FINITE AXIOMATIZABILITY OF CONGRUENCE RICH VARIETIES

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0. Introduction

In this paper we introduce the notion of a congruence rich variety of algebras, and investigate which locally finite subvarieties of such a variety are relatively finitely based. We apply the results obtained to investigate the finite axiomatizability of an interesting variety generated by a particular five-element directoid.

Roughly speaking, congruence rich varieties are those in which all large finitely generated algebras have homomorphic images of moderate size. More precisely, a variety $\mathcal{V}$ is **congruence rich** provided for each positive integer n, there is a positive integer m such that every finitely generated algebra in $\mathcal{V}$ with more than n elements has a homomorphic image with more than n elements but no more than m elements.

In this paper we restrict our attention to varieties of finite similarity types—that is, the algebras in the varieties we investigate are always assumed to have only finitely many basic operations.

Among varieties of finite similarity type, congruence rich varieties are not uncommon. Any finitely generated congruence modular variety is congruence rich, as is any locally finite variety with a finite upper bound on the cardinalities of its finite subdirectly irreducible algebras. Any variety of directoids, which were introduced in [9] as algebraic renderings of up-directed sets, is congruence rich. Below we introduce varieties orderable-by-divisibility. These are also congruence rich, and the variety of directoids is a special case.

Let $\mathbf{A}$ be an algebra. By an $\boldsymbol{HS}$**-reduct** of $\mathbf{A}$ we mean any algebra in $\boldsymbol{HS}(\mathbf{A})$, i.e., any algebra which is a homomorphic image of a subalgebra of $\mathbf{A}$. By a **proper** $\boldsymbol{HS}$**-reduct** of $\mathbf{A}$ we mean one that is not isomorphic to $\mathbf{A}$ itself. In case $\mathbf{A}$ is a finite algebra, its proper $\boldsymbol{HS}$-reducts are those with cardinality smaller than the cardinality of $\mathbf{A}$. Let $\mathcal{U}$ be a variety. A finite algebra $\mathbf{A}$ is said to be **critical for the variety** $\mathcal{U}$ provided $\mathbf{A}$ does not belong to $\mathcal{U}$ but all of the proper $\boldsymbol{HS}$-reducts of $\mathbf{A}$ belong to $\mathcal{U}$. $\mathbf{A}$ is called **critical** provided it is critical for some variety. Evidently, every critical algebra is subdirectly irreducible. Critical groups have played a role in the theory of varieties of groups. The understanding of algebras critical for locally finite subvarieties of congruence rich varieties is a key to our results.

In Section 1 we present several conditions which characterize congruence richness. We provide, for each locally finite subvariety of a congruence rich variety, a "forbidden $\boldsymbol{HS}$-reduct" characterization in terms of critical algebras. We also present some examples of congruence rich varieties and some results on how to obtain new congruence rich varieties from those already at hand.

In Section 2 we focus on finite axiomatizability. We give a necessary and sufficient condition for a locally finite subvariety $\mathcal{U}$ of a congruence rich variety $\mathcal{V}$ to be finitely based relative to $\mathcal{V}$. Perhaps the most useful formulation is that only finitely many finite algebras, up to isomorphisn, in $\mathcal{V}$ should be critical for $\mathcal{U}$. Likewise, we characterize those locally finite subvarieties which are inherently nonfinitely based relative to a congruence rich variety. This characterization also depends on critical algebras.

Understanding finite axiomatizability of locally finite varieties, or even of varieties generated by a finite algebra, has proven to be very challenging. Roger Lyndon, in [12] gave the earliest example of a finite algebra which generates a variety that is not finitely axiomatizable. Such algebras are said to be nonfinitely based. On the other hand, all finite groups ([20]), all finite rings ([13] and [11]), and all finite lattices ([14]) are known to be finitely based. McKenzie in [15], extending the celebrated result of Kirby Baker [1], proved that every finite algebra with only finitely many fundamental operations which generates a residually small congruence modular variety must be finitely based. On the other hand, intriguing examples of finite algebras which are not finitely based—and which are even inherently nonfinitely based—can be found in [19], [21], [7], [22], [17], [2], [10], [8] and [24]. Most spectacularly, Ralph McKenzie [16] has recently proven that there is no algorithm for determining which finite algebras are finitely based (or inherently nonfinitely based). This theorem of McKenzie's means that effective conditions which are either necessary or sufficient for finite axiomatizability have added significance. The same applies to conditions which are necessary and sufficient in some restricted domain.

In the last part of our paper, we apply our results to analyze the variety generated by a particular finite directoid. A **directoid** is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ for which there is an upward directed partial ordering on A such that $a \cdot b$ is an upper bound of $\{a, b\}$ and must be the maximum of $\{a, b\}$ whenever $\{a, b\}$ is a chain.

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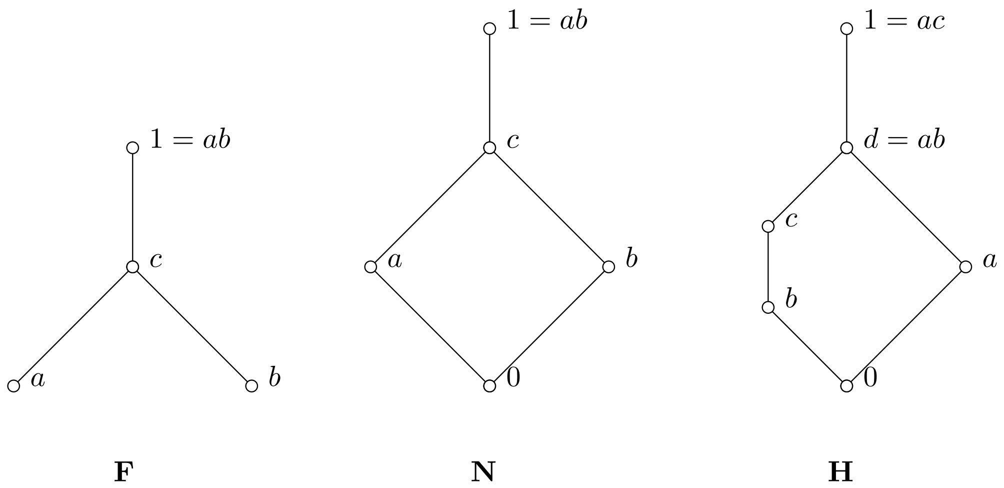

Figure 1

Figure 1 provides diagrams of the commutative directoids **F**, **N**, and **H**. **F** is called the fork and was shown in [9] to be finitely based. **H** was shown in [6] to be inherently nonfinitely based. It is the directoid **N** which we investigate. In Section 3, we prove that **N** is not finitely based. In Section 4, we show, on the other hand, that **N** fails to be inherently nonfinitely based.

For information about algebras and varieties of algebras, [5], [3], and [18] are good references. In this paper, we follow [18] in terminology and notation most closely. A similarity type is a function ρ whose domain is a set Δ of operation symbols F, each of which is associated with a nonnegative integer called the arity of F and denoted by $\rho(F)$. Groupoids are denoted multiplicatively. $a \cdot b$ is often written as ab, and, to avoid the build up of parentheses, $ab \cdot c$ means $(ab)c$. An equation is written as (u, v), or sometimes as $u = v$. $\mathbf{S}(t)$ denotes the support of a term t, the set of variables occurring in t. Semilattices are always join semilattices.

1. Congruence rich varieties

Our first theorem will be just a reformulation of the definition, the second will be the "forbidden ***HS***-reduct" characterization theorem, and then the rest of the section will contain mostly examples.

1.1. Theorem. *For a variety $\mathcal{V}$ of a finite similarity type, the following three conditions are equivalent:*

(1) *$\mathcal{V}$ is congruence rich.*
(2) *For every finite set F of finite $\mathcal{V}$-algebras there is a finite set F' of finite $\mathcal{V}$-algebras such that any finitely generated algebra from $\mathcal{V} - \boldsymbol{I}(F)$ has a homomorphic image in $F' - \boldsymbol{I}(F)$.*
(3) *For any finite $\mathcal{V}$-algebra $\mathbf{A}$ there is a finite set F' of finite $\mathcal{V}$-algebras such that every finitely generated algebra from $\mathcal{V} - \boldsymbol{H}(\mathbf{A})$ has a homomorphic image in $F' - \boldsymbol{H}(\mathbf{A})$.*

Proof. (1) implies (2). Given an F, denote by n the maximum of the cardinalities of the algebras in F; take the corresponding integer $m > n$; it is easy to verify that the set F' of the $\mathcal{V}$-algebras of cardinality at most m serves well for the given set F.

(2) implies (1). Given an n, take F to be the set of isomorphic copies of all the $\mathcal{V}$-algebras having at most n elements; by (2), there exists another finite set F' of finite $\mathcal{V}$-algebras; then one may take m to be the maximum of the cardinalities of the algebras in F' and prove easily that this number m has the desired property.

The equivalence of (2) with (3) follows from the fact that each finite collection of finite algebras can be replaced with its direct product to obtain an equally useful single finite algebra. □

1.2. Theorem. *Let $\mathcal{V}$ be a congruence rich variety of a finite similarity type and let $\mathcal{U}$ be a locally finite subvariety of $\mathcal{V}$. For every positive integer n there are, up to isomorphism, only finitely many n-generated $\mathcal{V}$-algebras critical for $\mathcal{U}$. An algebra $\mathbf{A} \in \mathcal{V}$ belongs to $\mathcal{U}$ if and only if no $\mathcal{V}$-algebra critical for $\mathcal{U}$ is an **HS**-reduct of $\mathbf{A}$.*

Proof. Given an n, there are up to isomorphism only finitely many n-generated algebras in $\mathcal{U}$. So, by the congruence richness of $\mathcal{V}$, there is a finite set F' of finite $\mathcal{V}$-algebras such that every n-generated algebra from $\mathcal{V} - \mathcal{U}$ has a homomorphic image in $F' - \mathcal{U}$. (A homomorphic image of an n-generated algebra is, of course, n-generated itself.) Now it is clear that every $\mathcal{V}$-algebra critical for $\mathcal{U}$ belongs to F'.

The direct implication in the second part of the theorem is clear. Let $\mathbf{A} \in \mathcal{V} - \mathcal{U}$. Then $\mathbf{A}$ has a finitely generated subalgebra $\mathbf{A}'$ not belonging to $\mathcal{U}$. If $\mathbf{A}'$ is infinite, then by the congruence richness of $\mathcal{V}$ there is a finite homomorphic image $\mathbf{A}''$ of $\mathbf{A}$, so big that $\mathbf{A}'' \notin \mathcal{U}$. (Note that $\mathbf{A}''$ is k-generated if $\mathbf{A}'$ is, and the variety $\mathcal{U}$, being locally finite, cannot contain arbitrarily large k-generated algebras.) If $\mathbf{A}'$ is finite, let $\mathbf{A}'' = \mathbf{A}'$. In each case, there exists a finite algebra $\mathbf{A}'' \in \boldsymbol{HS}(\mathbf{A})$ not belonging to $\mathcal{U}$. Easily, an ***HS***-reduct of $\mathbf{A}''$ is an algebra critical for $\mathcal{U}$. □

1.3. Theorem. *Every finitely generated, congruence modular variety of a finite similarity type is congruence rich.*

Proof. Let $\mathcal{V}$ be such a variety, generated by a finite algebra $\mathbf{A}$. By Theorem 16 of [4], $|B| \leq |A|^n$ for any algebra $\mathbf{B} \in \mathcal{V}$ whose congruence lattice is of length n. From this it follows that, for any positive integer n, any finitely generated (which is then finite) algebra from $\mathcal{V}$ with more than $|A|^n$ elements has a proper homomorphic image with at least n elements. □

1.4. Theorem. *Every locally finite variety of a finite similarity type, which contains only finitely many nonisomorphic finite subdirectly irreducible algebras, is congruence rich.*

Proof. Let c be a finite upper bound for cardinalities of finite subdirectly irreducible algebras in a locally finite variety $\mathcal{V}$ of a finite similarity type. Let n be a positive integer, and take $m = nc$.

Let $\mathbf{D} \in \mathcal{V}$ be a finitely generated algebra with more than m elements. Of course, $\mathbf{D}$ is finite. By the subdirect representation theorem, $\mathbf{D}$ can be represented as a subalgebra of the direct product $\mathbf{A}_1 \times \ldots \times \mathbf{A}_k$, for a family of $\mathcal{V}$-algebras $\mathbf{A}_i$ with $|A_i| \leq c$; let k be a minimal number for which such a representation of $\mathbf{D}$ exists. Denote by f the canonical homomorphism of $\mathbf{A}_1 \times \ldots \times \mathbf{A}_k$ onto $\mathbf{A}_1 \times \ldots \times \mathbf{A}_{k-1}$. The subalgebra $f(D)$ of $\mathbf{A}_1 \times \ldots \times \mathbf{A}_{k-1}$ is a proper homomorphic image of $\mathbf{D}$. For any $b \in f(D)$ we have $|f^{-1}(b)| \leq c$, and so $|f(D)| \geq |D|/c > n$. $\Box$

1.5. Example. Not every finitely generated variety of algebras of a finite type is congruence rich. In her thesis [23], Caroline Shallon provided examples of finite algebras which generate varieties containing arbitrarily large finite simple algebras. The details of some of these examples can be found in [17] as well. Such varieties cannot be congruence rich.

In particular, one of her examples is the three-element groupoid with multiplication table

	0	a	b
0	0	0	0
a	0	0	a
b	0	b	b

which was found to be inherently nonfinitely based in [19]. It is proved in [23] and [17] that the variety generated by this groupoid contains simple objects of any cardinality larger or equal 3.

1.6. Example. As it is easy to see, for any congruence rich variety $\mathcal{V}$ there is an upper bound for the finite cardinalities of simple algebras in $\mathcal{V}$. Not every variety with such an upper bound for the finite cardinalities of simple algebras is congruence rich. For example, consider the variety of groupoids satisfying

$$(xy \cdot z)u = xy \cdot z = u(xy \cdot z).$$

This variety has no simple groupoids with more than two elements. On the other hand, the variety is not congruence rich. For every positive integer n define a groupoid $\mathbf{G}_n$ with the underlying set $\{0, x_0, \ldots, x_n\}$ as follows: $x_0 x_i = x_{i+1}$ for $0 \leq i < n$; $x_0 x_n = x_1$; $xy = 0$ in all other cases. Clearly, the groupoid $\mathbf{G}_n$ belongs to the variety, and if n is a prime number, it has only three congruences: the two trivial ones, and the congruence $\{0, x_1, \ldots, x_n\}^2 \cup \mathrm{id}_{G_n}$.

The authors were not able to provide an example of a finitely generated variety with an upper bound for the cardinalities of finite simple algebras, which is not congruence rich.

An element a of an algebra $\mathbf{A}$ is said to divide an element $b \in A$ if $b = f(a)$ for a unary polynomial f not involving the nullary operations of $\mathbf{A}$. The divisibility relation is a quasiordering on any algebra. An algebra is called **orderable-by-divisibility** if its divisibility relation is a partial ordering. Equivalently, an algebra $\mathbf{A}$ is orderable-by-divisibility if and only if there exists a partial ordering $\leq$ on A such that $a_i \leq F(a_1, \ldots, a_n)$ for any operation symbol F of any arity $n \geq 1$ in the similarity type, any $i \in \{1, \ldots, n\}$ and any $a_1, \ldots, a_n \in A$. With respect to $\leq$, A is then an updirected set, provided only that the similarity type contains at least one at least binary symbol.

It is easy to find an infinite collection of quasiidentities, characterizing the algebras orderable-by-divisibility of a specified similarity type. For a given similarity type, the class of the algebras orderable-by-divisibility is a quasivariety. It is not, however, a variety, if the type contains operation symbols of positive arity: the algebra of nonnegative integers, with the operations defined by $F(a_1, \ldots, a_n) = a_1 + \ldots + a_n + 1$, is orderable-by-divisibility, while its factor through the congruence modulo 2 is not.

A variety is called orderable-by-divisibility if all its algebras are.

1.7. Lemma. *Let* $\mathbf{A}$ *be a finitely generated algebra of a finite type, which is orderable-by-divisibility, and let* $\leq$ *be any partial ordering of* A*, extending the divisibility relation. Then for any positive integer* $n \leq |A|$*, the algebra* $\mathbf{A}$ *has an order ideal (with respect to* $\leq$*) of cardinality* n*.*

Proof. Let X be a finite generating subset of $\mathbf{A}$. Clearly, at least one of the elements of X must be a minimal element of A, so A has an order ideal of cardinality 1. Let I be a finite order ideal such that $I \neq A$. The finite set

$$(X \cup \{F(a_1, \dots, a_{\rho(F)}) : \ F \in \Delta, \ \ a_1, \dots, a_{\rho(F)} \in I\}) - I$$

(where $\rho(F)$ denotes the arity of F) is nonempty, because it cannot be a subalgebra containing X. Take a minimal element a in this finite set. It is not difficult to see that $I \cup \{a\}$ is an order ideal of cardinality $|I| + 1$. $\square$

1.8. Theorem. *In any finite similarity type, every variety* $\mathcal{V}$ *orderable-by-divisibility is congruence rich.*

Proof. Let $\mathbf{A} \in \mathcal{V}$ be a finitely generated algebra, and let $\leq$ be any partial ordering of A extending the divisibility relation. If F is an order filter of $\mathbf{A}$, then it is clear that $F^2 \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$. Since the complement of an order ideal is an order filter, it follows from Lemma 1.7 that $\mathbf{A}$ has a homomorphic image of any finite cardinality $\leq |A|$. $\square$

Let ρ be a finite similarity type such that its domain Δ of operation symbols contains at least one at least binary symbol. Then we can take a pair F, S consisting of a symbol $F \in \Delta$ of arity $n \geq 2$ and a nonempty proper subset S of $\{1, \dots, n\}$. In the following, let $F(a : b)$ stand for $F(a_1, \dots, a_n)$ where $a_i = a$ for $i \in S$ and $a_i = b$ for $i \in \{1, \dots, n\} - S$.

By an (F, S)**-directed algebra** of type ρ we mean an algebra $\mathbf{A}$ for which there exists a partial ordering $\leq$ on A such that $a_i \leq G(a_1, \dots, a_{\rho(G)})$ for any $G \in \Delta$, any $i \in \{1, \dots, \rho(G)\}$ and any $a_1, \dots, a_{\rho(G)} \in A$, and such that $F(a : b)$ is the larger of a and b whenever a, b are two comparable elements of A.

1.9. Theorem. *The class of* (F, S)*-directed algebras of type* ρ *is a finitely based, congruence rich variety. It can be described by the following finite set of equations:*

(1) $F(x, \dots, x) = x$,
(2) $F(F(x : y) : x) = F(x : y)$,
(3) $F(x : F(F(x : y) : z)) = F(F(x : y) : z)$,
(4) $F(x_i : G(x_1, \dots, x_{\rho(G)})) = G(x_1, \dots, x_{\rho(G)})$ *for any* $G \in \Delta$ *and any* $i \in \{1, \dots, \rho(G)\}$*.*

Proof. Obviously, every (F, S)-directed algebra satisfies all these equations. Let $\mathbf{A}$ be an algebra satisfying the four groups of equations. Define a binary relation $\leq$ on A by $a \leq b$ if and only if $F(a : b) = b$. Due to (1), this relation is reflexive. Due to (2), we also have $a \leq b$ if and only if $F(b : a) = b$, and this implies that the relation is antisymmetric. The equation (3) is a reformulation of transitivity, and (4) says that $a_i \leq G(a_1, \dots, a_{\rho(G)})$. By Theorem 1.8, the variety is congruence rich. $\square$

The variety of (F, S')-algebras, where S' is the complement of S in $\{1, \dots, n\}$, clearly coincides with the variety of (F, S)-algebras.

If ρ is the type of groupoids, there is only one choice for the symbol F, and the two choices for S give the same result. In that case, (F, S)-directed algebras are called directoids, according to [9]. The above general result yields an equational base consisting of five equations for the variety of directoids; one of these five equations is a consequence of the other ones.

1.10. Example. Consider the commutative directoid $\mathbf{D}$ with the underlying set $\mathbb{Z} \times \{0,1\}$, with the order relation defined by

$$\begin{aligned} &(i,0) \leq (j,0) \quad \text{iff} \quad (i,1) \leq (j,1) \quad \text{iff} \quad i \leq j, \\ &(i,0) \leq (j,1) \quad \text{iff} \quad i+2 \leq j, \\ &(i,1) \leq (j,0) \quad \text{iff} \quad i+1 \leq j \end{aligned}$$

and multiplication determined by

$$\begin{aligned} (i,0) \cdot (i,1) &= (i+2,0), \\ (i,0) \cdot (i+1,1) &= (i+3,1). \end{aligned}$$

It can be easily verified that the one-element directoid is the only finite homomorphic image of $\mathbf{D}$. Of course, $\mathbf{D}$ is then not finitely generated. Consequently, the variety of directoids does not satisfy a stronger congruence richness condition, namely, one in which the existence of a homomorphic image of cardinality in a specified finite interval of positive integers would be required for all, not only finitely generated algebras.

1.11. Example. Let us use Theorem 1.9 to show that there are uncountably many congruence rich varieties. According to that theorem, the variety of semilattices (and, more generally, the variety of directoids) with an arbitrary finite number of unary operators F satisfying $x \leq Fx$ is a congruence rich variety. Our algebras will be semilattices with two operators F and G, satisfying $x \leq Fx$ and $x \leq Gx$. For any subset S of ω let $\mathbf{A}_S$ be the chain $\{o\} \cup \omega \cup \{\infty_1, \infty_2\}$ (with $o < i < \infty_1 < \infty_2$ for any $i \in \omega$), with

$$\begin{aligned} &Fo = 0, \quad F\infty_1 = F\infty_2 = \infty_2, \\ &Fi = \infty_2 \text{ for } i \in S, \\ &Fi = \infty_1 \text{ for } i \notin S, \\ &Go = 0, \quad Gi = i+1, \quad G\infty_1 = G\infty_2 = \infty_2. \end{aligned}$$

Then it is easy to see that an equation

$$FG^iFx \geq y$$

is satisfied in $\mathbf{A}_S$ if and only if $i \in S$. Consequently, there are uncountably many varieties of semilattices with two such operators, all of them congruence rich.

It seems very likely that there are uncountably many varieties of directoids, but we have no proof for this.

Let $\mathcal{V}$ be a variety of ρ-algebras, where ρ is a finite type. For any ρ-algebra $\mathbf{A}$, denote by $\mathbf{A}^+$ the set of the elements of A that are a result of an operation, i.e., $\mathbf{A}^+$ is the union of the ranges of all (including nullary) fundamental operations of $\mathbf{A}$. Clearly, $\mathbf{A}^+$ is a subalgebra of $\mathbf{A}$ (or the empty set, but only if ρ is empty). Denote by $\mathcal{V}^+$ the class of the ρ-algebras $\mathbf{A}$ such that $\mathbf{A}^+ \in \mathcal{V}$. It is easy to see that $\mathcal{V}^+$ is a variety, and a finitely based variety whenever $\mathcal{V}$ is.

By an absorption equation we shall mean an equation (x,t) where x is a variable and t is a term with $\mathbf{S}(t) = \{x\}$, such that $t \neq x$. By an absorption term for a variety $\mathcal{V}$ we shall mean a term t such that, with respect to a variable x, the equation (x,t) is an absorption equation satisfied in $\mathcal{V}$. Given a variety $\mathcal{V}$ of a finite type ρ and an absorption term $t = t(x)$ for $\mathcal{V}$, denote by $\epsilon_t(\mathcal{V})$ the class of the algebras from $\mathcal{V}^+$ satisfying, for any $F \in \Delta$ and any $i \in \{1, \dots, \rho(F)\}$, the equation

$$F(x_1, \dots, x_{i-1}, t(x_i), x_{i+1}, \dots, x_{\rho(F)}) = F(t(x_1), \dots, t(x_{\rho(F)})).$$

Clearly, $\epsilon_t(\mathcal{V})$ is a variety, and a finitely based one whenever $\mathcal{V}$ is. We have $\mathcal{V} \subset \epsilon_t(\mathcal{V}) \subseteq \mathcal{V}^+$; the first inclusion is proper, because $\epsilon_t(\mathcal{V})$ does not satisfy $(x, t(x))$.

1.12. Lemma. *Let $\mathcal{V}$ be a congruence rich variety of a finite type. Then $\epsilon_t(\mathcal{V})$ is also congruence rich.*

Proof. Let n be a positive integer. Since $\mathcal{V}$ is congruence rich, there exists an $m > n$ such that every finitely generated algebra from $\mathcal{V}$ with more than n elements has a homomorphic image with more than n, but at most m elements. Put $m' = m + n$, and let $\mathbf{A}$ be a finitely generated algebra from $\epsilon_t(\mathcal{V})$ with more than n elements. If X is a finite set of generators of $\mathbf{A}$, then one can easily see that the subalgebra $\mathbf{A}^+$ is generated by

$$(X \cap A^+) \cup \{t(a) : a \in X\} \cup \{F(a_1, \dots, a_{\rho(F)}) : \ F \in \Delta, \ a_1, \dots, a_{\rho(F)} \in X\}.$$

So, $\mathbf{A}^+$ is finitely generated.

We need to prove that $\mathbf{A}$ has a homomorphic image with more than n, but at most m elements. If $|A - A^+| > n$, take a subset K of cardinality n; the relation $r = (A - K)^2 \cup \mathrm{id}_A$ is a congruence and the factor $\mathbf{A}/r$ is of cardinality $n + 1$.

So, we can suppose that $|A - A^+| \leq n$. If $|A^+| \leq m$, then $|A| \leq n + m = m'$ and we are through. So, we can also suppose that $|A^+| > m$. Since $\mathbf{A}^+ \in \mathcal{V}$ is finitely generated, there is a congruence θ of $\mathbf{A}^+$ with $n < |A^+/\theta| \leq m$. It is easy to see that $\theta \cup \mathrm{id}_A$ is a congruence of $\mathbf{A}$; the corresponding factor is of cardinality $|A^+/\theta| + |A - A^+|$, and so of cardinality between n and m'. □

1.13. Theorem. *Let $\mathcal{V}$ be a variety of a finite similarity type, satisfying an absorption equation. There exists a least variety $\mathcal{V}'$ containing $\mathcal{V}$ and satisfying no absorption equation. This variety $\mathcal{V}'$ covers $\mathcal{V}$ and is finitely based whenever $\mathcal{V}$ is. Also, $\mathcal{V}'$ is congruence rich whenever $\mathcal{V}$ is.*

Proof. Take an absorption term t for $\mathcal{V}$, and let $\epsilon'_t(\mathcal{V})$ be the variety of the algebras from $\mathcal{V}^+$ satisfying the equation

$$F(x_1, \dots, x_{\rho(F)}) = F(t(x_1), \dots, t(x_{\rho(F)}))$$

for any $F \in \Delta$ and any $i \in \{1, \dots, \rho(F)\}$. Clearly, this variety is finitely based if $\mathcal{V}$ is, and because $\mathcal{V} \subset \epsilon'_t(\mathcal{V}) \subseteq \epsilon_t(\mathcal{V})$, we can use Lemma 1.12 to see that the variety is congruence rich if $\mathcal{V}$ is.[1] One can easily prove by induction on the length of a term p that if $\mathbf{S}(p) \subseteq \{x_1, \dots, x_n\}$ and p is not a variable, then the variety $\epsilon'_t(\mathcal{V})$ satisfies

$$p(x_1, \dots, x_n) = p(t(x_1), \dots, t(x_n)).$$

From this it follows that $t(a) = u(a)$ for all $a \in A$, whenever u is an absorptive term for $\mathcal{V}$. Consequently, the definition of $\epsilon'_t(\mathcal{V})$ is independent of the choice of the absorption term t; the variety can be denoted by $\mathcal{V}'$. Also, it is now clear that every equation (u, v) of $\mathcal{V}$, such that u and v are both non-variables, is satisfied in $\mathcal{V}'$, so that $\mathcal{V}'$ is the least variety containing $\mathcal{V}$ and satisfying no absorption equation. Clearly, $\mathcal{V}$ is covered by $\mathcal{V}'$. □

[1] Actually, a straightforward proof of the congruence richness for $\mathcal{V}'$ could be given; we have based this proof on Lemma 1.12 just from the reason that the lemma provides us with slightly more examples of congruence rich varieties.

2. Finite axiomatizability of congruence rich varieties

The aim of this section is to prove that in congruence rich varieties, it is useful to know the critical algebras for a locally finite subvariety if we need to decide whether the subvariety is finitely based or, perhaps, inherently nonfinitely based. In the later sections this technique will be exploited to decide these questions for one particular variety of directoids.

Recall that a subvariety $\mathcal{U}$ of a variety $\mathcal{V}$ is said to be finitely based relative to $\mathcal{V}$ provided there is a finite set Σ of equations such that an algebra $\mathbf{A}$ from $\mathcal{V}$ belongs to $\mathcal{U}$ if and only if Σ is true in $\mathbf{A}$. If $\mathcal{V}$ is finitely based, then $\mathcal{U}$ is finitely based relative to $\mathcal{V}$ if and only if $\mathcal{U}$ is finitely based.

2.1. Theorem. *Let $\mathcal{V}$ be a congruence rich variety of a finite similarity type. For a locally finite subvariety $\mathcal{U}$ of $\mathcal{V}$, the following five conditions are equivalent:*

(1) *$\mathcal{U}$ is finitely based relative to $\mathcal{V}$.*
(2) *There exists a positive integer k such that the equational theory of $\mathcal{U}$ has a basis, relative to $\mathcal{V}$, consisting entirely of identities in at most k variables.*
(3) *There exists a positive integer k such that a $\mathcal{V}$-algebra belongs to $\mathcal{U}$ if and only if all of its k-generated subalgebras belong to $\mathcal{U}$.*
(4) *There is a finite collection $\mathbf{A}_1, \dots, \mathbf{A}_r$ of finite algebras in $\mathcal{V}$ such that a $\mathcal{V}$-algebra $\mathbf{B}$ belongs to $\mathcal{U}$ if and only if none of the algebras $\mathbf{A}_1, \dots, \mathbf{A}_r$ is a $\boldsymbol{HS}$-reduct of $\mathbf{B}$.*
(5) *There are, up to isomorphism, only finitely many algebras in $\mathcal{V}$ critical for the variety $\mathcal{U}$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial. Let us prove that (3) implies (4). Let k be a number with the property given in (3). Denote by F the set of (isomorphic copies of) the k-generated algebras in $\mathcal{U}$. (By a k-generated algebra we mean an algebra having a set of at most k generators.) Then F is a finite set of finite algebras in the congruence rich variety $\mathcal{V}$, so that there is a finite set F' of finite algebras in $\mathcal{V}$ such that every finitely generated algebra in $\mathcal{V} - \boldsymbol{I}(F)$ has a homomorphic image in $F' - \boldsymbol{I}(F)$. Denote the algebras in $F' - \mathcal{U}$ by $\mathbf{A}_1, \dots, \mathbf{A}_r$. If $\mathbf{B} \in \mathcal{U}$, then, because the algebras $\mathbf{A}_i$ do not belong to $\mathcal{U}$, none of them is an $\boldsymbol{HS}$-reduct of $\mathbf{B}$. If $\mathbf{B} \in \mathcal{V} - \mathcal{U}$, then $\mathbf{B}$ has a k-generated subalgebra $\mathbf{B}'$ not belonging to $\mathcal{U}$, $\mathbf{B}'$ belongs to $\mathcal{V} - \boldsymbol{I}(F)$, and $\mathbf{B}'$ has a homomorphic image in $F' - \boldsymbol{I}(F)$. The homomorphic image is also k-generated, so it does not belong to $\mathcal{U}$ by the definition of F, and consequently is isomorphic to one of the algebras $\mathbf{A}_1, \dots, \mathbf{A}_r$; that algebra is then an $\boldsymbol{HS}$-reduct of $\mathbf{B}$.

(5) implies (4) by Theorem 1.2, so we need only to prove that (4) implies (1). Let $\mathbf{A}_1, \dots, \mathbf{A}_r$ be a finite collection of finite $\mathcal{V}$-algebras with the property given in (4). For each $i = 1, \dots, r$ there is an identity (u_i, v_i) of $\mathcal{U}$ not satisfied in $\mathbf{A}_i$. This gives a finite collection of identities, which serves together with the identities of $\mathcal{V}$ as a basis for $\mathcal{U}$: if an algebra $\mathbf{B} \in \mathcal{V}$ satisfies all these identities, then the same is true for any algebra from $\boldsymbol{HS}(\mathbf{B})$, so that the algebras $\mathbf{A}_1, \dots, \mathbf{A}_r$ do not belong to $\boldsymbol{HS}(\mathbf{B})$ and, consequently, $\mathbf{B} \in \mathcal{U}$. □

A locally finite subvariety $\mathcal{U}$ of a variety $\mathcal{V}$ is said to be **inherently nonfinitely based relative to** $\mathcal{V}$ if there is no locally finite variety $\mathcal{U}'$ which is finitely based relative to $\mathcal{V}$ with $\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathcal{V}$. If, moreover, $\mathcal{U} = \boldsymbol{V}(\mathbf{A})$ for an algebra $\mathbf{A}$, we also say that the algebra $\mathbf{A}$ is inherently nonfinitely based relative to $\mathcal{V}$.

Note that if $\mathcal{U}$ is inherently nonfinitely based relative to $\mathcal{V}$, then $\mathcal{U}$ is inherently nonfinitely based in the absolute sense. In the event that $\mathcal{V}$ is finitely based, the converse also holds—namely, every locally finite subvariety of $\mathcal{V}$ which is inherently nonfinitely based is also inherently nonfinitely based relative to $\mathcal{V}$.

For a variety $\mathcal{U}$ and a positive integer n, denote by $\mathcal{U}^{(n)}$ the variety determined by the at most n-variable identities of $\mathcal{U}$. An algebra $\mathbf{A}$ belongs o $\mathcal{U}^{(n)}$ if and only if all the n-generated subalgebras of $\mathbf{A}$ belong to $\mathcal{U}$. Clearly, $\mathcal{U}$ is the intersection of the varietal chain $\mathcal{U}^{(1)} \supseteq \mathcal{U}^{(2)} \supseteq \ldots$. By a result of G. Birkhoff, if $\mathcal{U}$ is a locally finite variety of a finite similarity type, then $\mathcal{U}^{(n)}$ is finitely based for any n. From this it follows that for any variety $\mathcal{V}$ of a finite similarity type and any locally finite subvariety $\mathcal{U}$ of $\mathcal{V}$, the following three conditions are equivalent:

(1) $\mathcal{U}$ is inherently nonfinitely based relative to $\mathcal{V}$.
(2) $\mathcal{U}^{(n)} \cap \mathcal{V}$ is not locally finite for any positive integer n.
(3) For any positive integer n there exists an infinite but finitely generated algebra $\mathbf{A} \in \mathcal{V}$ such that every n-generated subalgebra of $\mathbf{A}$ belongs to $\mathcal{U}$.

The following is a characterization theorem for inherently nonfinitely based subvarieties of a congruence rich variety. It will be more convenient to formulate the equivalent conditions in their negative forms.

2.2. Theorem. *Let $\mathcal{V}$ be a congruence rich variety of a finite similarity type. The following three conditions are equivalent for a locally finite subvariety $\mathcal{U}$ of $\mathcal{V}$:*

(1) *$\mathcal{U}$ is not inherently nonfinitely based relative to $\mathcal{V}$.*
(2) *There is a finite set F of finite $\mathcal{V}$-algebras such that $F \cap \mathcal{U}$ is empty and $\boldsymbol{HS}(\mathbf{A}) \cap F$ is nonempty for every infinite but finitely generated algebra $\mathbf{A} \in \mathcal{V}$.*
(3) *There is a finite set F of $\mathcal{V}$-algebras critical for $\mathcal{U}$ such that every infinite but finitely generated algebra from $\mathcal{V}$ has an $\boldsymbol{HS}$-reduct in F.*

Proof. Let $\mathcal{U}$ be not inherently nonfinitely based relative to $\mathcal{V}$, so that $\mathcal{U}^{(n)} \cap \mathcal{V}$ is locally finite for some n. Denote by F_0 the finite collection of isomorphic copies of the n-generated algebras from $\mathcal{U}$. By the congruence richness of $\mathcal{V}$, there is a finite set F' of finite $\mathcal{V}$-algebras such that every n-generated algebra from $\mathcal{V} - \mathcal{U}$ has a homomorphic image in $F' - \boldsymbol{I}(F_0)$, and hence in $F' - \mathcal{U}$. If $\mathbf{A} \in \mathcal{V}$ is infinite but finitely generated, then $\mathbf{A} \notin \mathcal{U}^{(n)}$, so that some n-generated subalgebra $\mathbf{A}'$ of $\mathbf{A}$ does not belong to $\mathcal{U}$. Consequently, a homomorphic image $\mathbf{A}''$ of $\mathbf{A}'$ belongs to $F' - \mathcal{U}$. The choice of $F = F' - \mathcal{U}$ completes the proof of (1) $\Rightarrow$ (2).

Since (2) $\Rightarrow$ (3) is evident, it remains to show that (3) implies (1). Take n so large that $F \cap \mathcal{U}^{(n)}$ is empty; the existence of such an n follows from the finiteness of F and the fact that $\mathcal{U}$ is the intersection of $\mathcal{U}^{(1)} \supseteq \mathcal{U}^{(2)} \supseteq \ldots$. If there is an infinite but finitely generated algebra $\mathbf{A} \in \mathcal{U}^{(n)} \cap \mathcal{V}$, then $\mathbf{A}$ has an $\boldsymbol{HS}$-reduct in F, but this reduct belongs to $\mathcal{U}^{(n)}$, a contradiction. Hence there is no such algebra $\mathbf{A}$, which means that $\mathcal{U}^{(n)} \cap \mathcal{V}$ is locally finite and then $\mathcal{U}$ is not inherently nonfinitely based relative to $\mathcal{V}$. $\square$

3. The variety generated by N

Let $\mathbf{N}$ be the commutative directoid with the underlying set $\{0, a, b, c, 1\}$, defined by $0 < a < c < 1$, $0 < b < c$ and $ab = 1$. A diagram of $\mathbf{N}$ is displayed in Figure 2.

The equational theory of $\mathbf{N}$ has been described in [9] in the following way. An equation $u \leq v$ belongs to the theory if and only if $\boldsymbol{S}(u) \subseteq \boldsymbol{S}(v)$ and for any subterm $pq \subseteq u$ and any pair of nonempty subsets $P \subseteq \boldsymbol{S}(p) - \boldsymbol{S}(q)$, $Q \subseteq \boldsymbol{S}(q) - \boldsymbol{S}(p)$ there is a term $p'q' \subseteq^* v$ (which means that either $p'q' \subseteq v$ or $q'p' \subseteq v$) with $\boldsymbol{S}(p') \subseteq \boldsymbol{S}(pq) - Q$, $\boldsymbol{S}(q') \subseteq \boldsymbol{S}(pq) - P$ and $\boldsymbol{S}(p') \cap P \neq \emptyset \neq \boldsymbol{S}(q') \cap Q$.

The subgroupoid generated by a subset S in a given groupoid will be denoted by $[S]$.

3.1. Lemma. *Let $\mathbf{D}$ be a commutative directoid. Then $\mathbf{D} \in \boldsymbol{V}(\mathbf{N})$ if and only if for any elements $a, b, c \in D$ with $a < c$, $b < c$, $ab \not\leq c$ and any subsets $K \subseteq {\downarrow}a$, $L \subseteq {\downarrow}b$ with $a \in [K]$, $b \in [L]$ there are nonempty subsets $U \subseteq K - L$, $V \subseteq L - K$ such that $ef \not\leq c$ for any elements $e \in [(K \cup L) - V]$, $f \in [(K \cup L) - U]$ such that e is above an element of U and f is above an element of V.*

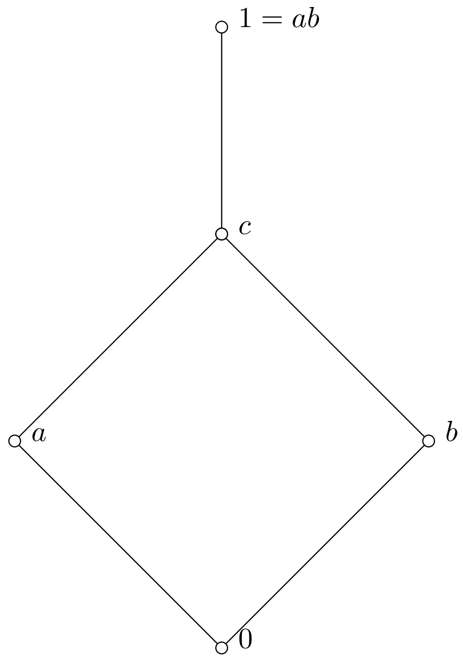

Figure 2: **N**

Proof. Let $\mathbf{D} \in \boldsymbol{V}(\mathbf{N})$ and let a, b, c, K, L be as above. Take a one-to-one mapping $i \mapsto x_i$ of D into X and let α be an interpretation in $\mathbf{D}$ with $\alpha(x_i) = i$ for all $i \in D$. There is a finite set S of terms such that whenever $k \in I \subseteq D$ and $k \leq j \in [I]$, then $j = \alpha(t)$ for a term $t \in S$ with $x_k \in \boldsymbol{S}(t) \subseteq \{x_i : i \in I\}$. There is a term p with $\boldsymbol{S}(p) = \{x_i : i \in K\}$ and $\alpha(p) = a$. Similarly, there is a term q with $\boldsymbol{S}(q) = \{x_i : i \in L\}$ and $\alpha(q) = b$. Put

$$v = x_c \cdot p \cdot q \cdot (t_1 u_1) \cdot \ldots \cdot (t_r u_r)$$

where $(t_1, u_1), \ldots, (t_r, u_r)$ are all the pairs $(t, u) \in S^2$ with $\alpha(tu) \leq c$. Clearly $\alpha(v) = c$, so that the equation $pq \leq v$ is not satisfied in D and thus does not belong to the equational theory of $\mathbf{N}$. Clearly, $\boldsymbol{S}(pq) \subseteq \boldsymbol{S}(v)$. Hence, by the description of the equational theory of $\mathbf{N}$, there are two nonempty subsets $P \subseteq \boldsymbol{S}(p) - \boldsymbol{S}(q)$, $Q \subseteq \boldsymbol{S}(q) - \boldsymbol{S}(p)$ such that $p'q' \not\subseteq v$ whenever $\boldsymbol{S}(p') \subseteq \boldsymbol{S}(pq) - Q$, $\boldsymbol{S}(q') \subseteq \boldsymbol{S}(pq) - P$ and $\boldsymbol{S}(p') \cap P \neq \emptyset \neq \boldsymbol{S}(q') \cap Q$. Put $U = \alpha(P)$ and $V = \alpha(Q)$, so that U is a nonempty subset of $K - L$ and V is a nonempty subset of $L - K$. Let $e \in [(K \cup L) - V]$, $f \in [(K \cup L) - U]$, $e \geq e_0 \in U$ and $f \geq f_0 \in V$. There are terms $p', q' \in S$ with $x_{e_0} \in \boldsymbol{S}(p') \subseteq \{x_i : i \in (K \cup L) - V\} = \boldsymbol{S}(pq) - Q$, $x_{f_0} \in \boldsymbol{S}(q') \subseteq \boldsymbol{S}(pq) - P$, $\alpha(p') = e$ and $\alpha(q') = f$. Since $x_{e_0} \in \boldsymbol{S}(p')$, we have $\boldsymbol{S}(p) \cap P \neq \emptyset$. Similarly, $\boldsymbol{S}(q') \cap Q \neq \emptyset$. If $ef \leq c$ then, by the construction of v, $p'q' \subseteq v$, a contradiction. Hence $ef \not\leq c$.

Conversely, assume that for any a, b, c, K, L there are U, V as above. We need to prove that any equation $u \leq v$ from the equational theory of $\mathbf{N}$ is satisfied in $\mathbf{D}$. Suppose that $u \leq v$ is not satisfied, so that $\alpha(u) \not\leq \alpha(v)$ for some interpretation α in $\mathbf{D}$. Let u' be a minimal subterm of u with $\alpha(u') \not\leq \alpha(v)$. Of course, the equation $u' \leq v$ also belongs to the equational theory of $\mathbf{N}$. Since $\boldsymbol{S}(u) \subseteq \boldsymbol{S}(v)$, $u' \notin X$. Hence $u' = pq$ for some terms p, q. Put $a = \alpha(p)$, $b = \alpha(q)$ and $c = \alpha(v)$. We have $ab \not\leq c$. By the minimality of u', $a \leq c$ and $b \leq c$. Of course, then $a < c$ and $b < c$. Put $K = \{\alpha(x) : x \in \boldsymbol{S}(p)\}$ and $L = \{\alpha(x) : x \in \boldsymbol{S}(q)\}$, so that $K \subseteq {\downarrow}a$, $L \subseteq {\downarrow}b$, $a \in [K]$ and $b \in [L]$. By the assumption, there are nonempty subsets $U \subseteq K - L$ and $V \subseteq L - K$ such that $ef \not\leq c$ for any elements $e \in [(K \cup L) - V]$, $f \in [(K \cup L) - U]$ such that e is above an element of U and f is above an element of V. Put $P = \{x \in \boldsymbol{S}(p) : \alpha(x) \in U\}$ and $Q = \{x \in \boldsymbol{S}(q) : \alpha(x) \in V\}$. Then P is a nonempty subset of $\boldsymbol{S}(p) - \boldsymbol{S}(q)$ and Q is a nonempty subset of $\boldsymbol{S}(q) - \boldsymbol{S}(p)$. In order to finish the proof, we need to show that $u \leq v$ does not belong to the equational theory of $\mathbf{N}$, and for this it is sufficient to prove that if p', q' are terms with $\boldsymbol{S}(p') \subseteq \boldsymbol{S}(pq) - Q$, $\boldsymbol{S}(q') \subseteq \boldsymbol{S}(pq) - P$ and $\boldsymbol{S}(p') \cap P \neq \emptyset \neq \boldsymbol{S}(q') \cap Q$, then $\alpha(p'q') \not\leq c$. Put $e = \alpha(p')$ and $f = \alpha(q')$. Then $e \in [(K \cup L) - V]$, $f \in [(K \cup L) - U]$, e is above an element of U and f is above an element of V. Hence $ef \not\leq c$, i.e., $\alpha(p'q') \not\leq c$. $\square$

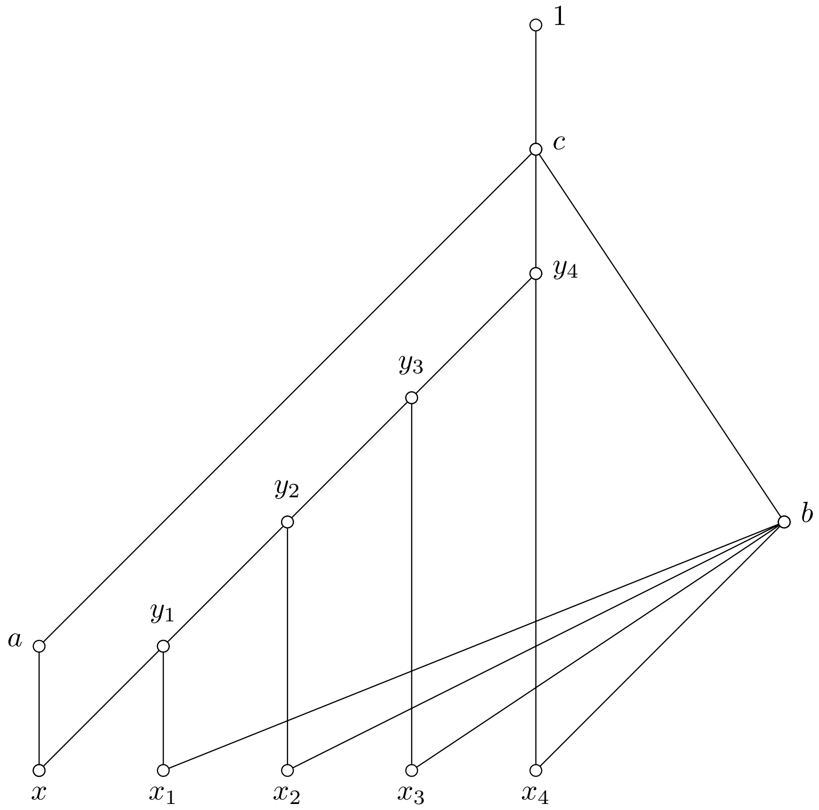

Figure 3: $\mathbf{D}_4$

For every integer $n \geq 2$ we define a commutative directoid $\mathbf{D}_n$ with the underlying set

$$D_n = \{x, x_1, \ldots, x_n, y_1, \ldots, y_n, a, b, c, 1\},$$

the order relation given as the transitive and reflexive closure of the relations

$$x < y_1 < y_2 < \ldots < y_n < c < 1, \quad x < a < c, \quad x_i < y_i, \quad x_i < b < c$$

and multiplication defined by

$$\begin{aligned} &xx_1 = x_1x = y_1, \\ &y_i x_{i+1} = x_{i+1} y_i = y_{i+1}, \\ &ay_n = y_n a = c, \\ &uv = u \vee v \quad \text{whenever } u, v \text{ are comparable}, \\ &uv = 1 \quad \text{otherwise}. \end{aligned}$$

3.2. Theorem. *For any $n \geq 2$, $\mathbf{D}_n$ is a directoid critical for the variety $\boldsymbol{V}(\mathbf{N})$. Consequently, $\mathbf{N}$ is not finitely based.*

Proof. One can easily verify that whenever v covers u in $\mathbf{D}_n$, then any congruence relating u with v relates also c with 1. Consequently, $\mathbf{D}_n$ is subdirectly irreducible, with the least nontrivial congruence $\theta = \{c, 1\}^2 \cup \mathrm{id}_{D_n}$.

Consider the pentuple $a, b, c, K = \{a, x\}, L = \{b, x_1, \ldots, x_n\}$, and take any pair of nonempty subsets $U \subseteq K - L$ and $V \subseteq L - K$. If $D_n \in \boldsymbol{V}(\mathbf{N})$, then, according to Lemma 3.1, we have $ef = 1$ for any elements $e \in [(K \cup L) - V]$ and $f \in [(K \cup L) - U]$ such that e is above an element of U and f is above an element of V. However, if $x \notin U$, then $ef \neq 1$ for $e = a$ and $f = xx_1 \ldots x_n = y_n$; if $x \in U$ and $x_i \in L$ for some i, take the least index i with this property and take $e = xx_1 \ldots x_{i-1}$ and $f = x_i$; and, finally, if $U = \{a, x\}$ and $L = \{b\}$, take $e = a(xx_1 \ldots x_n) = y_n$ and $f = b$. It follows that $\mathbf{D}_n$ does not belong to $\boldsymbol{V}(\mathbf{N})$.

It remains to show that each proper $\boldsymbol{HS}$-reduct $\mathbf{D}$ of $\mathbf{D}_n$ belongs to $\boldsymbol{V}(\mathbf{N})$. We need only to consider the case when $\mathbf{D}$ is either $\mathbf{D}_n/\theta$ or the subdirectoid obtained from $\mathbf{D}_n$ by deleting one of the elements $x, x_1, \ldots, x_n, a, b$; any other proper reduct is a reduct of one of these.

By Lemma 3.1, we must find subsets U, V to any pentuple a', b', c', K, L such that $a' < c'$, $b' < c'$, $a'b' \not\leq c'$, $K \subseteq \downarrow a'$, $L \subseteq \downarrow b'$, $a' \in [K]$ and $b' \in [L]$. Clearly, only the pentuples have to be considered for which $a'b' = 1$.

If $a' = x$ and $b' = x_i$ where $i > 1$, take $U = K$ and $V = L$.

If $a' = x_i$ and $b' = x_j$ where $i \neq j$, take $U = K$ and $V = L$.

If $a' = y_i$ and $b' = x_j$ where $j > i + 1$, take $U = K$ and $V = L$.

If $a' = x$ and $b' = b$, take $U = K$ and $V = \{b\}$.

If $a' = y_i$ and $b' = b$, take $U = K \cap \{x, y_1, \ldots, y_i\}$ and $V = \{b\}$.

If $a' = a$ and $b' = x_i$, take $U = \{a\}$ and $V = \{x_i\}$.

If $a' = a$ and $b' = y_i$ where $i \neq n$, take $U = \{a\}$ and $V = L - \{x\}$.

Finally, consider the case when $a' = a$ and $b' = b$. In that case, $\mathbf{D}$ can only be a subdirectoid obtained from $\mathbf{D}_n$ by deleting one of the elements $x, x_1, \ldots, x_n$. We have $a \in K \subseteq \{a, x\}$ and $b \in L \subseteq \{b, x_1, \ldots, x_n\}$. Take $U = K$ and $V = \{b\}$, and suppose that there are elements $e \in [(K \cup L) - V]$ and $f \in [(K \cup L) - U]$ such that e is above an element of U, f is above an element of V and $ef \neq 1$. Clearly, $f \in \{b, 1\}$ and it is sufficient to consider the case $f = b$. We have $e \in [K \cup \{x_1, \ldots, x_n\}]$ and $e \geq x$. But then $eb \neq 1$ is possible only if $e = c$. This yields $c \in [a, x, x_1, \ldots, x_n]$, which can be true only if all of the elements $x, x_1, \ldots, x_n$ belong to $\mathbf{D}$; we get a contradiction. □

Remark. If we extended the definition of $\mathbf{D}_n$ to include the case $n = 1$, then $\mathbf{D}_1$ would not be a directoid critical for $\boldsymbol{V}(\mathbf{N})$.

We are also going to show that the variety $\boldsymbol{V}(\mathbf{N})$ is not inherently nonfinitely based. Before being able to do it, let us first investigate one particular variety of commutative directoids.

4. The variety $\mathcal{W}$

We denote by $\mathcal{W}$ the variety of commutative directoids satisfying the following two equations:

(1) $xy \cdot z \leq xy \cdot xz$

(2) $xy \cdot z \leq ((xy \cdot u) \cdot xz) \cdot yz$

Also, $\mathcal{W}$ can be described as the class of commutative directoids satisfying the quasiequations

(1′) $x \leq y \to yz \leq y \cdot xz$

(2′) $(xy \leq u \ \ \& \ \ xz \leq u \ \ \& \ \ yz \leq u) \to xy \cdot z \leq u$

Indeed, it is easy to see that (1) is equivalent to (1′) and (2) is equivalent to (2′).

4.1. Lemma. *Let $\mathbf{D} \in \mathcal{W}$ and $x, y, z \in D$ be three elements such that $x \leq z$ and $y \leq z$. Then $z \cdot xy = z \vee xy$.*

Proof. Certainly, the element $z \cdot xy$ is above both z and xy. Let $c \in D$ be such that $c \geq z$ and $c \geq xy$. Since the product of any two of the three elements x, y, z is $\leq c$, by (2′) we get $z \cdot xy \leq c$. This means that $z \cdot xy$ is the join of z and xy. □

4.2. Lemma. *Let $\mathbf{D}$ be a commutative directoid satisfying (1) and*

$$(x \leq z \ \ \& \ \ y \leq z) \to z \cdot xy = z \vee xy\cdot$$

Then $ap = a \vee p$ for any elements $a, p \in D$ such that $p \in [\downarrow a]$. Also,

$$xy \cdot xz = xy \cdot (y \cdot xz) = xy \vee y \cdot xz$$

for any $x, y, z \in D$.

Proof. By induction on the length of a term t, we shall prove that whenever $a \in D$ and α is an interpretation of the variables of t in $\downarrow a$, then $a \cdot \alpha(t) = a \vee \alpha(t)$. If t is a variable, then clearly $a \cdot \alpha(t) = a = a \vee \alpha(t)$. If t is a product of two variables, we can use Lemma 4.1. The last case is when t can be expressed as $t = t_1 t_2$, where t_1 is not a variable. Take a variable y not contained in t_2, so that the term yt_2 is shorter than t. Let $p_1 = \alpha(t_1)$ and $p_2 = \alpha(t_2)$. By induction, $ap_1 = a \vee p_1$ and $ap_2 = a \vee p_2$. Now take $a' = ap_1$ and define an interpretation β by $\beta(y) = p_1$ and $\beta(x) = \alpha(x)$ for all variables x contained in t_2. By induction we have $a' \cdot \beta(yt_2) = a' \vee \beta(yt_2)$, i.e.,

$$ap_1 \cdot p_1 p_2 = ap_1 \vee p_1 p_2 = a \vee p_1 \vee p_1 p_2 = a \vee p_1 p_2$$

(yes, it works). By (1) we have $p_1 p_2 \cdot a \leq p_1 a \cdot p_1 p_2 = a \vee p_1 p_2$, i.e., $a \cdot \alpha(t) = a \vee \alpha(t)$.

In particular, for any elements $x, y, z \in D$ we have $xy \cdot (y \cdot xz) = xy \vee y \cdot xz$. Both xy and $y \cdot xz$ are below $xy \cdot xz$, the first element evidently and the second by (1). So, their join, the element $xy \cdot (y \cdot xz)$, is also below $xy \cdot xz$. The converse inequality can be obtained from (1) by the substitution $x \to y$, $y \to xy$, $z \to xz$. □

4.3. Lemma. *Let $\mathbf{D} \in \mathcal{W}$. Then $ap = a \vee p$ for any elements $a, p \in D$ such that $p \in [\downarrow a]$.*

Proof. It is a consequence of Lemmas 4.1 and 4.2. □

Three commutative directoids $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$, each of them (as it is mechanical to verify using Lemma 3.1) critical for the variety $\boldsymbol{V}(\mathbf{N})$, will play a special role in the next discussion. They are defined as follows.

$\mathbf{E}_1$: $E_1 = \{a, b, c, d, 1\}$; $d < a < c < 1$, $b < c$; $bd = c$.

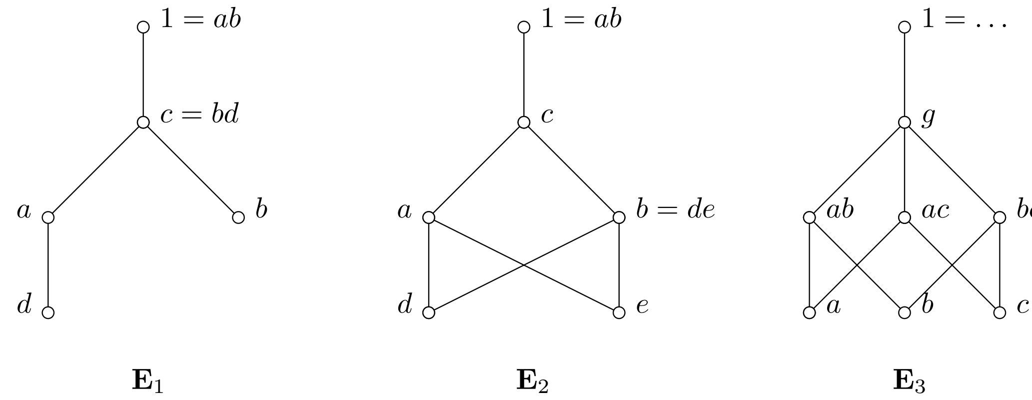

Figure 4

$\mathbf{E}_2$: $E_1 = \{a, b, c, d, e, 1\}$; $d < a < c < 1$, $e < b < c$, $e < a$, $d < b$; $de = b$.
$\mathbf{E}_3$: $E_3 = \{a, b, c, d, e, f, g, 1\}$; $a < d < g < 1$, $a < e < g$, $b < d$, $b < f < g$, $c < e$, $c < f$; $ab = d$, $ac = e$, $bc = f$.

In each definition, the order relation should be the reflexive and transitive closure of the given relations, and the product of any two incomparable elements x, y, if neither xy nor yx has been specified, equals 1.

4.4. Theorem. *A commutative directoid* $\mathbf{D}$ *belongs to* $\mathcal{W}$ *if and only if neither* $\mathbf{E}_1$ *nor* $\mathbf{E}_2$ *nor* $\mathbf{E}_3$ *is a* ***HS****-reduct of* $\mathbf{D}$.

Proof. With $x = d$, $y = a$ and $z = b$ we see that $(1')$ is not satisfied in $\mathbf{E}_1$. With $x = d$, $y = e$, $z = a$ and $u = c$ we see that $(2')$ is not satisfied in $\mathbf{E}_2$. With $x = a$, $y = b$, $z = c$ and $u = g$ we see that $(2')$ is not satisfied in $\mathbf{E}_3$. Consequently, the three directoids do not belong to $\mathcal{W}$ and if $\mathbf{D} \in \mathcal{W}$, then neither $\mathbf{E}_1$ nor $\mathbf{E}_2$ nor $\mathbf{E}_3$ can be an ***HS***-reduct of $\mathbf{D}$.

Let $\mathbf{D}$ be a commutative directoid such that neither $\mathbf{E}_1$ nor $\mathbf{E}_2$ nor $\mathbf{E}_3$ is an ***HS***-reduct of $\mathbf{D}$.

In order to verify $(1')$ for $\mathbf{D}$, let x, y, z be three elements with $x \leq y$. Denote by F the set of the elements $\not\leq y \cdot xz$, and by $\theta(F)$ the congruence $F^2 \cup \mathrm{id}_D$. If $yz \not\leq y \cdot xz$, then the set

$$G = \{x, y, z, xz, y \cdot xz, 1_{D/\theta(F)}\}$$

is a subgroupoid of $\mathbf{D}/\theta(F)$, the relation $R = \{xz, y \cdot xz\}^2 \cup \mathrm{id}_G$ is a congruence of G and the factor G/R is isomorphic to $\mathbf{E}_1$, a contradiction.

Next we shall prove that $\mathbf{D}$ satisfies $(x \leq z \;\&\; y \leq z) \rightarrow z \cdot xy = z \vee xy$. Suppose, on the contrary, that there is an element u with $z \leq u$ and $xy \leq u$, such that $z \cdot xy \not\leq u$. Let F be the set of the elements $\not\leq u$. Then the set

$$G = \{x, y, z, xy, u, 1_{D/\theta(F)}\}$$

is a subgroupoid of $D/\theta(F)$ isomorphic to $\mathbf{E}_2$, a contradiction.

It follows from Lemma 4.2 that $ap = a \vee p$ for any elements $a, p \in D$ such that $p \in [\downarrow a]$. Also, $xy \cdot xz = xy \cdot (y \cdot xz) = xy \vee y \cdot xz$ for any $x, y, z \in D$.

It remains to prove that $\mathbf{D}$ satisfies $(2')$. Let $xy \leq u$, $xz \leq u$, $yz \leq u$ and suppose that $xy \cdot z \not\leq u$. Denote by F the set of the elements $\not\leq u$, so that $xy \cdot z \in F$. By (1), $xy \cdot xz \in F$ and $xy \cdot yz \in F$. Since $xy \cdot xz = xy \cdot (y \cdot xz) = xy \vee y \cdot xz$, we have $y \cdot xz \in F$. Similarly, $x \cdot yz \in F$. Since $xz \cdot yz = xz \cdot (x \cdot yz) \geq x \cdot yz$, we have $xz \cdot yz \in F$. But then the set

$$G = \{x, y, z, xy, xz, yz, u, 1_{D/\theta(F)}\}$$

is a subgroupoid of $\mathbf{D}/\theta(F)$ isomorphic to $\mathbf{E}_3$, a contradiction. □

4.5. Lemma. *Let $\mathbf{D} \in \mathcal{W}$, $u \in D$ and K be a finite subset of D such that $xy \leq u$ for any $x, y \in K$. Then every element of $[K]$ is below u.*

Proof. Let $K = \{x_1, \ldots, x_k\}$. For any term t in the variables $x_1, \ldots, x_k$ we shall prove, by induction on the length of t, that $t \leq u$. If t is of length ≤ 2, this follows from the assumption. Let $t = t_1t_2 \cdot t_3$. By induction, $t_1t_2 \leq u$, $t_1t_3 \leq u$ and $t_2t_3 \leq u$. By $(2')$, we get $t_1t_2 \cdot t_3 \leq u$. □

4.6. Lemma. *Every finitely generated directoid from $\mathcal{W}$ has a largest element.*

Proof. Let K be a finite generating subset of a directoid $\mathbf{D} \in \mathcal{W}$. Certainly, there is an element $e \in D$ such that $xy \leq e$ for all $x, y \in K$ (e.g., the product of all the finitely many elements $xy \in X^2$, taken in any order). By Lemma 4.5, every element of D is $\leq e$. □

4.7. Lemma. *Suppose that there is a finitely generated, infinite directoid $\mathbf{D}$ in $\mathcal{W}$. Then $\mathbf{D}$ contains infinitely many elements a such that $\downarrow a$ is finite.*

Proof. This follows from Lemma 1.7. □

4.8. Theorem. *The variety $\mathcal{W}$ is locally finite.*

Proof. Suppose that $\mathcal{W}$ is not locally finite and take n to be the least positive integer for which there exists an infinite, n-generated directoid $\mathbf{D}$ in $\mathcal{W}$.

We may assume that $\mathbf{D}$ satisfies the minimal condition, i.e., every nonempty subset of D has a minimal element. Because if it does not, then we can replace it with its factor $\mathbf{D}/\theta(F)$, where F is the order filter of the elements $a \in D$ such that $\downarrow a$ is infinite; the factor is, of course, also n-generated and belongs to $\mathcal{W}$; by Lemma 4.7, the factor is infinite; and by definition, any principal order ideal in the factor except for the one generated by the largest element is finite, so that the factor satisfies the minimal condition.

Denote by X the set of n generators of $\mathbf{D}$, and by S the set of the elements $a \in D$ such that $a \geq x$ for all $x \in X$. By Lemma 4.3, $ap = a \vee p$ for any $a \in S$ and any $p \in D$. In particular, S is a subsemilattice of $\mathbf{D}$.

By the minimality of n, $[X - \{x\}]$ is finite for any $x \in X$. The complement $D - S$ is contained in the union of these finitely many finite sets. Consequently, $D - S$ is finite.

Denote by Y the set of the elements of S that are join irreducible in the semilattice S. Since S satisfies the minimal condition, S is generated by Y. But S is infinite and the variety of semilattices is locally finite, so Y is infinite.

Let $a \in Y - X$. Then $a = bc$ for some $b, c < a$ and, since a is join irreducible in S, at least one of the elements b, c belongs to $D - S$. Because $(D - S)(D - S)$ is finite, it follows that almost all elements $a \in Y$ can be expressed as $a = bc$ where $b \in S$, $b < a$ and $c \in D - S$. (By "almost all" we mean that the complement in Y is finite.) Let $a = bc$ be such a representation of a. Since $bp = b \vee p$ for any element p, we have $a = b \vee p$ for any p with $c \leq p < a$; but then $p \in D - S$ for any p with $c \leq p < a$, since a is join irreducible in S; among such elements p there is a maximal one, and we see that almost all elements $a \in Y$ can be expressed as $a = bc$ where $b \in S$, $b < a$, $c \in D - S$ and c is covered by a in D.

Consider now a situation when $a = bc$, $a \in Y$, $b \in S$, $b < a$, $c \in D - S$ and c is covered by a. With a and c fixed, we can assume that b is a minimal element with this property. Then we claim that b is a minimal element of S. Suppose, on the contrary, that there is an element $b' \in S$ with $b' < b$. We have $b'c = b' \vee c \leq a$; since c is covered by a and clearly $b'c \neq c$, we get $b'c = a$, a contradiction with the minimal property of b.

We see that almost all elements $a \in Y$ can be expressed as $a = bc$ where b is a minimal element of S and $c \in D - S$. Since $D - S$ is finite, it follows that the set of minimal elements of S is infinite.

But every minimal element of S which does not belong to X can be expressed as $b = b_1 b_2$ with $b_1 < b$ and $b_2 < b$, and necessarily $b_1 \in D - S$ and $b_2 \in D - S$. Since $X \cup (D - S)(D - S)$ is finite, there can be only finitely many minimal elements in S.

This contradiction proves that $\mathcal{W}$ is locally finite. □

4.9. Theorem. *The directoid* $\mathbf{N}$ *belongs to the variety* $\mathcal{W}$. *Consequently,* $\mathbf{N}$ *is not inherently nonfinitely based.*

Proof. It can be mechanically verified that $\mathbf{N}$ satisfies both equations (1) and (2). Since $\mathcal{W}$ is locally finite by Theorem 4.8 and clearly finitely based, $\mathbf{N}$ is not inherently nonfinitely based by definition. □

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