebra A with nominal expansion N:*

    (1) *for any congruence r of A there exists a unique equational theory E extending $Eq(N)$ such that $r = E \cap A^2$;*

    (2) *the mappings ε_1 and ε_2 defined above coincide and are an isomorphism of $Con(A)$ onto $L(N)$;*

    (3) *Every equation is A-good.*

Proof. The equivalence of (1) with (2) follows from Theorem 1; condition (3) is a reformulation. $\square$

An algebra A is said to be well-behaved if it satisfies the equivalent conditions of Theorem 2. So, if A is a well-behaved algebra then the congruence lattice of A is isomorphic to the lattice of equational theories extending $Eq(N)$ (and consequently dually isomorphic to the lattice of subvarieties of the variety generated by N).

1.3.PROPOSITION. *Let A be an algebra and N be its nominal expansion. Let $f(x)$ and $g(x)$ be two terms containing no other variable than x. The equation $f(x) \approx g(x)$ is $A - good$ iff it is a consequence of the set of equations $\{f(a) \approx g(a); a \in A\}$.*

Proof. Only the direct implication needs to be proved. Let (u,v) be a strictly constant consequence of $(f(x), g(x))$. Then there exists a derivation of this equation, a finite sequence of terms $u_0, \ldots, u_k$ such that $u_0 = u$, $u_k = v$ and if $i \in 1, \ldots, k$ then either $(u_{i-1}, u_i) \in Eq(N)$ or u_i can be obtained from u_{i-1} by replacing either an occurrence of a subterm $f(t)$ by $g(t)$ or an occurrence of $g(t)$ by $f(t)$, for a term t. Take a substitution φ mapping all the variables onto a constant from A. The sequence $\varphi(u_0), \ldots, \varphi(u_k)$ is a derivation with the following properties: $\varphi(u_0) = u$; $\varphi(u_k) = v$; if $i \in 1, \ldots, k$ then either $(\varphi(u_{i-1}), \varphi(u_i)) \in Eq(N)$ or $\varphi(u_i)$ can be obtained from $\varphi(u_{i-1})$ by replacing either an occurrence of a subterm $f(\varphi(t))$ by $g(\varphi(t))$ or an occurrence of $g(\varphi(t))$ by $f(\varphi(t))$. But $(\varphi(t), a) \in Eq(N)$ for some $a \in A$ and we have proved that each strictly constant consequence of $(f(x), g(x))$ is a consequence of the set $\{(f(a), g(a)); a \in A\}$ (with respect to $Eq(N)$). It follows

that if $(f(x), g(x))$ is a consequence of its strictly constant consequences then it is a consequence of the set $\{(f(a), g(a)); a \in A\}$. $\square$

2. Semilattices with operators

By a semilattice with operators we mean an algebra $A = A(\wedge, F)$ such that $A(\wedge)$ is a semilattice and F is a set of unary operations, acting as endomorphisms of $A(\wedge)$. If $A(\wedge)$ contains the least and the greatest elements then $A(\wedge, F)$ is said to be a 0,1-semilattice with operators; the two extreme elements are denoted by 0_A and 1_A, or only by 0 and 1. For $c \in A$, we denote by k_c the constant unary operation on A with value c, and by m_c the endomorphism $a \to a \wedge c$.

In the following let $A = A(\wedge, F)$ be a 0,1-semilattice with operators and N be its nominal expansion.

We denote by F' the least set of unary operations on A containing F, the identity, all the constant unary operations and closed under superposition and the forming of meets. Every element of F' is an endomorphism of $A(\wedge)$. For any $f \in F'$ and any variable x, the expression $f(x)$ can be considered as a term in an obvious way; this term is uniquely determined up to the equational theory of N.

2.1. PROPOSITION. *Let $A = A(\wedge, F)$ be a 0,1-semilattice with operators. Then A is well-behaved iff all the equations $f(x) \approx g(x)$, where $f, g \in F'$ and x is a variable, are A-good.*

Proof. The direct implication follows from Theorem 2. In order to prove the converse, let $(f(x), g(x))$ be good for any $f, g \in F'$ and let (u, v) be an arbitrary equation; we are going to prove that (u, v) is good. Since any term t is equivalent, modulo $Eq(N)$, to the term $t \wedge p(x)$ where p is the constant unary operation with value 1 and x is an arbitrary variable, we can suppose that $var(u) = var(v)$; moreover, we can suppose that $u = f_1(x_1) \wedge \cdots \wedge f_n(x_n)$ and $v = g_1(x_1) \wedge \cdots \wedge g_n(x_n)$ for some pairwise distinct variables $x_1, \ldots, x_n$ and endomorphisms $f, g \in F'$. For $i = 1, \ldots, n$ put $u_i = f_1(1) \wedge \cdots \wedge f_{i-1}(1) \wedge f_i(x_i) \wedge f_{i+1}(1) \wedge \cdots \wedge f_n(1)$ and $v_i = g_1(1) \wedge \cdots \wedge g_{i-1}(1) \wedge g_i(x_i) \wedge g_{i+1}(1) \wedge \cdots \wedge g_n(1)$. The equations (u_i, v_i) are clearly consequences of (u, v); each of them is a consequence of its own strictly constant consequences, since it is of the form $(p_i(x_i), q_i(x_i))$ for $p_i, q_i \in F'$; and (u, v) is a consequence of $(u_1, v_1), \ldots, (u_n, v_n)$, since the equations $(u, u_1 \wedge \cdots \wedge u_n)$ and $(v, v_1 \wedge \cdots \wedge v_n)$ belong to $Eq(N)$. $\square$

A pair (f, g) of operations from F' is said to be good if the equation $f(x) \approx g(x)$, where x is any variable, is good.

2.2. PROPOSITION. *Let $A = A(\wedge, F)$ be a 0,1-semilattice with operators and let f, g, h be three unary operations from F'. The following assertions are true:*

(1) *If either f or g is constant then (f, g) is good.*
(2) *If $(f \wedge g, f)$ and $(f \wedge g, g)$ are both good then (f, g) is good.*
(3) *If $f \le g \le h$ and if both (f, g) and (f, h) are good then (f, h) is good.*
(4) *If there exist elements $c, d \in A$ such that $c \approx d$ is a consequence of $f(x) \approx g(x)$ and*
$$f(a) = f(a) \wedge c,$$
$$g(a) = g(a) \wedge c,$$
$$f(a) \wedge d = g(a) \wedge d$$
for all $a \in A$ then (f, g) is good.
(5) *If there exist two elements $c, d \in A$ such that $c \approx d$ is a consequence of $f(x) \approx g(x)$ and $c \le f(a) \le d$ and $c \le g(a) \le d$ for all $a \in A$ then (f, g) is good.*
(6) *Let the range of f be a two-element set $\{a, b\}$ with $a < b$ and the range of g be a two-element set $\{c, d\}$ with $c < d$. If either $f^{-1}(1) \ne g^{-1}(1)$ or $a \wedge d = b \wedge c$ then (f, g) is a good pair.*

Proof. (1) If f is constant with value c then $f(x) \approx g(x)$ is equivalent to $c \approx g(0) \approx g(1)$. The assertions (2),(3),(4) and (5) are also obvious. Let us prove (6). If $f^{-1}(1) \ne g^{-1}(1)$ then clearly $f(x) \approx g(x)$ is a consequence of its own consequences $a \approx b \approx c \approx d$. Let $f^{-1}(1) = g^{-1}(1)$ and $a \wedge d = b \wedge c$. Then the equations $f(x) \wedge d \approx g(x) \wedge b$, $f(x) \wedge b \approx f(x)$ and $g(x) \wedge d \approx g(x)$ belong to $Eq(N)$, so that $f(x) \approx g(x)$ is a consequence of $b \approx d$. $\square$

3. Representing infinitely distributive algebraic lattices with compact 1

3.1. PROPOSITION. *Every infinitely distributive algebraic lattice with compact 1 is isomorphic to the congruence lattice of a well-behaved 0,1-semilattice with operators.*

Proof. It is well known that every infinitely distributive algebraic lattice is isomorphic to the lattice of hereditary subsets of a partially ordered set $P(\leq)$. If the greatest element of the lattice is compact then it is easy to see that P contains only a finite number of maximal elements $p_1, \ldots, p_n$ and that for every $a \in P$ there exists an i with $a \leq p_i$. Put

$$P_i = \{a \in P; a < p_i \& a \not\leq p_1 \& \ldots \& a \not\leq p_{i-1}\}$$

for any $i = 1, \ldots, n$, so that $P_1 = \{a \in P; a < p_1\}$ and $P = P_1 \cup P_2 \cup \cdots \cup P_n \cup \{p_1, \ldots, p_n\}$ is a disjoint union. Take a new element $p_0 \notin P$, put $L = P \cup \{p_0\}$ and define a partial ordering $\preceq$ on L as follows: p_0 is the least element of L with respect to $\preceq$; if $a \in P_i \cup \{p_i\}$ and $b \in P_j \cup \{p_j\}$ then $a \preceq b$ iff either $a = b$ or $i < j$ or $i = j$ and $b = p_i$. Clearly, L is a lattice; denote by $\wedge$ and $\vee$ its lattice operations.

For $b, c, d \in L$ define a mapping $h_{b,c,d}$ of L into itself by

$$h_{b,c,d}(a) = \begin{cases} d, & \text{for } b \preceq a, \\ a \wedge c & \text{otherwise.} \end{cases}$$

Denote by H the set of the mappings $h_{b,c,d}$ such that $d < b$ in P, $d \in P_i$ and $c = p_{i-1}$ for some $i \in \{1, \ldots, n\}$. It is easy to verify that H is a set of endomorphisms of $L(\wedge)$.

Claim 1: *For $h_{b,c,d} \in H$ and $a \in L$ we have*

 (1) $h_{b,c,d}(a) \wedge d = h_{b,c,d}(a)$,
 (2) $h_{b,c,d}(a) \wedge c = a \wedge c$,
 (3) $h_{b,c,d}(a \wedge b) = h_{b,c,d}(a)$.

Proof: It is easy.

Claim 2: *The congruence lattice of $L(\wedge, H)$ is isomorphic to the lattice of hereditary subsets of $P(\leq)$.*

Proof: For a congruence r of $L(\wedge, H)$ denote by $h(r)$ the set of the elements $a \in P$ such that $(a, b) \in r$ for an element b covered by a in $L(\preceq)$. Then $h(r)$ is a hereditary subset of P, since if $(a, b) \in r$ where b is covered by a in $L(\preceq)$ and $a' \in P$ is an element with $a' < a$ in P then we can take an element $b' \in L$ covered by a' and the operator $h_{a,b',a'}$ maps (a, b) onto (a', b'), so that $(a', b') \in r$ and we get $a' \in h(r)$.

Conversely, for any hereditary subset U of $P(\leq)$ denote by $c_0(U)$ the set of the ordered pairs $(a, b) \in L^2$ such that b is covered by a in L and $a \in U$; and denote by $c(U)$ the equivalence generated by $c_0(U)$. In order to prove that $c(U)$ is a congruence of $L(\wedge, H)$, it is sufficient to show that if $(a, b) \in c_0(U)$ then $(a \wedge z, b \wedge z) \in c(U)$ for any $z \in L$ and $(h_{p,q,r}(a), h_{p,q,r}(b)) \in id_L \cup c_0(U) \cup \{(a, b)\}$ for any $h_{p,q,r} \in H$. The first assertion is trivial in all cases except for the case when $a = p_i$, $b, z \in P_i$ and $b \neq z$; but then $(a \wedge z, b \wedge z) = (z, p_{i-1})$, p_{i-1} is covered by z in $L(\preceq)$ and $z \in H$ (since H is hereditary and $z < a \in H$)), so that $(z, p_{i-1}) \in c_0(U)$. In order to prove the second assertion, let $(h_{p,q,r}(a), h_{p,q,r}(b))$ belong to neither id_L nor $\{a, b\}$. Then $a = p$ and $(h_{p,q,r}(a), h_{p,q,r}(b)) = (r, q) \in c_0(U)$, since $r < a$ and H is hereditary.

The mappings h and c are clearly both order-preserving and it remains to prove $h(c(U)) = U$ and $c(h(r)) = r$. The first assertion follows from the fact (which is easy to prove) that if $(a, b) \in c(U)$ and b is covered by a in $L(\preceq)$ then $(a, b) \in c_0(U)$. In order to prove the second assertion, it is sufficient to show that if $(a, b) \in c_0(h(r))$ then $(a, b) \in r$. We have $a \in h(r)$ and so there exists an element b' covered by a in $L(\preceq)$ with $(a, b') \in r$. If $b = b'$ then we are through; so, let $b \neq b'$. Then $a = p_i$ for some $i > 0$; the operator $h_{a,p_{i-1},b'}$ maps a onto b' and b' onto p_{i-1}, so that $(b', p_{i-1}) \in r$; we get $(a, p_{i-1}) \in r$ and consequently $(a, b) \in r$. This ends the proof of Claim 1.

It follows that the originally given infinitely distributive algebraic lattice with compact 1 is isomorphic to the congruence lattice of $L(\wedge, H)$ and so it remains to prove that this semilattice with operators is well-behaved.

Denote by F the union of H with $\{k_c; c \in L\}$ and $\{m_c; c \in L\}$.

Claim 3: *F is closed under superposition.*

Proof: Take two operators $h_{b,c,d}$ and m_e from F. If $d \preceq e$ then $h_{b,c,d}(a) \wedge e = h_{b,c,d}(a) \wedge d \wedge e = h_{b,c,d}(a) \wedge d = h_{b,c,d}(a)$, so that $m_e \circ h_{b,c,d} = h_{b,c,d}$. In the contrary case we have $e \wedge d \preceq c$ and so $h_{b,c,d}(a) \wedge e = h_{b,c,d}(a) \wedge d \wedge e = h_{b,c,d}(a) \wedge c \wedge e = a \wedge c \wedge e$, so that $m_e \circ h_{b,c,d} = m_{c \wedge e}$.

We have $h_{b,c,d}(a \wedge e) = h_{b,c,d}(a) \wedge h_{b,c,d}(e)$ and so it follows from what we have already proved that $h_{b,c,d} \circ m_e \in F$.

Now take two operators $h_{b,c,d}$ and $h_{b',c',d'}$ from F. If $b \not\preceq d'$ then $h_{b,c,d}(h_{b',c',d'}(a)) = h_{b,c,d}(h_{b',c',d'}(a) \wedge d') = h_{b,c,d}(h_{b',c',d'}(a)) \wedge h_{b,c,d}(d') = h_{b,c,d}(h_{b',c',d'}(a)) \wedge c \wedge d' = h_{b',c',d'} \wedge c \wedge d'$, so that $h_{b,c,d} \circ h_{b',c',d'} \in F$ by the above finished part of the proof. If $b = d'$ then $h_{b,c,d} \circ h_{b',c',d'} = h_{b',c,d}$, since for $b' \preceq a$ we have $h_{b,c,d}(h_{b',c',d'}(a)) = h_{b,c,d}(d') = h_{b,c,d}(b) = d$ and otherwise $h_{b,c,d}(h_{b',c',d'}(a)) = h_{b,c,d}(a \wedge c') = a \wedge c' \wedge c = a \wedge c$. If $b \prec d'$ then $b \preceq c'$ and $h_{b,c,d}(h_{b',c',d'}(a)) = h_{b,c,d}(h_{b',c',d'}(a) \wedge b) = h_{b,c,d}(h_{b',c',d'}(a) \wedge c' \wedge b) = h_{b,c,d}(a \wedge c' \wedge b) = h_{b,c,d}(a \wedge b) = h_{b,c,d}(a)$, so that $h_{b,c,d} \circ h_{b',c',d'} = h_{b,c,d}$.

Claim 4: *F is closed under meets.*

Proof: Let $h_{b,c,d}$ and $h_{b',c',d'}$ be two elements of H and e be an element of L. It is easy to prove, using Claim 1, that $h_{b,c,d} \wedge k_e$ belongs to F. Further, we have $h_{b,c,d} \wedge id_L = h_{b\vee d,c,d}$ and it remains to consider the meet $h_{b,c,d} \wedge h_{b',c',d'}$. If $d \prec d'$ then $d \preceq c'$ and $h_{b,c,d}(a) \wedge h_{b',c',d'}(a) = h_{b,c,d}(a) \wedge d \wedge h_{b',c',d'}(a) = h_{b,c,d}(a) \wedge d \wedge c' \wedge h_{b',c',d'}(a) = h_{b,c,d}(a) \wedge d \wedge c' \wedge a = h_{b,c,d}(a) \wedge d \wedge a = h_{b,c,d}(a) \wedge a$, so that $h_{b,c,d} \wedge h_{b',c',d'} = h_{b,c,d} \wedge id_L \in F$. If $d = d'$ then $c = c'$ and $h_{b,c,d} \wedge h_{b',c',d'} = h_{b\vee b',c,d}$. If d, d' are incomparable then $d \wedge d' = c = c'$ and so $h_{b,c,d}(a) \wedge h_{b',c',d'}(a) = h_{b,c,d}(a) \wedge h_{b',c',d'}(a) \wedge d \wedge d' = a \wedge c$, so that $h_{b,c,d} \wedge h_{b',c',d'} = id_l \wedge k_c$.

Claim 5: *$H' = F$.*

Proof: It follows from Claims 3 and 4.

Claim 6: *$L(\wedge, F)$ is well-behaved.*

Proof: By 2.1 it is sufficient to prove that the equation $f(x) \approx g(x)$, for any $f, g \in F$, is good. Denote by C the set of strictly constant consequences of this equation; we need to prove that $f(x) \approx g(x)$ belongs to the equational theory generated by the union of C with the set of equations satisfied in the nominal expansion of $L(\wedge, F)$.

Consider first the case when $f = h_{b,c,d}$ and $g = h_{b',c',d'}$. We have $(d, d') = (f(p_n), g(p_n)) \in C$. If d, d' are incomparable in $L(\preceq)$ then $c = c' = d \wedge d'$ and so the equations $c \approx d \approx c' \approx d'$ belong to C. If $d \prec d'$ then $c \prec d \prec c' \prec d'$, $(c, d) = (f(d), g(d)) \in C$ and so the equations $c \approx d \approx c' \approx d'$ belong to C again. If $d' \prec d$, we can proceed similarly. If $d = d'$ then $c = c'$; we can suppose that $b \not\preceq b'$; then $(c, d) = (f(b'), g(b')) \in C$ and we see that the equations $c \approx d \approx c' \approx d'$ belong to C in any case. On the other hand, each of the following equations is either satisfied in the nominal expansion of $L(\wedge, F)$ or is an immediate consequence of $c \approx d \approx c' \approx d'$:

$$h_{b,c,d}(x) \approx h_{b,c,d}(x) \wedge d \approx h_{b,c,d}(x) \wedge c \approx x \wedge c \approx x \wedge c' \approx \ldots \approx h_{b',c',d'}(x).$$

It remains to consider the case when $f = h_{b,c,d}$ and $g = m_a$. Since $(d, a) = (f(p_n), g(p_n))$, the equations $a \approx d$ and $a \wedge d \approx d$ belong to C. We further have $(c, a \wedge d) = (f(d), g(d))$ and so $c \approx d \approx a$ belong to C. On the other hand, each of the following equations is either satisfied in the nominal expansion of $L(\wedge, F)$ or is an immediate consequence of $c \approx d \approx a$:

$$h_{b,c,d}(x) \approx h_{b,c,d}(x) \wedge d \approx h_{b,c,d}(x) \wedge c \approx x \wedge c \approx x \wedge a.$$

This ends the proof of Proposition 3.1. $\square$

4. An auxiliary construction

In this section let $S(\wedge, F)$ be a 0,1-semilattice with operators such that $F = F'$. Put $T = S \cup \{1_T\}$ where 1_T is a new element not belonging to S (the greatest element of S will be denoted by 1_S). Define a semilattice operation $\wedge$ on T in such a way that $S(\wedge)$ becomes a subsemilattice of $T(\wedge)$ and 1_T becomes the greatest element of $T(\wedge)$. For every $f \in F$ define an endomorphism f^* of $T(\wedge)$ by $f^* \supseteq f$ and $f^*(1_T) = f(1_S)$. For every (possibly empty) filter X of $S(\wedge)$ define an endomorphism e_X of $T(\wedge)$ by

$$e_X(a) = \begin{cases} 1_T & \text{for } a \in X \cup \{1_T\}, \\ 1_S & \text{otherwise.} \end{cases}$$

Put $F^* = \{f^*; f \in F\}$ and denote by G the union of F^* with $\{id_T\}$ and the set of the operators e_X, where X runs over arbitrary filters of $L(\wedge)$.

4.1.PROPOSITION. *The congruence lattice of $T(\wedge, G)$ is isomorphic to the congruence lattice of $S(\wedge, F)$ with a new least element added. We have $G' = G$ and $T(\wedge, G)$ is well-behaved whenever $S(\wedge, F)$ is.*

Proof. For a congruence r of $S(\wedge)$ denote by r^* the equivalence on T having the same blocks as r with the exception of the block B containing 1_S, which is replaced by $B \cup \{1_T\}$. It is easy to verify that r^* is a congruence of $T(\wedge, G)$ and that every congruence of $T(\wedge, G)$, with the exception of id_T, is of this form. The first assertion follows and the rest of the proof will be divided into several parts.

Claim 1: $G' = G$.

Proof: The set G is closed under the meets and superposition, as

$$f^* \wedge g^* = (f \wedge g)^*,$$
$$f^* \wedge id_T = (f \wedge id_S)^*,$$
$$f^* \wedge e_X = f^*,$$
$$id_T \wedge e_X = id_T,$$
$$e_X \wedge e_Y = e_{X \cap Y},$$
$$f^* \circ g^* = (f \circ g)^*,$$
$$f^* \circ e_X = k_{f(1_S)},$$
$$e_X \circ f^* = e_{f^{-1}(u)} \quad \text{for} \quad f(1_S) \in X,$$
$$e_X \circ f^* = k_{1_S} \quad \text{for} \quad f(1_S) \notin X,$$
$$e_X \circ e_Y = 1_{k_T} \quad \text{for} \quad u \neq \emptyset,$$
$$e_X \circ e_Y = e_Y \quad \text{for} \quad u = \emptyset.$$

Moreover, G contains the identity and all constants. This ends the proof of Claim 1.

Denote by τ the type of the nominal expansion of $S(\wedge, F)$ and by τ^* the type of the nominal expansion of $T(\wedge, G)$. For any τ-term t denote by t^* the term obtained from t by replacing any $f \in F$ with f^* and denote by f^+ the term obtained from t^* by replacing any variable x with $x \wedge 1_S$.

If we say that an equation is satisfied in S, or in T, we mean that it is satisfied in the nominal expansion of $S(\wedge, F)$, or of $T(\wedge, G)$, respectively.

Claim 2: Let $t = t(x_1, \ldots, x_n)$ be a τ term. Then $t(a_1, \ldots, a_n) = t^*(a_1, \ldots, a_n)$ for any elements $a_1, \ldots, a_n \in S$. The equation $t^* \wedge 1_S \approx t^+$ is satisfied in T.

Proof: It is easy by induction on the length of t.

Claim 3: Let $t \approx u$ be an equation satisfied in S. Then $t^+ \approx u^+$ is satisfied in T.

Proof: It follows from Claim 2.

Claim 4: If $S(\wedge, F)$ is well-behaved then $T(\wedge, G)$ is well-behaved, as well.

Proof: Let p, q be two operators from G such that $p \leq q$. By 2.1 and 2.2(2) it is sufficient to prove that the equation $p(x) \approx q(x)$ is good (with respect to $T = T(\wedge, G)$). Denote by C the set of the strictly constant equations $p(a) \approx q(a)$ with $a \in T$ and $p(a) \neq q(a)$ and by Y the equational theory generated by C together with the equational theory of the nominal expansion of $T(\wedge, G)$. We need to prove that $p(x) \approx q(x)$ belongs to Y.

Consider first the case when $p = f^*$ and $q = g^*$ for some $f, g \in F$. Clearly, the set C coincides with the set of the strictly constant equations $f(a) \approx g(a)$ where a runs over S with $f(a) \neq g(a)$. Since $S(\wedge, F)$ is well-behaved, by 1.3 there exists a derivation of $f(x) \approx g(x)$ from C and the equations satisfied in S; this means that there exists a finite sequence $t_0, \ldots, t_m$ of τ-terms such that $t_0 = f(x)$, $t_m = g(x)$ and, for any $i \in \{1, \ldots, m\}$, either $t_{i-1} \approx t_i$ is satisfied in S or t_i can be obtained from t_{i-1} by replacing a constant b with a constant a, for an equation $a \approx b$ belonging to $C \cup C^{-1}$. If $t_{i-1} \approx t_i$ is satisfied in S then the equation $t_{i-1}^+ \approx t_i^+$ is satisfied in T by Claim 3 and thus belongs to Y. If t_i results from t_{i-1} by replacing a constant with a constant, then the same holds for the terms t_i^+ and t_{i-1}^+ and so $t_{i-1}^+ \approx t_i^+$ belongs to Y again. From this we get, by transitivity, that $f(x)^+ \approx g(x)^+$ belongs to Y. We have $f(x)^+ = f^*(x \wedge 1_S)$ and $g(x)^+ = g^*(x \wedge 1_S)$; since the equations $f^*(x) \approx f^*(x \wedge 1_S)$ and $g^*(x) \approx g^*(x \wedge 1_S)$ are satisfied in T, it follows that the equation $f^*(x) \approx g^*(x)$, i.e., the equation $p(x) \approx q(x)$ belongs to Y.

Next consider the case when $p = f^*$ for some $f \in F$ and $q = id_T$. Put $r = id_S^*$. We have $p \leq r \leq q$; by what we have already proved, the pair (p, r) is good and so, by 2.2(3), it is sufficient to prove that (r, q) is good. However, the equation $r(x) \approx q(x)$ is a consequence of its strictly constant consequence $1_S \approx 1_T$.

If $p = f^*$ and $q = e_X$ then $p(x) \approx q(x)$ is a consequence of its strictly constant consequence $f(0) \approx 1_T$.

If $p = e_X$ and $q = e_Y$ where X, Y are two distinct filters then $p(x) \approx q(x)$ is a consequence of its strictly constant consequence $1_S \approx 1_T$.

If $p = id_T$ and $q = e_X$ then $p(x) \approx q(x)$ is a consequence of its strictly constant consequence $0 \approx 1_T$.

This ends the proof of Proposition 4.1. □

5. The parallel join

In this section let $S_1(\wedge, F_1)$ and $S_2(\wedge, F_2)$ be two 0,1-semilattices with operators such that $F_1' = F_1$ and $F_2' = F_2$, and denote by $T_1(\wedge, G_1)$ and $T_2(\wedge, G_2)$ the operator semilattices constructed from S_1 and S_2 as in Section 4.

Put $U(\wedge) = T_1(\wedge) \times T_2(\wedge)$. If f_1 is an endomorphism of $T_1(\wedge)$ and f_2 is an endomorphism of $T_2(\wedge)$ then $f_1 \times f_2$ denotes (there is a little inconsistency) the endomorphism f of $U(\wedge)$ defined by $f(a, b) = (f_1(a), f_2(b))$; if Q_1 and Q_2 are two sets of endomorphisms on T_1 and T_2 then $Q_1 \times Q_2$ denotes the set $\{f_1 \times f_2; f_1 \in Q_1, f_2 \in Q_2\}$.

Recall that k_a denotes the constant endomorphism with value a; in order to avoid too many indices in formulas, we shall sometimes denote it simply by a. For $a, b \in S_i$ (where $i \in \{1, 2\}$) such that $a < b$ define an endomorphism $k_{a,b}$ of T_i by

$$k_{a,b}(c) = a \quad \text{for} \quad c \in S_i,$$

$$k_{a,b}(1_{T_i}) = b.$$

Put

$$
\begin{aligned}
H = &\{k_{a,b} \times k_c; a, b \in S_1, c \in S_2, a < b\} \cup \{k_c \times k_{a,b}; c \in S_1, a, b \in S_2, a < b\} \cup \\
&\{f \times id_{T_2}; f \in G_2\} \cup \{id_{T_1} \times f; f \in G_2\} \cup \\
&\{e_X \times 1_{t_2}; X \in \mathcal{F}_1\} \cup \{1_{T_1} \times e_X; X \in \mathcal{F}_2\}
\end{aligned}
$$

where $\mathcal{F}_i$ is the set of filters of $S_i(\wedge)$.

5.1. PROPOSITION. *The congruence lattice of $U(\wedge, H)$ is isomorphic to the parallel join of the congruence lattices of $S_1(\wedge, F_1)$ and $S_2(\wedge, F_2)$. If both $S_1(\wedge, F_1)$ and $S_2(\wedge, F_2)$ are well-behaved then $U(\wedge, H)$ is well-behaved, as well.*

Proof. For a congruence r of $S_1(\wedge, F_1)$ we can define a congruence s of $U(\wedge, H)$ by $((a, b), (c, d)) \in s$ iff $b = d$ and either $b \neq 1_{T_2}$ or $(a, b) \in r^*$. In a similar way each congruence on $S_2(\wedge, F_2)$ gives a congruence of $U(\wedge, H)$ and it is not difficult to prove that any nonextreme congruence of $U(\wedge, H)$ is of one of these two kinds. From this the first assertion follows. Let now both $S_i(\wedge, F_i)$ be well-behaved; we are going to show that $U(\wedge, H)$ is well-behaved, too.

Recall that H' denotes the closure, under superposition and the meets, of H together with the set of the constant operators and the identity on U. For $i = 1, 2$ denote by M_i the set of all endomorphisms of $T_i(\wedge)$, by N_i the set of the endomorphisms of $T_i(\wedge)$ mapping T_i into S_i, by E_i the set of the endomorphisms e_X for a filter X of $S_i(\wedge)$ and by K_i the union of E_i with the set of constant endomorphisms of T_i.

Claim 1: $H' \subseteq (M_1 \times N_2) \cup (G_1 \times \{id_{T_2}\}) \cup (K_1 \times E_2)$.

Proof: Denote the right side by H_0. Each of the three sets of operators is clearly closed under superposition and meets and H_0 contains H, all the constants and the identity on U. Further, we have

$$
\begin{aligned}
(M_1 \times N_2) \wedge H_0 &\subseteq M_1 \times N_2, \\
(G_1 \times \{id\}) \wedge (K_1 \times E_2) &\subseteq G_1 \times \{id\}, \\
(M_1 \times N_2) \circ H_0 &\subseteq M_1 \times N_2, \\
(G_1 \times \{id\}) \circ (M_1 \times N_2) &\subseteq M_1 \times N_2, \\
(K_1 \times E_2) \circ (M_1 \times N_2) &\subseteq (M_1 \times N_2) \cup (K_1 \times E_2), \\
(G_1 \times \{id\}) \circ (K_1 \times E_2) &\subseteq K_1 \times E_2, \\
K_1 \times E_2) \circ (G_1 \times \{id\}) &\subseteq K_1 \times E_2.
\end{aligned}
$$

Claim 2: *Let $f \in H'$. Then $f = p \times q$ for some p and q; if $q(1_{T_2}) = 1_{T_2}$ then $p \in G_1$ and either $q = id_{T_2}$ or $q = e_X$ for some filter X of $S_2(\wedge)$.*
Proof: It follows from Claim 1.

Let $p_1 \times q_1$ and $p_2 \times q_2$ be two operators from H' such that $p_1 \le p_2$ and $q_1 \le q_2$. By 2.2(2) and 2.1 it remains to prove that the equation $(p_1 \times q_1)(x) \approx (p_2 \times q_2)(x)$ is good. Denote by C the set of strictly constant consequences of this equation and by Z the equational theory of the nominal expansion of $U(\wedge, H)$. We need to prove that $(p_1 \times q_1)(x) \approx (p_2 \times q_2)(x)$ belongs to the equational theory generated by $C \cup Z$.

Consider first the case when $p_1 \ne p_2$ and $q_1 \ne q_2$. It follows from $p_1 \ne p_2$ that there exist elements $a, b \in T_1$ and $c \in T_2$ such that $a \ne b$ and $(a, c) \approx (b, c)$ belongs to C. Using the operator $e_X \times 1_{T_2}$ for an appropriate X we get from this that $(1_{S_1}, 1_{T_2}) \approx (1_{T_1}, 1_{T_2})$ belongs to C. Quite similarly, $(1_{T_1}, 1_{S_2}) \approx (1_{T_1}, 1_{T_2})$ belongs to C. Using the operators $k_{a,b} \times k_c$ and $k_c \times k_{a,b}$, it is now easy to see that $0_U \approx 1_U$ belongs to C. The equation $(p_1 \times q_1)(x) \approx (p_2 \times q_2)(x)$ (or any equation at all) is a consequence of $0_U \approx 1_U$.

Next consider the case when $p_1 \ne p_2$, $q_1 = q_2 = q$ and $q(1_{T_2}) \ne 1_{T_2}$. Similarly as in the previous case, the equation $(0_{T_1}, q(1_{T_2})) \approx (1_{T_1}, q(1_{T_2}))$ belongs to C. Further, the equations

$$(p_1 \times q)(x) \approx (p_1 \times q)(x) \wedge (1_{T_1}, q(1_{T_2})),$$

$$(p_2 \times q)(x) \approx (p_2 \times q)(x) \wedge (1_{T_1}, q(1_{T_2})),$$

$$(p_1 \times q)(x) \wedge (0_{T_1}, q(1_{T_2})) \approx (p_2 \times q)(x) \wedge (0_{T_1}, q(1_{T_2})) \approx (0_{T_1} \times q)(x)$$

belong to Z; now it is clear that the equation $(0_{T_1}, q(1_{T_2})) \approx (1_{T_1}, q(1_{T_2}))$ belongs to the equational theory generated by $C \cup Z$.

Since the case when $p_1 = p_2$ is analogous, it now remains to consider the case when $p_1 \ne p_2$, $q_1 = q_2 = q$ and $q(1_{T_2}) = 1_{T_2}$. By Claim 2 we have $p_1, p_2 \in G_1$ and $q \in \{id\} \cup E_2$.

Let $q \ne id$, so that $q \in E_2$. By Claim 1 we have $p_1, p_2 \in K_1$. Denote by a the least element in the range of p_1 and by b the greatest element in the range of p_2. Considering the various cases for $p_1, p_2 \in K_1$, it is easy to verify that $a \approx b$ is a strictly constant consequence of $p_1(x) \approx p_2(x)$; similarly, $(a, 1_{T_2}) \approx (b, 1_{T_2})$ is a strictly constant consequence of $(p_1 \times q)(x) \approx (p_2 \times q)(x)$. Since the equations

$$(p_1 \times q)(x) \approx (p_1 \times q)(x) \wedge (b, 1_{T_2}),$$

$$(p_2 \times q)(x) \approx (p_2 \times q)(x) \wedge (b, 1_{T_2}),$$

$$(p_1 \times q)(x) \wedge (a, 1_{T_2}) \approx (p_2 \times q)(x) \wedge (a, 1_{T_2}) \approx (a, 1_{T_2})$$

belong to Z, it follows that the equation $(p_1 \times q)(x) \approx (p_2 \times q)(x)$ belongs to the equational theory generated by $C \cup Z$.

It remains to consider the case when $q = id_{T_2}$. For every term t in the signature τ_1 of the nominal expansion of $T_1(\wedge, G_1)$ define a term t' in the signature τ of the nominal expansion of $U(\wedge, H)$ in the following way: t' is obtained from t by replacing any unary operation symbol $f \in G_1$ with $f \times id_{T_2}$ and any nullary symbol $a \in T_1$ with $(a, 1_{T_2})$. It is easy to prove by induction on the length of t that if t is a τ_1-term containing no other variable than x and if we are given an element (a, b) of U then $t'((a, b)) = (t(a), b)$ in the case when x occurs in t and $t'((a, b)) = (t(a), 1_{T_2})$ in the opposite case. Since (by Proposition 4.1) $T_1(\wedge, G_1)$ is well-behaved, the equation $p_1(x) \approx p_2(x)$ is good; by 1.3 there exists a finite sequence $u_0, \ldots, u_m$ of τ_1-terms such that $u_0 = p_1(x)$, $u_m = p_2(x)$ and such that for any $i \in \{1, \ldots, m\}$ either $u_{i-1} \approx u_i$ is satisfied in the nominal expansion of $T_1(\wedge, G_1)$ or u_i results from u_{i-1} by replacing an occurrence of b with a, for an equation $a \approx b$ belonging to $M \cup M^{-1}$ where $M = \{(p_1(c), p_2(c)); c \in T_1\}$. We can suppose that the terms $u_0, \ldots, u_m$ contain no other variables than x (since in the opposite case we could replace all the other variables by x) and that x occurs in every one of the terms $u_0, \ldots, u_m$ (since otherwise we could replace the derivation $u_0, \ldots, u_m$ by $u_0, u_0 \wedge k, u_1 \wedge k, \ldots, u_m \wedge k, u_m$ where k is the constant operator with value 1_{T_1}). If i is such that $u_{i-1} \approx u_i$ is satisfied in the nominal expansion of $T_1(\wedge, G_1)$ then $u'_{i-1} \approx u'_i$ belongs to Z, since for any $(a, b) \in U$ we have $u'_{i-1}((a, b)) = (u_{i-1}(a), b) = (u_i(a), b) = u'_i((a, b))$ by the assertion that was above proved by induction on the length of a term. If u_i results from u_{i-1} by replacement of a nullary symbol a with b, for $(a, b) \in M \cup M^{-1}$, then u'_i results from u'_{i-1} by replacement of $(a, 1_{T_2})$ with $(b, 1_{T_2})$, and $(a, 1_{T_2}) \approx (b, 1_{T_2})$ clearly belongs to C. We see that for any $i \in \{1, \ldots, m\}$ the equation $u'_{i-1} \approx u'_i$ belongs to the equational theory generated by $C \cup Z$.

Consequently, the equation $(p_1 \times q)(x) \approx (p_2 \times q)(x)$, which is nothing else than $u'_0 \approx u'_m$, belongs to the equational theory too. $\square$

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