Remarks on equational theories of semilattices with operators

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0. Introduction

Let us call a lattice L representable by a universal algebra A if it is isomorphic to the lattice of equational theories extending the equational theory of A (or, which is the same, antiisomorphic to the lattice of subvarieties of the variety generated by A). There are some restricting conditions on a lattice to be representable; cf. W. Lampe [3]. We have shown in [1] that in many simple cases when there is a hope for L to be representable, nominal semilattices with operators are good candidates for the algebras establishing the representation. (By "nominal" we mean that all the constants are added as fundamental nullary operations, and by an operator we mean an endomorphism of a semilattice.) For example, it is not difficult to represent the pentagon or, more generally, the parallel join of any two finite chains, by a finite nominal semilattice with operators. Of course, not every representable lattice can be represented in this way: it follows from D. Papert [5] that the congruence lattice of any algebra containing a semilattice operation among its fundamental operations is necessarily pseudodistributive.

For an algebra A denote by $\mathrm{Eq}(A)$ the lattice of equational theories extending the equational theory of A. An algebra A is said to be well-behaved if it is nominal and the lattice $\mathrm{Eq}(A)$ is canonically isomorphic to the congruence lattice of A. In [1] we were concerned with well-behaved semilattices with operators, and the concept was further discussed in [4]. In the present paper we are going to investigate (nominal) semilattices with operators that are not well-behaved.

An equation is said to be good with respect to a universal algebra A if it is a consequence of its own constant consequences together with the equations satisfied in A. Then A is well-behaved iff any equation of the appropriate similarity type is good with respect to A. In Lemma 1.2 we shall find an effective method to decide whether an equation is good with respect to a semilattice with operators. Using this criterion, it would be possible to reduce a little the length of several proofs in the paper [1].

In Section 2 we characterize the well-behaved chains with one operator (including the infinite ones).

In Section 3 we describe an effective method to find the lattice represented by a given finite nominal chain A with one operator. Moreover, every equational theory extending the equational theory of A is effectively described, given its generating equations.

1. Good equations in general semilattices with operators

Let $A = (A, \wedge, F)$ be a semilattice with operators. We denote by F' the set of unary term functions of A, i.e., the least monoid containing F and closed under meets. Further, we denote by F'' the set of unary polynomials of A, i.e., the least monoid containing F and all the constants and closed under meets.

This paper was written at the Technische Hochschule Darmstadt.

A pair of polynomials f, g is said to be good (with respect to A) if the equation $f(x) \approx g(x)$ belongs to the equational theory generated by the equations satisfied in the nominal expansion of A and the equations $f(a) \approx g(a)$ $(a \in A)$. If all the pairs of polynomials of A are good then A is said to be well-behaved.

For a pair f, g of polynomials of A we denote by $R(f, g)$ the congruence of A generated by the pairs $(f(a), g(a))$, with a running over the elements of A.

1.1. Lemma. *Let $A = (A, \wedge, F)$ be a semilattice with operators. A pair of polynomials $f, g \in F''$ is good with respect to A iff there exist a sequence $h_0, \dots, h_k$ $(k \geq 0)$ of polynomials and a sequence $(c_1, d_1), \dots, (c_k, d_k)$ of ordered pairs belonging to $R(f, g)$ such that $f = h_0$, $g = h_k$ and $h_{i-1} \leq c_i$, $h_i \leq d_i$ and $h_{i-1} \wedge d_i = h_i \wedge c_i$ for all $i \in \{1, \dots, k\}$.*

Proof. Let f, g be a good pair. Denote by E the equational theory of the nominal expansion of A and define a binary relation R on the set of terms (in the signature of the nominal expansion) as follows: $(u, v) \in R$ iff there exist a sequence $u_0, \dots, u_k$ $(k \geq 0)$ of terms and a sequence $(c_1, d_1), \dots, (c_k, d_k)$ of ordered pairs from $R(f, g)$ such that $(u, u_0) \in E$, $(v, u_k) \in E$ and whenever $i \in \{1, \dots, k\}$ then $(u_{i-1}, u_{i-1} \wedge c_i) \in E$, $(u_i, u_i \wedge d_i) \in E$ and $(u_{i-1} \wedge d_i, u_i \wedge c_i) \in E$. One can easily verify that R is a fully invariant congruence containing both E and $R(f, g)$. Since f, g is a good pair, we have $(f(x), g(x)) \in R$; for $u = f(x)$ and $v = g(x)$ there exist terms u_i and pairs (c_i, d_i) as above; and we can assume that the terms u_i contain no variables other than x, as they could otherwise be replaced with the terms $s(u_i)$, where s is the substitution sending any variable to x. Now the unary polynomials h_i corresponding to the terms u_i do the job. This proves the direct part of the iff statement, and the converse follows from

$$h_0(x) \approx h_0(x) \wedge c_1 \approx h_0(x) \wedge d_1 \approx h_1(x) \wedge c_1 \approx h_1(x) \wedge d_1 \approx h_1(x) \approx \dots \approx h_k(x). \quad \square$$

1.2. Lemma. *Let $A = (A, \wedge, F)$ be a semilattice with operators containing the largest element 1. A pair of polynomials $f, g \in F''$ is good iff there exists an element $c \in A$ such that $c \leq f(1) \wedge g(1)$, the elements $c, f(1), g(1)$ are all contained in one block of $R(f, g)$ and $f \wedge c = g \wedge c$.*

Proof. Let f, g be good, so that there exist h_i, c_i, d_i as in 1.1. Denote the congruence $R(f, g)$ by $\sim$. For $i \in \{1, \dots, k\}$ we have $h_{i-1}(1) = h_{i-1}(1) \wedge c_i \sim h_{i-1}(1) \wedge d_i = h_i(1) \wedge c_i \sim h_i(1) \wedge d_i = h_i(1)$. Put $c = h_0(1) \wedge \dots \wedge h_k(1)$, so that $c \sim f(1) \wedge g(1)$ and $c \leq f(1) \wedge g(1)$. For $i \in \{1, \dots, k\}$ we have $h_{i-1} \wedge c_i \wedge d_i = h_i \wedge c_i \wedge d_i$ and $c \leq c_i \wedge d_i$, so that $h_{i-1} \wedge c = h_i \wedge c$; consequently, $f \wedge c = g \wedge c$. The converse is clear. $\square$

1.3. Lemma. *Let $A = (A, \wedge, F)$ be a well-behaved semilattice with operators. Then A contains both the least and the greatest elements.*

Proof. Put $f = id_A$ and let g be an arbitrary constant. Since f, g is a good pair, there exist k and h_i, c_i, d_i as in 1.1. If $\mathrm{Card}(A) > 1$ then $k \geq 1$ and we have $id_A = h_0 \leq c_1$, so that c_1 is the largest element of A. Put $c = c_1 \wedge \dots \wedge c_k \wedge d_1 \wedge \dots \wedge d_k$. It it follows from $h_{i-1} \wedge c_i \wedge d_i = h_i \wedge c_i \wedge d_i$ that $f \wedge c = g \wedge c$. Hence $x \wedge c = g \wedge c$ for all $x \in A$, i.e., c is the least element of A. $\square$

2. Well-behaved semilattices with one operator

In this section let $A = (A, \wedge, f)$ be a semilattice with one operator f and greatest element 1. Put $F = \{f\}$. The set F' of unary term functions consists of the operators $f^{i_1} \wedge \dots \wedge f^{i_k}$ with $k \geq 1$ and $0 \leq i_1 < \dots < i_k$. The set F'' of unary polynomials consists of the operators $g \wedge c$ with $g \in F'$ and $c \in A$.

2.1. Lemma. *Let $g, h \in F'$ be two unary term functions and $x, y \in A$. Then $(x, y) \in R(g, h)$ iff there exists a sequence $x_0, \ldots, x_k$ ($k \geq 0$) such that $x = x_0$, $y = x_k$ and such that for any $i \in \{1, \ldots, k\}$ there are elements $a, d \in A$ with $\{x_{i-1}, x_i\} = \{g(a) \wedge d, h(a) \wedge d\}$.*

Proof. Denote by R the binary relation defined by $(x, y) \in R$ iff there exist $x_0, \ldots, x_k$ as above. It is clear that $R \subseteq R(g, h)$ and that R is an equivalence containing all the pairs $(g(x), h(x)$. Also, it is clear that $(x, y) \in R$ implies $(x \wedge a, y \wedge a) \in R$ for any $a \in A$. So, it remains to prove that if $(x, y) \in R$ then $(f(x), f(y)) \in R$. For this it is sufficient to prove that if $a, d \in A$ then $\{fg(a) \wedge d, fh(a) \wedge d\} = \{g(b) \wedge d, h(b) \wedge d\}$ for some $b \in A$. Since $fg = gf$ and $fh = hf$, we can put $b = f(a)$. (The fact that f commutes with any element of F' follows from $F = \{f\}$; notice that 2.1 is not necessarily true when F is arbitrary, or when g, h are unary polynomial functions instead of term functions.) $\square$

2.2. Lemma. *Let $g, h \in F'$ be two unary term functions and $a \in A$ be a constant. If g, h is a good pair then $g \wedge a, h \wedge a$ is a good pair too.*

Proof. As it easily follows from 1.2, we shall be done if we prove that if $(x, y) \in R(g, h)$ then $(x \wedge a, y \wedge a) \in R(g \wedge a, h \wedge a)$. Let $(x, y) \in R(g, h)$, so that there exist $x_0, \ldots, x_k$ as in 2.1. If $\{x_{i-1}, x_i\} = \{g(b) \wedge d, h(b) \wedge d\}$ then it is clear that $\{x_{i-1} \wedge a, x_i \wedge a\} = \{g(b) \wedge a \wedge d, h(b) \wedge a \wedge d\}$. From this we get $(x \wedge a, y \wedge a) \in R(g \wedge a, h \wedge a)$. $\square$

2.3. Lemma. *A is well-behaved iff all the pairs g, h of unary term functions such that $g \leq h$ are good.*

Proof. Only the converse implication needs to be proved, and by [1] it is sufficient to show that if $p, q \in F'$, $p = g \wedge a$, $q = h \wedge b$ where $g, h \in F'$ and if $p \leq q$ then the pair p, q is good. Since $g \wedge a = g \wedge a \wedge b$, we can assume that $a \leq b$. Then we have $g \wedge a \leq g \wedge b \leq h \wedge b$ and it remains to show that both the pairs $g \wedge a, g \wedge b$ and $g \wedge b, h \wedge b$ are good. The last pair is good by 2.2, as it follows from the assumption by [1] that all pairs of unary term functions are good. Using 1.2, it is easy to see that also the pair $g \wedge a, g \wedge b$ is good. $\square$

2.4. Lemma. *Let $g, h \in F'$ be two unary term functions. If the pair g, h is good then the pair fg, fh is good too.*

Proof. It is easy to see, using 2.1, that if $(x, y) \in R(g, h)$ then $(f(x), f(y)) \in R(fg, fh)$. Now we can apply 2.1 to get the result. $\square$

2.5. Lemma. *A is well-behaved iff all the pairs $g \wedge f^i, g$ such that $g = f^{i_1} \wedge \ldots \wedge f^{i_k} \in F'$, $0 \leq i_1 < \ldots < i_k$ and either $i = 0$ or $i_1 = 0$ are good.*

Proof. Let all these pairs be good. It follows from 2.4 that all the pairs $g \wedge f^i, g$ with $g \in F'$ and $i \geq 0$ are good. Let $g, h \in F'$ be such that $g \leq h$. Then $g = h \wedge f^{i_1} \wedge \ldots \wedge f^{i_k}$ for some $i_1, \ldots, i_k \geq 0$. As the pairs $(h \wedge f^{i_1} \wedge \ldots \wedge f^{i_k}, h \wedge f^{i_1} \wedge \ldots \wedge f^{i_{k-1}}), (h \wedge f^{i_1} \wedge \ldots \wedge f^{i_{k-1}}, h \wedge f^{i_1} \wedge \ldots \wedge f^{i_{k-2}}), \ldots, (h \wedge f^{i_1}, h)$ are good, the pair (g, h) is good. So, we can apply 2.3. $\square$

2.6. Lemma. *Let A be well-behaved. Then there exist elements $c, e \in A$ with the following properties:*

(1) e is the largest fixpoint of f; we have $1 > f(1) > f^2(1) > \ldots > f^k(1) = e$ for some $k \geq 0$.
(2) For $x \in A$, $x \geq c$ iff $f^k(x) = e$ for some k.
(3) $f(x) \wedge c = x \wedge c$ for all $x \in A$.
(4) $f(x) \geq x \wedge e$ for all $x \in A$.

Proof. Denote by M the set of the elements $x \in A$ for which there exist $i, j \geq 0$ with $f^i(x) \geq f^j(1)$. Evidently, M is a filter of A and we have $x \in M$ iff $f(x) \in M$. The relation R defined by $(x, y) \in R$ iff either $x, y \in M$ or $x, y \notin M$ is easily seen to be a congruence of A containing all the pairs $(x, f(x)$. Since the pair f, id_A is good, it follows from 1.2 that there is an element $c \in M$ such that (3) is true. By (3) we get $c \le f(c)$. Since $c \in M$, there are nonnegative integers i, j with $f^i(c) \ge f^j(1)$. We have $f^k(c) \le f^k(1)$ for all k and hence $f^k(c) \le f^l(1)$ for all k, l. Consequently, $f^i(c) = f^j(1)$; the element $e = f^i(c) = f^j(1)$ is clearly the largest fixpoint of f and (1) is true. If $x \ge c$ then $f^i(x) \ge f^i(c) = e$ and hence $f^{i+j}(x) = e$. Conversely, if $f^k(x) = e$ for some k then $f^k(x) \wedge c = c$; but (3) yields $f^l \wedge c = id_A \wedge c$; hence $x \wedge c = c$, i.e., $x \ge c$. We have proved (2) and it remains to prove (4). Put $R = R(id, f \wedge id)$. If x, y are elements such that $\{x, y\} = \{a \wedge d, f(a) \wedge a \wedge d\}$ for some a, d then $x \ge e$ implies $y \ge e$. Indeed, if $e \le a \wedge d$ then $e \le a$, $e = f(e) \le f(a)$ and so $e \le f(a) \wedge a \wedge d$. Now this means, applying 2.1, that if $(x, y) \in R$ then $x \ge e$ iff $y \ge e$. In particular, $(e, x) \in R$ implies $x \ge e$. On the other hand, it is clear that $(e, 1) \in R$. Consequently, the principal filter generated by e is a block of R. Since the pair $id_A, f \wedge id_A$ is good, by 1.2 there exists an element $c' \ge e$ with $x \wedge c' = f(x) \wedge x \wedge c'$ for all x; but then $x \wedge e = f(x) \wedge x \wedge e$ for all x and (4) is true. □

2.7. Lemma. *Let there exist elements c, e with the four properties formulated in 2.6. Let $g = f^{i_1} \wedge \ldots \wedge f^{i_k}$ where $i_1 < \ldots < i_k$ and let $i > i_k$. Then the pair $g \wedge f^i, g$ is good.*

Proof. By (4) we have $f(x) \wedge x \wedge e = x \wedge e$ for all x. From this it is easy to prove $f^j(x) \wedge f^m(x) \wedge e = f^m(x) \wedge e$ for any j, m such that $m < j$. But then, $g(x) \wedge f^i(x) \wedge e = g(x) \wedge e$. Since $i > i_k$, it is easy to see that $(e, g(1)) \in R(g \wedge f^i, g)$. Now, it follows from 1.2 that the pair $g \wedge f^i, g$ is good. □

For the pairs not covered by Lemma 2.7 it seems that there is no uniform condition necessary and sufficient for their goodness. The next lemma is concerned with these pairs.

2.8. Lemma. *Let $g = f^{i_1} \wedge \ldots \wedge f^{i_k}$ where $i_1 < \ldots < i_k$ and let $0 \le i < i_k$. Denote by M the least subset of A such that $g(1) \in M$, if $g(x) \in M$ then $g(x) \wedge f^i(x) \in M$ and if $x \le y \le g(1)$ where $x \in M$ then $y \in M$. Then M is a filter in $\{x; x \le g(1)\}$ and we have $x \in M$ iff there exists a sequence $x_0, x_1, \ldots, x_k$ ($k \ge 0$) such that $x_0 = g(1)$, $x \ge x_k$ and whenever $i \in \{1, \ldots, n\}$ thene there is an element a with $g(a) \ge x_{i-1}$ and $x_i = g(a) \wedge f^i(a)$. The pair $g \wedge f^i, g$ is good iff there exists an element $d \in M$ such that $g(x) \wedge f^i(x) \wedge d = g(x) \wedge d$ for all $x \in A$; this element d is then the least element of M.*

Proof. The set of the elements x for which there exists a sequence as above is easily seen to be a filter in the principal ideal generated by $g(1)$ with the property of M, so that it coincides with M. It follows easily from 2.1 that M is just the block of $R(g \wedge f^i, g)$ containing $g(1)$. By 1.2, this means that the equation is good iff there exists an element $d \in M$ with $g(x) \wedge f^i(x) \wedge d = g(x) \wedge d$; it is easy to prove that then d must be the least element of M. □

2.9. Lemma. *Let A be a chain with one operator. Then A is well-behaved iff there exist elements $c, e \in A$ with the following properties:*

(1) e is the largest fixpoint of f; we have $1 > f(1) > f^2(1) > \ldots > f^k(1) = e$ for some $k \ge 0$.
(2) $c < f(c) < \ldots < f^l(c) = e$ for some $l \ge 0$.
(3) $f(x) = x$ for any $x < c$.
(4) Either $c = e$ or $e = f(1)$.

Proof. Let A be well-behaved, so that there exist elements c, e as in 2.6. The relation R defined by $(x, y) \in R$ iff either $x, y \le e$ or $x = y$ is easily seen to be a congruence of A containing all the pairs $(f(x) \wedge x, f(x))$. Since the pair $f \wedge id, f$ is good, by 1.2 there exists an element d in the block of R containing $f(1)$ such that $f(x) \wedge x \wedge d = f(x) \wedge d$ for all x. We have either $f(1) = e$ or $d = f(1)$. In the latter case $f(x) \leq x$ for all x; but then $e = c$. It is clear that all the elements less than c are fixpoints of f.

Now, let there exist elements c, e with the four properties. Consider first the case $c = e$. Then $id_A \geq f \geq f^2 \geq \dots$ and we have $F' = \{id_A, f, f^2, \dots$. By 2.5 and 2.7 it is enough to prove that id_A, f^m is a good pair for any $m > 0$. Clearly, $(c, 1) \in R(id, f)$ and $f^m(x) \wedge c = x \wedge c$ for all x, so that we can apply 1.2.

Next consider the case $e = f(1)$. If $e = 1$ then $id_A \leq f \leq f^2 \leq \dots$ and the proof is as simple as in the case $c = e$. So, let $c < e < 1$. Then F' consists of the mappings id_A, $f \wedge id_A$ and f^i $(i > 0)$. By 2.5 and 2.7 it is enough to prove that the pair $id_A \wedge f, f^m$ is good for any $m \geq 1$. Clearly, $(c, e) \in R(id_A \wedge f, f^m)$ and $e = f^m(1)$. Since $f(x) \wedge c = x \wedge c$, we have $x \wedge f(x) \wedge c = f^m(x) \wedge c$ for all x. Now we can apply 1.2. □

3. Equational theories of a finite chain with one operator

In this section let $A = (A, \wedge, f)$ be a finite chain with one operator f; denote by 0 the least and by 1 the largest element of A.

By an equation we shall mean an ordered pair of term functions (rather than terms) of the nominal expansion of A, i.e., a pair of polynomials of A. There are just two kinds of polynomials: the constants and the polynomials $g = f^i \wedge f^j \wedge a$ where $i \leq j$ and $a \leq f^j(1)$ (so that $a = g(1)$). (The reason for it is that $f^i \wedge f^j \wedge f^k = f^i \wedge f^k$ whenever $i \leq j \leq k$.) The polynomials of the second kind will be called composed. An equation (g, h) is called trivial if $g = h$ (i.e., if $g(x) = h(x)$ for all $x \in A$).

Let R be a congruence of A. An equation (g, h) is said to be R-valid if $(g(x) = h(x)) \in R$ for all $x \in A$.

An element $a \in A$ is called R-reduced if there is no $b < a$ with $(b, a) \in R$. For any element a denote by a^* the only R-reduced element such that $(a, a^*) \in R$. (This notation will be used only when R is fixed.) An equation (g, h) is called R-reduced if $g(1) = h(1)$ and $g(1)$ is an R-reduced element.

By an R-special equation we shall mean a nontrivial R-valid R-reduced equation (g, h), with $a = g(1) = h(1)$, which is either of the form $(g, h) = (f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a)$ with $i < j$ or of the form $(g, h) = (f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a)$ with $i \leq j$.

By an R-special set we shall mean a set S of R-special equations satisfying the following conditions:

(1) if $(f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a) \in S$ and $a \leq f^{j+1}(1)$ then the equation $(f^i \wedge f^{j+1} \wedge a, f^{i+1} \wedge f^{j+1} \wedge a)$ is either trivial or belongs to S;

(2) if $(f^{i+1} \wedge f^j \wedge a, f^{i+1} \wedge f^{j+1} \wedge a) \in S$ then $(f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a)$ is either trivial or belongs to S;

(3) if $(f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a) \in S$ and $(f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a) \in S$ then $(f^{i+1} \wedge f^j \wedge a, f^{i+1} \wedge f^{j+1} \wedge a)$ is either trivial or belongs to S;

(4) if $(f^i \wedge f^{j+1} \wedge a, f^{i+1} \wedge f^{j+1} \wedge a) \in S$ and $(f^{i+1} \wedge f^j \wedge a, f^{i+1} \wedge f^{j+1} \wedge a) \in S$ then $(f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a)$ is either trivial or belongs to S;

(5) if $(f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a) \in S$ and $a \leq f^{j+1}(1)$ then $(f^{i+1} \wedge f^{j+1} \wedge a, f^{i+2} \wedge f^{j+1} \wedge a)$ is either trivial or belongs to S;

(6) if $(f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a) \in S$ and $a \leq f^{j+2}(1)$ then $(f^{i+1} \wedge f^{j+1} \wedge a, f^{i+1} \wedge f^{j+2} \wedge a)$ is either trivial or belongs to S;

(7) if $(f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a) \in S$ and $b \leq a$ is an R-reduced element then $(f^i \wedge f^j \wedge b, f^{i+1} \wedge f^j \wedge b)$ is either trivial or belongs to S;

(8) if $(f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a) \in S$ and $b \leq a$ is R-reduced then $(f^i \wedge f^j \wedge b, f^i \wedge f^{j+1} \wedge b)$ is either trivial or belongs to S.

In the following let R be a congruence of A and S be an R-special set.

Denote by E_0 the union of S with the set of trivial equations.

Denote by E_1 the set of the R-valid R-reduced equations $(f^i \wedge f^j \wedge a, f^k \wedge f^j \wedge a)$ such that $i \leq j$, $k \leq j$ and $(f^c \wedge f^j \wedge a, f^{c+1} \wedge f^j \wedge a) \in E_0$ for any c with $\min(i,k) \leq c < \max(i,k)$.

Denote by E_2 the set of the R-valid R-reduced equations $(f^i \wedge f^j \wedge a, f^i \wedge f^k \wedge a)$ such that $i \leq j$, $i \leq k$ and $(f^i \wedge f^c \wedge a, f^i \wedge f^{c+1} \wedge a) \in E_0$ for any c with $\min(j,k) \leq c < \max(j,k)$.

Denote by E_3 the set of the R-valid R-reduced equations $(f^i \wedge f^j \wedge a, f^k \wedge f^l \wedge a)$ $(i \leq j,\ k \leq l)$ such that either $i \leq k$ and the equations $(f^i \wedge f^j \wedge a, f^i \wedge f^l \wedge a)$ and $(f^i \wedge f^l \wedge a, f^k \wedge f^l \wedge a)$ belong to $E_1 \cup E_2$ or else $k \leq i$ and the equations $(f^k \wedge f^l \wedge a, f^k \wedge f^j \wedge a)$ and $(f^k \wedge f^j \wedge a, f^i \wedge f^j \wedge a)$ belong to $E_1 \cup E_2$.

It is clear that both E_1 and E_2 are equivalences on the set of the composed polynomials g such that $g(1)$ is an R-reduced element. Also, the relation E_3 is symmetric and reflexive on this set. We need to prove that E_3 is transitive. For this sake, the element a can be considered fixed; we shall write $[i,j,k,l]$ instead of $(f^i \wedge f^j \wedge a, f^k \wedge f^l \wedge a) \in E_3$. (When this equation belongs to E_3, it is obvious that it belongs to E_1 if $j = l$ and to E_2 if $i = k$.) So, for $i \leq k$ we have $[i,j,k,l]$ iff $[i,j,i,l]$ and $[i,l,k,l]$.

It is useful first to realize that if $[i,j,i,k]$ then $[i',j,i',k]$ for any $i' \leq i$; and if $[i,j,k,j]$ then $[i,j',k,j']$ for any $j' \geq j$ such that $a \leq f^{j'}(1)$. These two facts follow from (1) and (2).

From (3) and (4) we get: if $[i,j,i',j]$ and $[i,j,i,j']$ where $i < i'$ and $j < j'$ then $[i,j',i',j']$; and if $[i,j,i',j]$ and $[i,j,i,j']$ where $i' < i$ and $j' < j$ then $[i,j',i',j']$.

The pairs i,j can be imagined as points in the plane, and the assertion $[i,j,k,l]$ paraphrased as "the points (i,j) and (k,l) are connected". Then the definition of $[i,j,k,l]$ can be stated as follows: two points are connected iff they are connected in both the horizontal the vertical direction with the third vertex of the left-side rectangular triangle which they determine. And the last two remarks imply that any two connected points lying on a vertical line can be shifted to the left; any two connected points lying on a horizontal line can be shifted up; if in a rectangle the left and the bottom vertices are connected then so are the opposite vertices too; and if the right and upper vertices are connected then so are the left and bottom ones. (Notice that a rectangle can be completed also if the bottom and the right vertices are connected; thus the only bad case is when the left and upper vertices are connected.) Finally, notice that the relation of connectedness is transitive on any vertical as well as on any horizontal line. Taking these remarks into account and distinguishing several cases, it is not difficult to see that the relation of connectedness is transitive on the plane. One can reduce the number of the cases a little by taking the following observation also into the account. In order to prove that $[i,j,k,l]$ and $[k,l,p,q]$ imply $[i,j,p,q]$, it is sufficient to prove the same under the assumption that either $k = p$ or $l = q$.

So, we can consider the transitivity of E_3 to be established. Now denote by E the set of the R-valid equations (g,h) such that either one of the polynomials g,h is constant or else $(g \wedge a, h \wedge a) \in E_3$ where $a = (g(1))^* = (h(1))^*$. Since E_3 is an equivalence, E is an equivalence on the set of all polynomials; we have $R = E \cap A^2$. We are now going to prove that E is an equational theory (i.e., a fully invariant congruence on the algebra of polynomials).

Using (7) and (8), it is easy to prove for $i = 1,2,3$ that if $(g,h) \in E_i$ and $b \leq g(1) = h(1)$ is an R-reduced element then $(g \wedge b, h \wedge b) \in E_i$. Consequently, $(g,h) \in E$ implies $(g \wedge a, h \wedge a) \in E$ for any $a \in A$.

By (1),(2) and (7) we get the following: if $(g,h) \in S$ and c is a nonnegative integer then $(g \wedge f^c \wedge b, h \wedge f^c \wedge b) \in E_0$, where $b = (g(1) \wedge f^c(1))^*$. From this we get for $i = 1,2,3$ that $(g,h) \in E_i$ implies $(g \wedge f^c \wedge b, h \wedge f^c \wedge b) \in E_i$ and we can conclude that if $(g,h) \in E$ then $(g \wedge f^c, h \wedge f^c) \in E$ for any c.

Similarly, using (5) and (6) one can show that $(g,h) \in E$ implies $(fg, fh) \in E$.

We have proved that E is a congruence. It is not difficult to verify that this congruence is fully invariant. Clearly, E is just the fully invariant congruence generated by the union $R \cup S$.

Conversely, if E is a fully invariant congruence of the algebra of polynomials such that $E \cap A^2 = R$ then E is uniquely determined by its intersection with the set of R-special equations, and this intersection is an R-special set. For example, let us prove that (4) is satisfied. Let $(f^i \wedge f^j \wedge a, f^{i+1} \wedge f^j \wedge a) \in E$ and $(f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a) \in E$. The first equation gives us $(f^i \wedge f^{j+1} \wedge a, f^{i+1} \wedge f^{j+1} \wedge a) \in E$; by transitivity we get $(f^{i+1} \wedge f^j \wedge a, f^{i+1} \wedge f^{j+1} \wedge a) \in E$. The conditions (5) and (6) can be proved by substituting $f(x)$ for x; this is better than to apply the congruence property with respect to f, since the latter approach could change the element a.

Given a congruence R, the corresponding interval in the lattice of fully invariant congruences of the algebra of polynomials is thus isomorphic to the lattice of R-special sets.

Let R, R' be two congruences of A. Further, let S be an R-special set and S' be an R'-special set. We shall write $(R,S) \leq (R',S')$ iff $R \subseteq R'$ and the following is true: whenever $(g,h) \in S$ and a is the least element of A with $(a, g(1)) \in R'$ then $(g \wedge a, h \wedge a)$ is either trivial or belongs to S'. It is easy to see that $(R,S) \leq (R',S')$ iff the fully invariant congruence generated by $R \cup S$ is contained in the fully invariant congruence generated by $R' \cup S'$.

Strictly speaking, equational theories are sets of ordered pairs of terms (in arbitrary variables) rather than of polynomials. However, it is easy to see that the lattice of equational theories extending the equational theory of the nominal expansion of A is isomorphic to the lattice of fully invariant congruences of the algebra of polynomials. Summarizing what has been proved and said, we get:

3.1. Theorem. *Let $A = (A, \wedge, f)$ be a finite chain with one operator and A' be the nominal expansion of A. The lattice of equational theories extending the equational theory of A' is isomorphic to the lattice of the ordered pairs (R,S) where R is a congruence of A and S is an R-special set, with respect to the ordering described above.* □

To obtain a picture of the lattice, one can proceed in the following way. First, draw a picture of the congruence lattice of A. (This is a distributive lattice; by [2], it belongs to the smallest class of lattices containing the two-element lattice and closed under finite products and ordinal sums with finite chains placed at the top; and any lattice from this class can be represented in this way.) Then replace any element of this lattice (it corresponds to a congruence R) with a picture of the lattice of R-special sets; and connect elements in the resulting various blocks according to the above described relation $\leq$.

In the special case when A contains a single fixpoint, there are no R-special equations of the form $(f^i \wedge f^j \wedge a, f^i \wedge f^{j+1} \wedge a)$. Consequently, some of the conditions (1)-(8) are empty in this case. Most significantly, the conditions (3) and (4) are empty. But then, for a given R, the union of any two R-special sets is again R-special, which means that the interval in the lattice of equational theories corresponding to R is a distributive lattice. We get:

3.2. Corollary. *Let $A = (A, \wedge, f)$ be a finite chain with one operator containing a single fixpoint. The lattice of equational theories extending the equational theory of the nominal expansion of A is distributive-by-distributive.* □

On the other hand, the following example shows that if A contains two fixpoints then the lattice of equational theories need not be distributive-by-distributive.

3.3. EXAMPLE. Let A be the five-element chain $\{0, 1, 2, 3, 4\}$ with the endomorphism $f : (0, 1, 2, 3, 4) \mapsto (1, 1, 1, 2, 4)$. The lattice of equational theories has 54 elements and is pictured in Fig. 1. In this picture two elements have the same label iff the corresponding equational theories intersect A^2 in the same congruence.

3.4. EXAMPLE. Let A be the four-element chain with the endomorphism $f : (0, 1, 2, 3) \mapsto (1, 1, 2, 2)$. The lattice of equational theories has 10 elements and is pictured in Fig. 2.

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