THE EXISTENCE OF FINITELY BASED LOWER COVERS FOR FINITELY BASED EQUATIONAL THEORIES

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By an equational theory we mean a set of equations from some fixed language which is closed with respect to logical consequences. We regard equations as universal sentences whose quantifier-free parts are equations between terms. In our notation, we suppress the universal quantifiers. Once a language has been fixed, the collection of all equational theories for that language is a lattice ordered by set inclusion. The meet in this lattice is simply intersection; the join of a collection of equational theories is the equational theory axiomatized by the union of the collection. In this paper we prove, for languages with only finitely many fundamental operation symbols, that any nontrivial finitely axiomatizable equational theory covers some other finitely axiomatizable equational theory. In fact, our result is a little more general.

There is an extensive literature concerning lattices of equational theories. These lattices are always algebraic. Compact elements of these lattices are the finitely axiomatizable equational theories. We also call them **finitely based**. The largest element in the lattice is compact; it is the equational theory based on the single equation $x \approx y$. The smallest element of the lattice is the **trivial theory** consisting of tautological equations. For all but simplest languages, the lattice of equational theories is intricate. R. McKenzie in [6] was able to prove in essence that the underlying language can be recovered from the isomorphism type of this lattice. A key to McKenzie's main result involved understanding first order definability within this lattice. Our knowledge of definability in this lattice was substantially advanced in the work of Ježek [3].

Given an equational theory T, by $\mathbf{L}_T$ we denote the sublattice of the lattice of equational theories of the given language comprised of all equational theories which include T. Thus if T were the equational theory of semigroups, then $\mathbf{L}_T$ would be the lattice of all equational theories of semigroups and one of the members of $\mathbf{L}_T$ would be the equational theory of commutative semigroups. Almost any algebraic lattice appears as an interval in the lattice of all equational theories, subject to the obvious cardinality restrictions, see Ježek [2] and Pigozzi and Tardos [17]. It is nevertheless true that the lattices of the form $\mathbf{L}_T$ have not yet been clearly understood. Lampe in [4] found a series of first order conditions, each actually a universal Horn sentence, that each of the lattices $\mathbf{L}_T$ must satisfy. As a result, even a simple lattice of five elements can be shown not representable in the form $\mathbf{L}_T$.

For the basics of equational logic the reader is referred to either McNulty [11] or Taylor [18], and for the basics about varieties of algebras, to [8].

We are going to investigate the existence of covering in the lattice $\mathbf{L}_T$, where T is a term finite equational theory. We say that an equational theory T is **term finite** provided that for each term t, the set $\{s : t \approx s \in T\}$ is finite. Theorem 6, which is our main result, asserts that for every term finite equational theory T of a finite language, every equational theory properly extending T and finitely based relative to T has a lower cover extending T, which is again finitely based relative to T. In particular, if T is the trivial theory, this means that every nontrivial finitely based equational theory of a finite language has a finitely based lower cover in the lattice of equational theories. The construction is not effective; however, we can effectively construct a finite base for an equational theory E', properly contained in E and such that the number of the equational theories between E and E' is finite and can be effectively estimated by an upper bound.

Our proof takes much from McKenzie [6], where among other things he proved that any nontrivial equational theory has a lower cover. In McNulty [9] it was proven that if T is an equational theory such that $x \approx t \in T$ for a variable x and a term t containing either an operation symbol of arity ≥ 2 or at least two distinct unary operation symbols, then T has at least $2^{\aleph_0}$ lower covers in the lattice of equational theories. In particular, this means that many finitely based equational theories have lower covers that are not finitely based. Other results on coverings in the lattice of equational theories are contained in McNulty [10], Pollák [16] and Trahtman [19].

Let ρ be a fixed language, with no predicate symbols. Elements of the domain of ρ are called **operation symbols**. For an operation symbol F, $\rho(F)$ is the arity of F. We fix a countably infinite set of variables and consider the algebra of terms over this set. The **support** of a term t, i.e., the set of variables occurring in t, will be denoted by $\boldsymbol{S}(t)$. A **substitution** can be most easily defined as an endomorphism of the term algebra. By an **elementary lift** we mean a mapping

$$L(t) = F(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_n)$$

where F is an operation symbol of some arity n and $i \in \{1, \ldots, n\}$. A composition of a finite (possibly empty) sequence of elementary lifts is called a **lift**. An **equation** $u \approx v$ can be naturally identified with an ordered pair (u, v) of terms; an equational theory is then nothing else than a fully invariant congruence of the term algebra. An equation $u \approx v$ is called **regular** if $\boldsymbol{S}(u) = \boldsymbol{S}(v)$; an equational theory is regular if it contains regular equations only. An equation $u \approx v$ is called **trivial** if $u = v$; a set of equations is trivial if it contains trivial equations only. The least equational theory of the language ρ, which is the set of trivial equations, will be denoted by O_ρ.

Given two terms u and v, we write $u \leq v$ if $v = Lf(u)$ for a lift L and a substitution f. This is a quasiordering on the set of terms. Two terms u, v are called (**literally**) **similar** if $u \leq v$ and $v \leq u$; we then write $u \sim v$. Also, $u \sim v$ if and only if $v = \alpha(u)$ for an automorphism α of the term algebra. Factored through this equivalence, the set of terms becomes a partially ordered set every principal ideal of which is finite. Two equations $u \approx v$ and $p \approx q$ are called similar if there is an automorphism α of the term algebra with $p = \alpha(u)$ and $q = \alpha(v)$. We write $u < v$ if $u \leq v$ but $u \not\sim v$.

In the following we shall assume that T is a given term finite equational theory. Clearly, T is regular and if $u \approx v \in T$ and $u \leq v$, then $u \sim v$. For two terms u and v, write $u \leq_T v$ if there is a term w with $u \leq w$ and $w \approx v \in T$. It is easy to see that this relation is a quasiordering on the set of terms, and because T is term finite, for any term u there are, up to similarity, only finitely many terms v with $v \leq_T u$. Let us write $u \sim_T v$ if $u \leq_T v$ an $v \leq_T u$. Again, using the term finiteness of T we can see that $u \sim_T v$ if and only if there is a term w with $u \sim w$ and $w \approx v \in T$. We shall write $u <_T v$ if $u \leq_T v$ and $v \not\leq_T u$, i.e., if $u \leq_T v$ and $u \not\sim_T v$. Clearly, $u < v$ implies $u <_T v$.

By an **immediate T-consequence** of an equation $u \approx v$ we mean any equation $u' \approx v'$ for which there exist a lift L and a substitution f with $u' \approx Lf(u) \in T$ and $v' \approx Lf(v) \in T$. By a **proof based on B modulo T**, where B is a set of equations, we mean a finite sequence $u_0, \ldots, u_j$ of terms such that for any $i \in \{1, \ldots, j\}$, $u_{i-1} \approx u_i$ is an immediate T-consequence of an equation from $B \cup B^{-1}$. By a proof of $u \approx v$ based on B, modulo T, we mean one such that $u_0 = u$ and $u_j = v$. This notion of proof is complete in the sense that the equations $u \approx v$ which are provable based on B modulo T are exactly the equations true in every model of $B \cup T$. This sort of completeness is an easy consequence of one of the common systems of proof for equational logic, described for example in Birkhoff [1]. By a **minimal proof** of $u \approx v$ based on B modulo T we mean one which is of the shortest possible length. Given an equational theory $E \supset T$, a subset B of E is said to be a **T-base** for E if every equation from E has a proof based on B modulo T. An equational theory is said to be **finitely based relative to** T if it has a finite T-base.

Let E be an equational theory properly extending T. A term m is said to be **T-minimal** for E if there exists a term w with $m \approx w \in E - T$, but there is no equation $m' \approx w' \in E - T$ with $m' <_T m$. It is easy to see that any equational theory properly extending T has a T-minimal term.

Also, as it is easy to see, if m is a T-minimal term for E and B is any T-base for E, then there always exists an equation $u \approx v \in B$ such that either $m \sim_T u$ or $m \sim_T v$. Given a term m which is T-minimal for E, we denote by $\boldsymbol{C}_{m,T}(E)$ the set of the equations $u \approx v \in E$ such that either $u \approx v \in T$ or $u \not\sim_T m \not\sim_T v$. One can easily verify that $\boldsymbol{C}_{m,T}(E)$ is the union of T with the set of the equations $u \approx v \in E$ such that $u \not\leq_T m$ and $v \not\leq_T m$, and then it follows that $\boldsymbol{C}_{m,T}(E)$ is an equational theory properly contained in E, and extending T.

All the above definitions, with the parameter T omitted, refer to the case $T = O_\rho$.

The following theorem has been essentially proved by McKenzie [6] in the special case $T = O_\rho$, and by Trahtman [19] in the case when T is a balanced equational theory and the language contains no symbols of arity less than 2; related results are contained in Pollák [16]. An equational theory T is called **balanced** if for every equation $u \approx v \in T$ and every variable x, the number of the occurrences of x in u is the same as that in v. Clearly, every balanced equational theory of a language containing no symbols of arity less than 2 is term finite.

Theorem 1. *Let T be a term finite equational theory and E be an equational theory properly extending T. Let m be a term T-minimal for E. Then the interval $E/\boldsymbol{C}_{m,T}(E)$ in the lattice of equational theories of the given language is finite. In particular, every equational theory properly extending T has a lower cover extending T.*

Proof. Denote by K the set of the terms t with $t \leq_T m$, and by Q the set $K \times K$ (which is a set of equations). Since K is, due to the term finiteness of T, finite up to (absolute) similarity, also Q is finite up to similarity and thus it is sufficient to show that every equational theory U with $\boldsymbol{C}_{m,T}(E) \subseteq U \subset E$ is uniquely determined by a subset of Q. We shall show, more strongly, that $U = \boldsymbol{C}_{m,T}(E) \cup (U \cap Q)$.

Suppose, on the contrary, that there is an equation $u \approx v \in U - T$ such that $u \leq_T m$ and $v \not\leq_T m$. (One can easily see that this is just the opposite case.) There are a substitution f and a lift L such that $Lf(u) \approx m \in T$. Put $w = Lf(v)$, so that $w \approx Lf(u) \in T$ and, consequently, $w \approx m \in U$. Also, $w \not\leq_T m$.

Since $U \subset E$, there is an equation $p \approx q \in E - U$. If both $p \not\leq_T m$ and $q \not\leq_T m$, then $p \approx q \in \boldsymbol{C}_{m,T}(E) \subseteq U$, a contradiction. So, without loss of generality, we assume that $p \leq_T m$. By the minimality of m, we have $\alpha(p) \approx m \in T$ for an automorphism α of the term algebra. Clearly,

$$w \approx m \in U \subseteq E, \quad m \approx \alpha(p) \in T \subseteq U \subseteq E, \quad \alpha(p) \approx \alpha(q) \in E.$$

From this it follows that $w \approx \alpha(q) \in E$. Here we have $w \not\leq_T m$. If also $\alpha(q) \not\leq_T m$, then $w \approx \alpha(q) \in \boldsymbol{C}_{m,T}(E) \subseteq U$, so that also $\alpha(p) \approx \alpha(q) \in U$ and then $(p \approx q) = \big(\alpha^{-1}\alpha(p) \approx \alpha^{-1}\alpha(q)\big) \in U$, a contradiction. Hence $\alpha(q) \leq_T m$, i.e., $q \leq_T m$. By the minimality of m we get $q \sim_T m$ and so there is an automorphism β of the term algebra with $\beta(q) \approx m \in T$. We have $w \approx \beta(q) \in U$, and consequently $\alpha\beta^{-1}(w) \approx \alpha(q) \in U$. But also $w \approx \alpha(q) \in E$, so $w \approx \alpha\beta^{-1}(w) \in E$; both sides of this equation are $\not\leq_T m$, which means that the equation belongs to $\boldsymbol{C}_{m,T}(E) \subseteq U$. This was just the missing link to obtain $\alpha(q) \approx \beta(q) \in U$. But also $\alpha(p) \approx \beta(q) \in U$, since both sides are T-related to m, and so we get $\alpha(p) \approx \alpha(q) \in U$. This yields $p \approx q \in U$, a contradiction. $\square$

A variety V is said to have the **cover property** if every proper subvariety of V has a cover in the lattice of subvarieties of V. As an easy corollary to Theorem 1, we get: *Every term finite variety has the cover property.*

Lemma 2. *Suppose that the language ρ contains at least one symbol of nonzero arity. Let T be a term finite equational theory and E be an equational theory properly extending T. Then there exists a T-base B for E such that B is finite if E is finitely based relative to T and whenever $u \approx v \in B$ is an equation at least one side of which is a term T-minimal for E, then $u \approx v$ is regular.*

Proof. Take one operation symbol F of arity $n > 0$, and for any ρ-term t define $t^* = F(t, \dots, t)$ (where t appears n times). Let us call an equation $u \approx v$ bad if $\boldsymbol{S}(u) \neq \boldsymbol{S}(v)$ and at least one of the terms u and v is T-minimal for E. Given a T-base for E, we can construct a new T-base by replacing each of its bad equations with a triple of good equations in the following way.

Consider first the case when the sets $\boldsymbol{S}(u) - \boldsymbol{S}(v)$ and $\boldsymbol{S}(v) - \boldsymbol{S}(u)$ are both nonempty. Denote by f the substitution with $f(x) = x^*$ for any $x \in (\boldsymbol{S}(u) - \boldsymbol{S}(v)) \cup (\boldsymbol{S}(v) - \boldsymbol{S}(u))$ and $f(x) = x$ for any other variable x. Remove the equation $u \approx v$ from B and replace it with the following three equations:

$$u \approx f(u), \qquad f(u) \approx f(v), \qquad f(v) \approx v.$$

Each of them is a consequence of $u \approx v$: the first equation because $f(u) \approx v$ is an immediate consequence, the last because $u \approx f(v)$ is an immediate consequence and the middle is an immediate consequence itself. Clearly, the triple of equations is equivalent to $u \approx v$; the first and the last equations are regular, while the middle one has both sides non-T-minimal for E.

The next case is when $\boldsymbol{S}(u)$ is a proper subset of $\boldsymbol{S}(v)$. Denote by f and g the substitutions with $f(x) = x^*$ and $g(x) = u^*$ for $x \in \boldsymbol{S}(v) - \boldsymbol{S}(u)$ and $f(x) = g(x) = x$ for any other variable x. Remove the equation $u \approx v$ from B and replace it with the following three equations:

$$u \approx g(v), \qquad g(v) \approx f(v), \qquad f(v) \approx v.$$

Again, the resulting new base contains three good equations in place of one which was bad.

One can proceed symmetrically if $\boldsymbol{S}(v)$ is a proper subset of $\boldsymbol{S}(u)$. $\square$

Lemma 3. *Suppose that the language ρ contains at least one symbol of nonzero arity. Let T be a term finite equational theory and let E be a an equational theory properly extending T and finitely based relative to T. Then there exists a finite T-base B for E with the following two properties:*

(1) *if $u \approx v \in B$ is an equation at least one side of which is a term T-minimal for E, then $u \approx v$ is regular;*

(2) *if $u_0, \dots, u_j$ is a proof based on B modulo T and u_i is a term T-minimal for E with $i \in \{1, \dots, j-1\}$, then either $u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_j$ is also a proof based on B modulo T or $u_{i-1} \approx u_{i+1} \in T$.*

Proof. Let B_0 be a finite T-base for E; by Lemma 2, we can suppose that B_0 already has the property formulated in (1). Denote by K the set of the terms u for which there exists a v with $u \approx v \in B_0 \cup B_0^{-1}$. Let $k = \max\{|\boldsymbol{S}(u)| : u \in K\}$, and take pairwise distinct variables $x_1, \dots, x_k$. Denote by K' the set of the terms u such that $\boldsymbol{S}(u) \subseteq \{x_1, \dots, x_k\}$ and $u \sim_T v$ for a term $v \in K$. Denote by B^* the equivalence on K' generated by the equations $u \approx v \in K' \times K'$ that are similar to regular equations from $B_0 \cup T$, and put $B = (B_0 \cup B^*) - T$.

Clearly, B is a finite T-base for E. Since the equations in $B - B_0$ are all regular, B also satisfies (1). It is easy to see that if $u \approx v$ is an equation which is similar to a regular equation from B and $\boldsymbol{S}(u) \subseteq \{x_1, \dots, x_k\}$, then $u \approx v \in B$.

Let $u_0, \dots, u_j$ be a proof based on B modulo T and let u_i, with $0 < i < j$, be a term T-minimal for E. There exist two equations $p \approx q$ and $r \approx s$ in $B \cup B^{-1}$, two lifts L, M and two substitutions f, g such that

$$u_{i-1} \approx Lf(p) \in T, \quad u_i \approx Lf(q) \in T, \quad u_i \approx Mg(r), \in T \quad u_{i+1} Mg(s) \in T.$$

Since $q \leq_T u_i$, $p \approx q \in E - T$ and the term u_i is T-minimal, we have $q \sim_T u_i$. Similarly, $r \sim_T u_i$. But then both L and M are necessarily identical mappings and the substitutions f and g coincide with some automorphisms of the term algebra when restricted to $\boldsymbol{S}(q)$ and $\boldsymbol{S}(r)$

respectively. Moreover, the equations $p \approx q$ and $r \approx s$ are both regular by (1), since the terms q and r are T-minimal for E. We get

$$u_{i-1} \approx f(p) \in T, \quad u_i \approx f(q) \in T, \quad u_i \approx g(r) \in T, \quad u_{i+1} \approx g(s) \in T$$

for some automorphisms f and g of the term algebra. Since the equations $p \approx q$ and $f^{-1}g(r) \approx f^{-1}g(s)$ are both regular and each is similar to an equation from B, and since $q \approx f^{-1}g(r) \in T$ (which follows from $f(q) \approx u_i \in T$ and $u_i \approx g(r) \in T$), it follows from the construction of B that also the regular equation $p \approx f^{-1}g(s)$ belongs to $B \cup T$. Now $u_{i-1} \approx f(p) \in T$ and $u_{i+1} \approx ff^{-1}g(s) = u_{i+1} \approx g(s) \in T$, so that $u_{i-1} \approx u_{i+1}$ is an immediate T-consequence of $p \approx f^{-1}g(s)$. If $p \approx f^{-1}g(s)$ does not belong to T, then the sequence $u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_j$ is a proof based on B modulo T; otherwise, $u_{i-1} \approx u_{i+1} \in T$. □

Let the language ρ be given and let the set of variables be arranged into an infinite sequence. Let $u \approx v$ be a regular equation and let $z_1, z_2, \cdots$ be the subsequence of the sequence of all variables, consisting of the variables not belonging to $\boldsymbol{S}(u)$. By a **primitive substitution** for the equation $u \approx v$ we mean a substitution f such that there is precisely one variable x with $f(x) \neq x$, this variable x belongs to $\boldsymbol{S}(u)$ and either $f(x) = F(z_1, \dots, z_n)$ for an operation symbol F of some arity n or $f(x) = y$ for a variable $y \in \boldsymbol{S}(u) - \{x\}$. By a **primitive lift** for $u \approx v$ we mean a lift L such that there exist an operation symbol F of some nonzero arity n and an index $i \in \{1, \dots, n\}$ with $L(t) = F(z_1, \dots, z_{i-1}, t, z_{i+1}, \dots, z_n)$ for all terms t.

If the language ρ is finite, then clearly for any regular equation there are only finitely many primitive substitutions and only finitely many primitive lifts.

Lemma 4. *Let T be a term finite equational theory and let $u \approx v$ be a regular equation. If f is either a primitive substitution or a primitive lift for $u \approx v$, then $u < f(u)$ and $v < f(v)$ and, consequently, $u <_T f(u)$ and $v <_T f(v)$. If $u' \approx v'$ is an immediate T-consequence of $u \approx v$ with either $u <_T u'$ or $v <_T v'$, then $u' \approx v'$ is an immediate T-consequence of an equation $f(u) \approx f(v)$ where f is either a primitive substitution or a primitive lift for $u \approx v$.*

Proof. The first statement is obvious. Let $u' \approx v'$ be an immediate T-consequence of $u \approx v$, so that there exist a substitution h and a lift H with $u' \approx Hh(u) \in T$ and $v' \approx Hh(v) \in T$. If either $u <_T u'$ or $v <_T v'$, then either H is not the identical lift or the restriction of h to $\boldsymbol{S}(u)$ is not a one-to-one mapping into the set of variables. It is clear that in the first case $u' \approx v'$ is an immediate T-consequence of $L(u) \approx L(v)$ for a lift L which is primitive for $u \approx v$. In the second case, if H is the identity, there exists a variable $x \in \boldsymbol{S}(u)$ such that either $h(x)$ is not a variable or $h(x) = h(y)$ for a variable $y \in \boldsymbol{S}(u) - \{x\}$; then h can be expressed as $h = gf$ where f is a primitive substitution for $u \approx v$ and g is some other substitution, and $u' \approx v'$ is an immediate T-consequence of $f(u) \approx f(v)$. □

Theorem 5. *Let T be a term finite equational theory of a finite language ρ, let E be an equational theory properly extending T and finitely based with respect to T, and let m be a term T-minimal for E. Then $\boldsymbol{C}_{m,T}(E)$ is also finitely based with respect to T.*

Proof. The situation is clear if ρ is a very simple language, so we can suppose that ρ contains at least one operation symbol of nonzero arity. There exists a finite T-base B for E with the two properties stated in Lemma 3. Denote by B' the union of two sets: the set of the equations $u \approx v \in B$ such that $u \not\sim_T m \not\sim_T v$; and the set of the equations of the form $f(u) \approx f(v)$ where $u \approx v \in B$ is an equation with either $u \sim_T m$ or $v \sim_T m$ and f is either a primitive substitution or a primitive lift for $u \approx v$.

Evidently, B' is a finite set of equations belonging to $\boldsymbol{C}_{m,T}(E)$. If $u \approx v$ is an equation from $\boldsymbol{C}_{m,T}(E)$ such that $u \approx v \notin T$, then by Lemma 3 there exists a $u_0, \dots, u_j$ of $u \approx v$ based on B

modulo T such that $u_i \not\sim_T m$ for all i; for example, any minimal proof of the equation based on B modulo T has this property. Let $i \in \{1, \ldots, j\}$, so that $u_{i-1} \approx u_i$ is an immediate T-consequence of an equation $v_{i-1} \approx v_i \in B \cup B^{-1}$. If this last equation does not belong to $B' \cup B'^{-1}$, then either $v_{i-1} \sim_T m$ or $v_i \sim_T m$ and the equation is regular, so that either $v_{i-1} <_T u_{i-1}$ or $v_i <_T u_i$ and then, according to Lemma 4, $u_{i-1} \approx u_i$ is an immediate T-consequence of an equation $f(v_{i-1}) \approx f(v_i)$ for some f which is either a primitive substitution or a primitive lift for $v_{i-1} \approx v_i$. In any case, $u_{i-1} \approx u_i$ is an immediate T-consequence of an equation from $B' \cup B'^{-1}$, and the proof $u_0, \ldots, u_j$ is a proof of $u \approx v$ based on B' modulo T. We see that B' is a finite T-base for $\boldsymbol{C}_{m,T}(E)$. □

Theorem 6. *Let T be a term finite equational theory of a finite language ρ. Every equational theory properly extending T and finitely based relative to T has a lower cover in the lattice of equational theories of the language ρ which extends T and is again finitely based relative to T.*

In particular and in other words, every finitely based variety other than the variety of all ρ-algebras has a finitely based cover in the lattice of varieties of ρ-algebras.

Proof. It follows directly from Theorem 1 and Theorem 5. □

Theorem 7. *There is an algorithm producing for any nontrivial finite set B_0 of equations of a finite language ρ another finite set B' of equations of the language ρ and a positive integer n such that the equational theory E' based on B' is properly contained in the equational theory E based on B_0, and the interval E/E' contains at most n equational theories.*

Proof. We are dealing with the case $T = O_\rho$, and the parameter T can thus be disregarded. The first step is to find a term m minimal for E, which can be done effectively, since one can search for it among the finitely many terms that are a left or a right side of an equation from B_0. The next step is to find a finite base B for E, having the two properties formulated in Lemma 3; the construction given in the proof of Lemma 3 can easily be seen to be effective. The last step, to construct a finite base B' for the equational theory $E' = \boldsymbol{C}_m(E)$, has been carried out, also effectively, in the proof of Theorem 5. □

Remark. Theorem 7 remains true with respect to any decidable term finite equational theory of a finite language. Such theories can be seen to be those term finite equational theories which have a recursive set of equational axioms.

Remark. Taking, moreover, the proof of Theorem 1 into account, one would be tempted to say that there is also an algorithm producing for any nontrivial finite base B_0 another finite base B'' such that E'', the equational theory based on B'', is covered by E. Let us distinguish two cases.

The first case is when there exists a term m minimal for B_0 such that $u \approx v \in B_0 \cup B_0^{-1}$ and $u \sim m$ imply $u \sim v$ and $\boldsymbol{S}(u) = \boldsymbol{S}(v)$. One can construct the finite base B' for $E' = \boldsymbol{C}_m(E)$ as in Theorem 7, and then add as many equations $u \approx v$ with $u \sim m \sim v$ and $u \approx v \in E$ as necessary to obtain the new base B'' with the desired property, which can be done effectively in this case.

In the remaining case, when there is no term m as above, one can take an arbirary minimal term m for B_0 and still construct the finite base B' for $E' = \boldsymbol{C}_m(E)$ effectively, following the proof of Theorem 7. In this case we are, moreover, able to see easily that the finite interval E/E' contains a unique equational theory covered by E; its base can be obtained from B' by adding the equations $m \approx v$ such that $m \sim v$, $\boldsymbol{S}(m) = \boldsymbol{S}(v)$ and $m \approx v$ is a consequence of B_0. However, if we were to recover this finite base effectively, we would need to decide, for every B_0, which of the equations $u \approx v$ with $u \sim m \sim v$ are its consequences. In a very particular case, one would need to have an algorithm conflicting with the following assertion.

Assertion. *Consider the language ρ consisting of three unary operation symbols and one binary symbol for multiplication, and denote by $\mathcal{S}$ the set of the finite sets of ρ-equations, for which the* *term xy serves as a minimal term. There is no algorithm deciding for any $B \in \mathcal{S}$, whether the commutative law, the equation $xy \approx yx$, is a consequence of B.*

Proof. As it has been proved in Mal'tsev [5], there exists an undecidable, finitely based and regular equational theory T of the language containing two unary symbols p and q. Denote by ρ the language containing p, q, one more unary symbol f and a binary symbol for multiplication. For any pair s, t of finite sequences of elements of $\{p, q\}$ denote by $M_{s,t}$ the equational theory based on (the equations of a finite base for) T and, moreover, the following two equations:

$$fs(xy) \approx fs(yx),$$
$$ft(xy) \approx xy.$$

Clearly, it is sufficient to prove that $xy \approx yx$ is a consequence of $M_{s,t}$ if and only if $s(x) \approx t(x) \in T$. The converse implication is evident, so let $s(x) \approx t(x) \notin T$. Since also $t(x) \approx t(y) \notin T$, there is a model P for T, containing two elements a and b such that $s(a) \neq t(a)$ and $t(a) \neq t(b)$. It is then clear that in P^ω, which is also a model for T, there are infinitely many elements c such that $s(c) \neq t(c)$, with the elements $t(c)$ pairwise different. So, there is a countably infinite model Q of T having an infinite subset S such that $Q - S$ is also infinite, t is one-to-one if restricted to S and the sets $s(S)$ and $t(S)$ are disjoint. Define a multiplication on Q in such a way that Q becomes an absolutely free groupoid, freely generated by $Q - S$; and define a unary operation f on Q in such a way that f is constant on $s(S)$ and $f(t(a)) = a$ for any $a \in S$. Clearly, Q then becomes a model of $M_{s,t}$; it does not, of course, satisfy $xy \approx yx$.

This proof follows the general pattern of undecidability resuts for equational theories established in McNulty [12] and [13], Murskiĭ [14] and Perkins [15]. □

So, we were not able to decide the following question; there is still a chance for its positive solution, although it seems more likely that the answer will be negative.

Problem. Can one effectively construct, for any nontrivial finite set B of equations of a finite language, another finite set B'' of equations such that the equational theory based on B'' is covered by the equational theory based on B?

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