BIJECTIVE REFLEXIONS AND COREFLEXIONS OF COMMUTATIVE UNARS

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Abstract. In this paper we investigate bijective reflexions and coreflexions of commutative unars.

V tomto článku jsou vyšetřovány bijektivní reflexe a koreflexe komutativních unárů.

1. Introduction and preliminaries

Let n be a positive integer. By an n-unar we mean an algebra $\mathbf{A}$ with n unary operations, i.e., $\mathbf{A} = (A, f_1, \dots, f_n)$ where A is a nonempty set and $f_1, \dots, f_n$ are transformations of A. The n-unar is said to be

— commmutative if $f_i f_j = f_j f_i$ for all $i, j \in \{1, \dots, n\}$,
— cancellative if the transformations f_i are all injective,
— divisible if the transformations f_i are all surjective,
— bijective if the unary operations f_i are all permutations.

We denote by $\mathcal{C}_n$ the variety of commutative n-unars, by $\mathcal{CC}_n$ the class of cancellative commutative n-unars, by $\mathcal{DC}_n$ the class of divisible commutative n-unars and by $\mathcal{BC}_n$ the class of bijective commutative n-unars. Let us observe that the n-unar $\mathbf{A} = (A, f_1, \dots, f_n)$ is commutative if and only if f_i is an endomorphism of $\mathbf{A}$, for any i.

In this paper we are going to investigate reflexions and coreflexions of arbitrary commutative n-unars in the category of bijective commutative n-unars. If L is a subcategory of a category K and A is an object of K, then by a reflexion of A in L we mean an object B of L together with a morphism $\mu : A \to B$ such that for any morphism $\nu : A \to C$, where C is an object of L, there exists a unique L-morphism $\lambda : B \to C$ with $\nu = \lambda\mu$. (More simply, we can say that $\mu : A \to B$ is the reflexion.) Coreflexions are defined dually.

With respect to the subcategory of bijective commutative n-unars, the reflexions and coreflexions will be called bijective. We shall give the constructions and also investigate related concepts of bijective envelopes and bijective covers of cancellative and divisible commutative n-unars, respectively.

The construction of a bijective coreflexion of a divisible commutative 2-unar has been given in an earlier paper [2]; it was needed as an auxiliary result to prove that

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every medial division groupoid is a homomorphic image of a medial quasigroup, and turned out to be a two-dimensional version of the ergodic-theoretic construction of an automorphism on a measure space naturally extending an endomorphism; see Chapter 10, §4 of [1] for the entropic theory of dynamical systems. In Section 4 of the present paper we generalize this construction.

Let $\mathbf{A} = (A, f_1, \dots, f_n)$ be a commutative n-unar and denote by f the composition $f_1 \dots f_n$, so that f is an endomorphism of $\mathbf{A}$. One can easily see that $\mathbf{A}$ is cancellative if and only if f is injective, and $\mathbf{A}$ is divisible if and only if f is surjective. Consequently, $\mathbf{A}$ is bijective if and only if f is a permutation of A. We have

$$\mathrm{id}_A \subseteq \ker(f) \subseteq \ker(f^2) \subseteq \dots$$

and the relation $c_{\mathbf{A}} = \bigcup_{i=0}^{\infty} \ker f^i$ is the smallest cancellative congruence of $\mathbf{A}$. (By a cancellative congruence we mean such a congruence that the corresponding factor is cancellative.)

Subalgebras of n-unars will be called subunars. By a dense subunar of a commutative n-unar $\mathbf{A}$ we mean a subunar $\mathbf{B}$ such that for any element $a \in A$ there is a nonnegative integer i with $f^i(a) \in B$.

1.1. Lemma. *Let φ and ψ be two homomorphisms of a commutative n-unar $\mathbf{A}$ into a cancellative commutative n-unar $\mathbf{C}$ and let $\mathbf{B}$ be a dense subunar of $\mathbf{A}$. If φ and ψ coincide on B, then $\varphi = \psi$.*

Proof. Let $f = f_1 \dots f_n$ and $g = g_1 \dots g_n$, where $\mathbf{A} = (A, f_1, \dots, f_n)$ and $\mathbf{C} = (C, g_1, \dots, g_n)$. Let $a \in A$. Then $f^i(a) \in B$ for some i and we have

$$g^i \varphi(a) = \varphi f^i(a) = \psi f^i(a) = g^i \psi(a).$$

Hence $\varphi(a) = \psi(a)$, since g is injective. $\square$

Let $\mathbf{A} = (A, f_1, \dots, f_n)$ be a commutative n-unar and $f = f_1 \dots f_n$. We denote by $\mathrm{D}_{\mathbf{A}}$ the set of the elements of A for which there exists an infinite sequence $a_0, a_1, \dots$ with $a_i = f(a_{i+1})$ for all i. Clearly, $\mathrm{D}_{\mathbf{A}}$ is either empty or the underlying set of a subunar of $\mathbf{A}$, which is then denoted by $\mathbf{D}_A$; it is the largest divisible subunar of $\mathbf{A}$.

We denote by $\mathcal{C}_n^+$ the class of the commutative n-unars $\mathbf{A}$ with nonempty $\mathrm{D}_{\mathbf{A}}$. Clearly, the classes $\mathcal{BC}_n$, $\mathcal{DC}_n$ and the class of finite commutative n-unars are contained in $\mathcal{C}_n^+$.

If φ is a homomorphism of a commutative n-unar $\mathbf{A}$ into a commutative n-unar $\mathbf{B}$, then clearly $\varphi(\mathrm{D}_{\mathbf{A}}) \subseteq \mathrm{D}_B$. In particular, if $\mathbf{A} \in \mathcal{C}_n^+$, then $\mathbf{B} \in \mathcal{C}_n^+$.

1.2. Example. Define a unary operation f on the set Z of integers by

$$f(i) = i - 1 \text{ for } i \leq 0,$$
$$f(n^2 + i) = i \text{ for } n > 0 \text{ and } 0 \leq i \leq 2n.$$

For the 1-unar (Z, f) we have $\mathrm{D}_{(Z,f)} = \emptyset$, although the set $\bigcap_{i \geq 0} f^i(Z)$ is nonempty.

2. Bijective reflexions of commutative unars

Let $\mathbf{A} = (A, f_1, \dots, f_n)$ be a commutative n-unar. Put $f = f_1 \dots f_n$, $\hat{A} = A \times \omega$ where ω is the set of nonnegative integers and define transformations $\hat{f}_1, \dots, \hat{f}_n$ of $\hat{A}$ by

$$\hat{f}_i(a, k) = (f_i(a), k).$$

Clearly, the algebra $\hat{\mathbf{A}} = \hat{A}(\hat{f}_1, \dots, \hat{f}_n)$ is again a commutative n-unar and the mapping $\iota : a \mapsto (a, 0)$ is an injective homomorphism of $\mathbf{A}$ into $\hat{\mathbf{A}}$.

Denote by r the set of the ordered pairs $((a, k), (f(a), k + 1))$ where $a \in A$ and $k \in \omega$, so that r is a binary relation on $\hat{A}$; denote by $s_{\mathbf{A}}$ (or only by s) the transitive closure of $r \cup r^{-1} \cup \mathrm{id}_{\hat{A}}$. Then s is the equivalence on $\hat{A}$ generated by r.

2.1. Lemma. *Let (a, k) and (b, m) be two elements of $\hat{A}$ with $k \leq m$. Then $((a, k), (b, m)) \in s$ if and only if $f^j(a) = f^{j+k-m}(b)$ for an integer $j \geq m - k$.*

Proof. The 'if' part is clear, since $((a, k), (f^j(a), k+j)) \in s$ and $((b, m), (f^{j+k-m}, k+j)) \in s$ by the definition of s. Let $((a, k), (b, m)) \in s$. There is a finite sequence $x_0, \dots, x_q$ $(q \geq 0)$ of elements of $\hat{A}$ such that $x_0 = (a, k)$, $x_q = (b, m)$ and $(x_i, x_{i+1}) \in r \cup r^{-1}$ for $0 \leq i < q$. We can assume that q is the least nonnegative integer with respect to the existence of a finite sequence with these properties. Using the minimality of q, the obvious observation $r \cap r^{-1} = \emptyset$ and the fact that $(x_i, x_{i+1}) \in r^{-1}$ and $(x_{i+1}, x_{i+2}) \in r$ together imply $x_i = x_{i+2}$, it is easy to see that there is a $j \in \{0, \dots, q\}$ such that $(x_i, x_{i+1}) \in r$ for all $i < j$ and $(x_i, x_{i+i}) \in r^{-1}$ for all $i \geq j$. But then $x_j = (f^j(a), k + j)$ and, from the other side, $x_j = (f^{q-j}(b), m + q - j)$; we get $k + j = m + q - j$, which implies $q - j = j + k - m$, and $f^j(a) = f^{q-j}(b)$. $\square$

The relation r is clearly compatible with the unary operations $\hat{f}_i$. This implies that s is a congruence of $\hat{\mathbf{A}}$; s is just the congruence generated by r. Denote by $\tilde{\mathbf{A}} = (\tilde{A}, \tilde{f}_1, \dots, \tilde{f}_n)$ the factor $\hat{\mathbf{A}}/s$, by π the natural projection of $\hat{\mathbf{A}}$ onto $\tilde{\mathbf{A}}$ and put $\psi_{\mathbf{A}} = \pi\iota$, so that $\psi_{\mathbf{A}}$ is a homomorphism of $\mathbf{A}$ into $\tilde{\mathbf{A}}$.

2.2. Theorem. *$\tilde{\mathbf{A}}$ is a bijective commutative n-unar; together with $\psi_{\mathbf{A}}$, it is a reflexion of $\mathbf{A}$ in the category $\mathcal{BC}_n$. We have $\ker(\psi_{\mathbf{A}}) = c_{\mathbf{A}}$ and the range of $\psi_{\mathbf{A}}$ is a dense subunar of $\tilde{A}$ which is isomorphic to $\mathbf{A}/c_{\mathbf{A}}$.*

Proof. Let $\tilde{f} = \tilde{f}_1 \dots \tilde{f}_n$. In order to show that $\tilde{\mathbf{A}}$ is bijective, it is sufficient to verify that $\tilde{f}$ is a permutation of $\tilde{A}$. From

$$(a, k)/s = (f(a), k + 1)/s = \tilde{f}((a, k + 1)/s)$$

we see that $\tilde{f}$ is surjective. On the other hand, let $(a, k)/s$ and $(b, m)/s$ be two elements of $\tilde{A}$ with $k \leq m$ such that $\tilde{f}((a, k)/s) = \tilde{f}((b, m)/s)$. Then we have $((f(a), k), (f(b), m)) \in s$ and consequently $f^{j+1}(a) = f^{j+1+k-m}(b)$ for some $j \geq m - k$ by 2.1; a second application of 2.1 yields $(a, k)/s = (b, m)/s$. Hence $\tilde{f}$ is also injective.

It follows easily from 2.1 that $\ker(\psi_{\mathbf{A}}) = c_{\mathbf{A}}$. If $a \in A$ and $k \in \omega$, then $\tilde{f}((a, k)/s) = (a, 0)/s \in \psi_{\mathbf{A}}(A)$, so that the range of $\psi_{\mathbf{A}}$ is a dense subunar of $\tilde{\mathbf{A}}$; the subunar is isomorphic to $\mathbf{A}/c_{\mathbf{A}}$ by the homomorphism theorem.

It remains to prove that $\psi_{\mathbf{A}} : \mathbf{A} \to \tilde{\mathbf{A}}$ is a reflexion of $\mathbf{A}$ in $\mathcal{BC}_n$. Let ρ be a homomorphism of $\mathbf{A}$ into a bijective commutative n-unar $\mathbf{B} = (B, g_1, \dots, g_n)$. Put

$g = g_1 \dots g_n$, so that g is a permutation of B. Define a mapping $\lambda : \hat{A} \to B$ by $\lambda(a,k) = g^{-k}\rho(a)$. One may check easily that λ is a homomorphism of the n-unars and $\lambda\iota = \rho$. Since

$$\lambda(a,k) = g^{-k}\rho(a) = g^{-k-1}g\rho(a) = g^{-k-1}\rho f(a) = \lambda(f(a), k+1),$$

we have $s \subseteq \ker(\lambda)$. Consequently, λ induces a homomorphism $\tau : \tilde{\mathbf{A}} \to \mathbf{B}$ such that $\tau\pi = \lambda$. Then also $\tau\psi_{\mathbf{A}} = \tau\pi\iota = \lambda\iota = \rho$. Moreover, τ is unique by 1.1, because the range of $\psi_{\mathbf{A}}$ is dense in $\tilde{\mathbf{A}}$. □

2.3. Proposition. *If* $\mathbf{A}$ *is cancellative, then* $\psi_{\mathbf{A}}$ *is injective. If* $\mathbf{A}$ *is divisible, then* $\psi_{\mathbf{A}}$ *is surjective. If* $\mathbf{A}$ *is finite, then* $\tilde{\mathbf{A}}$ *is also finite and* $\psi_{\mathbf{A}}$ *is surjective.*

Proof. If $\mathbf{A}$ is cancellative, then $\ker(\psi_{\mathbf{A}}) = c_{\mathbf{A}} = \mathrm{id}_A$ by Theorem 2.2. If $\mathbf{A}$ is divisible, then the range of $\psi_{\mathbf{A}}$ is a divisible subunar of the bijective n-unar $\tilde{\mathbf{A}}$, and consequently bijective itself; but it is also a dense subunar and thus coincides with $\tilde{\mathbf{A}}$. If $\mathbf{A}$ is finite, then the image of $\psi_{\mathbf{A}}$ is a finite subunar of the bijective n-unar $\tilde{\mathbf{A}}$, so that again it is bijective and coincides with $\tilde{\mathbf{A}}$. □

The existence of reflexions implies that there is a functor Φ of the category $\mathcal{C}_n$ into $\mathcal{BC}_n$: if $\varphi : \mathbf{A} \to \mathbf{B}$ is a homomorphism of two commutative n-unars, then $\Phi(\varphi) : \tilde{\mathbf{A}} \to \tilde{\mathbf{B}}$ is the only homomorphism with $\Phi(\varphi)\psi_{\mathbf{A}} = \psi_{\mathbf{B}}\varphi$. It follows easily from 2.1 that $\Phi(\varphi)$ can be defined by $\Phi(\varphi)((a,k)/s_{\mathbf{A}}) = (\varphi(a),k)/s_{\mathbf{B}}$, and that $\Phi(\varphi)$ is injective (or surjective, respectively) whenever φ is.

3. Bijective envelopes of cancellative commutative unars

By a bijective envelope of a commutative n-unar $\mathbf{A}$ we mean a bijective commutative n-unar $\mathbf{B}$ such that $\mathbf{A}$ is a dense subunar of $\mathbf{B}$.

3.1. Theorem. *A commutative n-unar* $\mathbf{A}$ *has a bijective envelope if and only if it is cancellative; in that case the bijective envelope is unique up to isomorphism over* $\mathbf{A}$ *and is isomorphic with* $\tilde{\mathbf{A}}$*. If* $\mathbf{A}$ *is a subunar of a bijective commutative n-unar* $\mathbf{B}$*, then* $\mathbf{B}$ *is a bijective envelope of* $\mathbf{A}$ *if and only if* $\mathbf{B}$ *is the only bijective subunar of* $\mathbf{B}$ *containing* $\mathbf{A}$*.*

Proof. It is an easy combination of the construction and results given in Section 2. □

4. Bijective coreflexions of commutative unars

Let $\mathbf{A} = (A, f_1, \dots, f_n)$ be a commutative n-unar and put $f = f_1 \dots f_n$. Denote by $\bar{A}$ the set of the mappings $\alpha : \omega \to A$ such that $f(\alpha(i+1)) = \alpha(i)$ for all $i \in \omega$. Define unary operations $\bar{f}_1, \dots, \bar{f}_n$ on $\bar{A}$ by

$$\bar{f}_j(\alpha)(i) = f_j(\alpha(i));$$

the correctness follows from

$$f(\bar{f}_j(\alpha)(i+1)) = f(f_j(\alpha(i+1))) = f_j(f(\alpha(i+1))) = f_j(\alpha(i)) = \bar{f}_j(\alpha)(i).$$

One can easily check that that the unary operations $\bar{f}_j$ commute. Hence, if $\bar{A}$ is nonempty, then $\bar{\mathbf{A}} = (\bar{A}, \bar{f}_1, \dots, \bar{f}_n)$ is again a commutative n-unar. Define a mapping $\phi_{\mathbf{A}} : \bar{A} \to A$ by $\phi_{\mathbf{A}}(\alpha) = \alpha(0)$.

If $\mathbf{A} \notin \mathcal{C}_n^+$, then there is no bijective commutative n-unar with a homomorphism into $\mathbf{A}$; in particular, $\mathbf{A}$ has no coreflexion in the category of bijective commutative n-unars.

4.1. Theorem. *Let* $\mathbf{A} \in \mathcal{C}_n^+$. *Then* $\phi_{\mathbf{A}} : \bar{\mathbf{A}} \to \mathbf{A}$ *is a coreflexion of* $\mathbf{A}$ *in the category of bijective commutative* n*-unars. The range of* $\phi_{\mathbf{A}}$ *coincides with* $\mathrm{D}_{\mathbf{A}}$.

Proof. Since $\mathbf{A} \in \mathcal{C}_n^+$ (i.e., $\mathrm{D}_{\mathbf{A}} \neq \emptyset$), the set $\bar{A}$ is nonempty. We are going to prove that the unary operations $\bar{f}_j$ are permutations of $\bar{A}$.

If $\bar{f}_j(\alpha) = \bar{f}_j(\beta)$ for some $\alpha, \beta \in \bar{A}$, then $f_j(\alpha(i)) = f_j(\beta(i))$ for all $i \in \omega$. Consequently,

$$f(\alpha(i)) = f_1 \dots f_{j-1} f_{j+1} \dots f_n f_j(\alpha(i)) = f_1 \dots f_{j-1} f_{j+1} \dots f_n f_j(\beta(i)) = f(\beta(i))$$

for all $i \in \omega$, and hence also

$$\alpha(i) = f(\alpha(i+1)) = f(\beta(i+1)) = \beta(i),$$

i.e., $\alpha = \beta$. We see that $\bar{f}_j$ is injective.

If $\alpha \in \bar{A}$, then we can define a mapping $\beta : \omega \to A$ by

$$\beta(i) = f_1 \dots f_{j-1} f_{j+1} \dots f_n(\alpha(i+1)).$$

One can easily verify that $f(\beta(i+1)) = \beta(i)$, i.e., $\beta \in \bar{A}$. We have $\bar{f}_j(\beta)(i) = f_j(\beta(i)) = f(\alpha(i+1)) = \alpha(i)$, so that $\bar{f}_j(\beta) = \alpha$. We see that $\bar{f}_j$ is surjective, and $\mathbf{A}$ is a bijective commutative n-unar.

Obviously, $\phi_{\mathbf{A}} \bar{f}_j = f_j \phi_{\mathbf{A}}$ for every j, and hence $\phi_{\mathbf{A}}$ is a homomorphism of $\bar{\mathbf{A}}$ into $\mathbf{A}$. Clearly, $\mathrm{D}_{\mathbf{A}}$ is just the range of $\phi_{\mathbf{A}}$.

Let $\mathbf{B} = (B, g_1, \dots, g_n)$ be a bijective commutative n-unar and $\rho : \mathbf{B} \to \mathbf{A}$ be a homomorphism. Put $g = g_1 \dots g_n$. For every $x \in B$ define a mapping $\lambda(x) : \omega \to A$ by $\lambda(x)(i) = \rho(g^{-i}(x))$. Then

$$f(\lambda(x)(i+1)) = f(\rho(g^{-i-1}(x))) = \rho(g g^{-i-1}(x)) = \rho(g^{-i}(x)) = \lambda(x)(i)$$

and this means that $\lambda(x) \in \bar{A}$. Consequently, λ is a mapping of B into $\bar{A}$. It is a homomorphism of $\mathbf{B}$ into $\bar{\mathbf{A}}$, since

$$\lambda(g_j(x))(i) = \rho(g^{-i}(g_j(x))) = f_j \rho(g^{-i}(x)) = f_j(\lambda(x)(i)) = \bar{f}_j(\lambda(x))(i),$$

i.e., $\lambda(g_j(x)) = \bar{f}_j(\lambda(x))$. For $x \in B$ we have $\phi_{\mathbf{A}} \lambda(x) = \lambda(x)(0) = \rho(x)$, and hence $\phi_{\mathbf{A}} \lambda = \rho$.

It remains to prove the uniqueness of a homomorphism λ with this property. Let $\lambda' : \mathbf{B} \to \bar{\mathbf{A}}$ be a homomorphism with $\phi_{\mathbf{A}} \lambda' = \rho$. Put $\bar{f} = \bar{f}_1 \dots \bar{f}_n$. One can easily check that $\bar{f}(\alpha)(i) = f(\alpha(i))$ and $\bar{f}^i(\alpha)(i) = \alpha(0)$ for all $\alpha \in \bar{A}$ and $i \in \omega$. If $x \in B$, then

$$\begin{aligned} \lambda'(x)(i) &= (\bar{f}^{-i}(\lambda'(x)))(0) = \lambda'(g^{-i}(x))(0) = \phi_{\mathbf{A}}(\lambda'(g^{-i}(x))) \\ &= \rho(g^{-i}(x)) = \lambda(x)(i) \end{aligned}$$

for every $i \in \omega$, so that $\lambda'(x) = \lambda(x)$. $\square$

4.2. Proposition. *If* $\mathbf{A}$ *is divisible, then* $\phi_{\mathbf{A}}$ *is surjective. If* $\mathbf{A} \in \mathcal{C}_n^+$ *is cancellative, then* $\phi_{\mathbf{A}}$ *is injective. If* $\mathbf{A}$ *is finite, then* $\bar{\mathbf{A}}$ *is finite.*

Proof. If $\mathbf{A}$ is divisible, then the range $\mathrm{D}_{\mathbf{A}}$ of $\phi_{\mathbf{A}}$ coincides with A. If $\mathbf{A}$ is injective, then f is injective and it is easy to see that for any element $a \in A$ there is at most one element $\alpha \in \bar{A}$ with $a = \phi_{\mathbf{A}}(\alpha)$. If $\mathbf{A}$ is finite, then $\mathbf{D}_{\mathbf{A}}$ is bijective and the rest is clear. □

From the existence of coreflexions it follows that there is a functor Ψ of the category $\mathcal{C}_n^+$ into $\mathcal{BC}_n$: if $\varphi : \mathbf{A} \to \mathbf{B}$ is a homomorphism and $\mathbf{A} \in \mathcal{C}_n^+$ (which implies that also $\mathbf{B} \in \mathcal{C}_n^+$), then $\Psi(\varphi) : \bar{\mathbf{A}} \to \bar{\mathbf{B}}$ is the only homomorphism with $\phi_{\mathbf{B}}\Psi(\varphi) = \varphi\phi_{\mathbf{A}}$. It follows easily from 4.1 that $\Psi(\varphi)$ can be defined by $\Phi(\varphi)(\alpha)(i) = \varphi(\alpha(i))$, and that $\Psi(\varphi)$ is injective whenever φ is injective. On the other hand, the following example shows that $\Psi(\varphi)$ is not necessarily surjective if φ is surjective.

4.3. Example. Consider the 1-unars $\mathbf{Z} = (Z, f)$ and $\mathbf{N} = (N, g)$, where Z is the set of all integers, N is the set of nonnegative integers, $f(i) = i - 1$ for all i, $g(i) = i-1$ for $i > 0$, and $g(0) = 0$. The mapping $\varphi : Z \to N$, defined by $\varphi(i) = i$ for $i \geq 0$ and $\varphi(i) = 0$ for $i \leq 0$, is a homomorphism of $\mathbf{Z}$ onto $\mathbf{N}$. Since $\mathbf{Z}$ is bijective, $\bar{\mathbf{Z}}$ is isomorphic to $\mathbf{Z}$. On the other hand, $\mathbf{N}$ is only divisible. As it is easy to see, $\bar{\mathbf{N}}$ contains a subunar $\mathbf{C}$ isomorphic with $\mathbf{Z}$ and an element e not belonging to C, with $\bar{g}(e) = e$; we have $\bar{N} = C \cup \{e\}$. Clearly, $\Psi(\varphi)$ is an isomorphism of $\bar{\mathbf{Z}}$ onto $\mathbf{C}$; it is an injective, but not surjective homomorphism of $\bar{\mathbf{Z}}$ into $\bar{\mathbf{N}}$.

5. Bijective covers of divisible commutative unars

By a bijective cover of a divisible commutative n-unar $\mathbf{A}$ we mean a surjective homomorphism $\rho : \mathbf{B} \to \mathbf{A}$ such that $\mathbf{B}$ is a bijective commutative n-unar and there is no cancellative congruence of $\mathbf{B}$, contained in $\ker(\rho)$, other than id_B.

Such a bijective cover is said to be strong if, moreover, there is no proper bijective subunar of $\mathbf{B}$, the ρ-image of which would give the whole of A.

5.1. Theorem. *Let* $\mathbf{A}$ *be a divisible commutative* n*-unar. A homomorphism* $\rho : \mathbf{B} \to \mathbf{A}$ *is a bijective cover of* $\mathbf{A}$ *if and only if there is an injective homomorphism* $\sigma : \mathbf{B} \to \bar{\mathbf{A}}$ *with* $\rho = \phi_{\mathbf{A}}\sigma$.

In other words, all representative examples of bijective covers of $\mathbf{A}$ *can be obtained by taking a bijective subunar* $\mathbf{C}$ *of* $\bar{\mathbf{A}}$*, such that* $\phi_{\mathbf{A}}(C) = A$*, and restricting the homomorphism* $\phi_{\mathbf{A}} : \bar{\mathbf{A}} \to \mathbf{A}$ *to* C*.*

Proof. First, let $\rho : \mathbf{B} \to \mathbf{A}$ be a bijective cover. By Theorem 4.1, $\rho = \phi_{\mathbf{A}}\sigma$ for a homomorphism $\sigma : \mathbf{B} \to \bar{\mathbf{A}}$. Then $\ker(\sigma)$ is a cancellative congruence of $\mathbf{B}$ contained in $\ker(\rho)$, so that $\ker(\sigma) = \mathrm{id}_B$ and σ is injective.

Next, let $\rho = \phi_{\mathbf{A}}\sigma$ for an injective homomorphism $\sigma : \mathbf{B} \to \bar{\mathbf{A}}$ and let c be a cancellative congruence of $\mathbf{B}$ contained in $\ker(\rho)$. Denote by $\mathbf{B}'$ the image of $\mathbf{B}$ under σ and by c' the image of c under σ, so that c' is a cancellative congruence of $\mathbf{B}'$. Then $d = c' \cup \mathrm{id}_{\bar{A}}$ is a cancellative congruence of $\bar{\mathbf{A}}$ and $d \subseteq \ker(\phi_{\mathbf{A}})$.

Let $(\alpha, \beta) \in d$. Since d is cancellative, $(\bar{f}^{-i}(\alpha), \bar{f}^{-1}(\beta)) \in d$ for any nonnegative integer i (we employ the notation of Section 4). But $d \subseteq \ker(\phi_{\mathbf{A}})$ and we get

$$\alpha(i) = \bar{f}^{-i}(\alpha)(0) = \bar{f}^{-i}(\beta)(0) = \beta(i).$$

This means that $\alpha = \beta$. We have proved that $d = \mathrm{id}_{\bar{A}}$; but then, $c = \mathrm{id}_B$. □

5.2. Example. In the notation of Example 4.3, the 1-unar $\mathbf{N} = (N, g)$ has two bijective covers, namely, $\phi_{\mathbf{N}} : \bar{\mathbf{N}} \to \mathbf{N}$ and $\phi : \mathbf{C} \to \mathbf{N}$, where ϕ is a restriction of $\phi_{\mathbf{N}}$. The first of these two covers is not strong, while the second is. By comparison with Section 3, we see that there are more possibilitites for bijective covers than for bijective envelopes.

5.3. Proposition. *Let* $\mathbf{A}$ *be a divisible commutative* n*-unar. There exists a bijective cover* $\rho : \mathbf{B} \to \mathbf{A}$ *of* $\mathbf{A}$ *with* $\mathrm{Card}(B) = \mathrm{Card}(A)$*. Moreover, we have* $\mathrm{Card}(B) = \mathrm{Card}(A)$ *for any strong bijective cover* $\rho : \mathbf{B} \to \mathbf{A}$ *of* $\mathbf{A}$.

Proof. With respect to Theorem 5.1, it is sufficient to prove that if $\mathbf{B}$ is a bijective subunar of $\bar{\mathbf{A}}$ with $\phi_{\mathbf{A}}(B) = A$, then there is a bijective subunar $\mathbf{C}$ of $\mathbf{B}$ with $\mathrm{Card}(C) = A$ and $\phi_{\mathbf{A}}(C) = A$. Clearly, there exists a subset S of B with $\mathrm{Card}(S) = A$ and $\phi_{\mathbf{A}}(S) = A$; then we can take $\mathbf{C}$ to be the bijective subunar generated by S. $\square$

5.4. Example. Denote by S the set of finite sequences of elements of $\{1, 2\}$; the empty sequence $\emptyset$ is not excluded. Consider the divisible 1-unar $\mathbf{S} = (S, f)$, where f is defined by $f(\emptyset) = \emptyset$ and $f(a_1, \dots, a_k) = (a_1, \dots, a_{k-1})$ for $k > 0$. If $u_1, u_2, \dots$ is an infinite sequence of elements of $\{1, 2\}$, then we can define an element α_u of $\bar{S}$ by taking $\alpha_u(i) = (u_1, \dots, u_i)$. In this way we obtain an injective mapping, and hence $\mathrm{Card}(\bar{S}) = 2^{\aleph_0}$. We are going to show that $\mathbf{S}$ *has no strong bijective cover.*

Let $\rho : \mathbf{T} \to \mathbf{S}$ be a bijective cover of $\mathbf{S}$, and $\mathbf{T} = (T, g)$. Our claim will be justified by proving that there is a proper bijective subunar of $\mathbf{T}$ which is mapped onto S by ρ. With respect to Theorem 5.1, we can assume that $\mathbf{T}$ is a subunar of $\bar{\mathbf{S}}$ (so that g is a restriction of $\bar{f}$) and that ρ is a restriction of $\phi_{\mathbf{S}}$. Since ρ is surjective, there exists an element α of T with $\rho(\alpha) = (1)$, i.e., $\alpha(0) = (1)$. Denote by P the set of the elements $\bar{f}^i(\alpha)$, where i runs over all integers. Then P is a bijective subunar of $\mathbf{T}$; but also its complement $Q = T - P$ is a proper bijective subunar of $\mathbf{T}$. We are going to show that $\rho(Q) = S$.

Let $a = (a_1, \dots, a_k) \in S$. As it is easy to see, at most one of the elements $b_1 = (a_1, \dots, a_k, 1)$ and $b_2 = (a_1, \dots, a_k, 2)$ belongs to the set $\{\alpha(0), \alpha(1), \dots\}$ and thus we can take a number $j \in \{1, 2\}$ with $b_j \notin \{\alpha(0), \alpha(1), \dots\}$. Since ρ is surjective, there is an element β of T with $\rho(\beta) = b_j$, i.e., $\beta(0) = b_j$. By the choice of j we have $\beta \notin P$, and so $\beta \in Q$. Then also $\bar{f}(\beta) \in Q$. We have $\rho(\bar{f}(\beta)) = \bar{f}(\beta)(0) = f(b_j) = a$. An arbitrary element a of S belongs to $\rho(Q)$.

6. Balanced divisible commutative 2-unars

Let $\mathbf{A} = (A, f, g)$ be a divisible commutative 2-unar. Put $\mathbf{A}_l = (A, f)$ and $\mathbf{A}_r = (A, g)$, so that both $\mathbf{A}_l$ and $\mathbf{A}_r$ are divisible commutative 1-unars and we have the bijective coreflexions $\phi_{\mathbf{A}_l} : \mathbf{B}_l \to \mathbf{A}_l$ and $\phi_{\mathbf{A}_r} : \mathbf{B}_r \to \mathbf{A}_r$, where $\mathbf{B}_l = \overline{(\mathbf{A}_l)}$ and $\mathbf{B}_r = \overline{(\mathbf{A}_r)}$. Of course, g is an endomorphism of $\mathbf{A}_l$ and thus there is a corresponding endomorphism $\bar{g}$ of $\mathbf{B}_l$; as it is easy to see, it can be defined by $\bar{g}(\alpha)(i) = g(\alpha(i))$. We say that $\mathbf{A}$ is left balanced if $\bar{g}$ is a surjective endomorphism of $\mathbf{B}_l$. Analogously, f is an endomorphism of $\mathbf{A}_r$ and $\mathbf{A}$ is called right balanced if the corresponding endomorphism $\bar{f}$ of $\mathbf{B}_r$ is surjective.

6.1. Proposition. *Let* $\mathbf{A} = (A, f, g)$ *be a divisible commutative 2-unar. Then* $\mathbf{A}$ *is both left and right balanced, provided that at least one of the following four conditions is satisfied:*

(1) $f = g$;

(2) *f is bijective;*
(3) *g is bijective;*
(4) *for any triple a, b, c of elements of A with $f(a) = b = g(c)$ there is an element d of A such that $g(d) = b$ and $f(d) = c$.*

Proof. The situation is clear under any of the first three conditioons. Let the assumptions of (4) be satisfied, and let $\alpha \in \mathbf{B}_l$. We need to find an element β with $\bar{g}(\beta) = \alpha$. The elements $\beta(i)$, $i = 0, 1, \dots$ can be costructed by induction on i as follows: $\beta(0)$ is an arbitrary element of A with $g(\beta(0)) = \alpha(0)$; if $\beta(i-1)$ has already been constructed in such a way that $g(\beta(i-1)) = \alpha(i-1)$, then there is an element d with $g(d) = \alpha(i)$ and $f(d) = \beta(i-1)$, and we can take $\beta(i) = d$. $\square$

Let us remark that if $\mathbf{A} = (A, f, g)$ is left balanced, then a 'by parts' construction of a 2-unar isomorphic to $\bar{\mathbf{A}}$ is possible and may in some cases turn out to be more advantageous: first we find the 1-unar $\mathbf{B}_l$ and consider it as a 1-unar $\mathbf{C}$ with respect to the unary operation $\bar{g}$; then we find the 1-unar $\bar{\mathbf{C}}$, to which we add the endomorphism, corresponding to the fundamental operation of $\mathbf{B}_l$, as the first fundamental unary operation (and keep the other operation).

6.2. Example. Let E be the set of the ordered pairs (i, j) of integers such that either $i \geq 2$ or else $0 \leq i \leq 1$ and $j \geq 0$. Define two binary operations f and g on E by

$$f(i,j) = \begin{cases} (i,j) & \text{if } i \in \{0,1\} \text{ and } j = 0, \\ (i, j-1) & \text{otherwise} \end{cases}$$

and

$$g(i,j) = \begin{cases} (i,j) & \text{if } i = 0, \\ (1,0) & \text{if } i = 2 \text{ and } j \leq 0, \\ (i-1, j) & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mathbf{E} = (E, f, g)$ is a divisible commutative 2-unar. Put $\mathbf{B}_l = \overline{(\mathbf{E}_l)}$ and $\mathbf{B}_r = \overline{(\mathbf{E}_r)}$.

Let us prove that $\mathbf{E}$ is not left balanced. Since $f(1,0) = (1,0)$, we have $\alpha \in \mathbf{B}_l$, where $\alpha(i) = (1,0)$ for all i. Suppose that $\alpha = \bar{g}(\beta)$ for some β. Then $g(\beta(i)) = (1,0)$ for all i, and so $\beta(i) = (2, k_i)$ for some $k_i \leq 0$. Let k_j be maximal among the numbers $k_0, k_1, \dots$; its existence is clear. We have

$$(2, k_j) = \beta(j) = f(\beta(j+1)) = f(2, k_{j+1}) = (2, k_{j+1} - 1),$$

so that $k_j = k_{j+1} - 1$, a contradiction with the maximality of k_j.

Let us prove that $\mathbf{E}$ is right balanced. Let $\alpha \in \mathbf{B}_r$; for any $i \geq 0$, denote $\alpha(i) = (p_i, q_i)$. If $p_i = 0$ for every i, then there is a q with $q_i = q$ for all i and we have $\alpha = \bar{f}(\beta)$, where $\beta(i) = (0, q+1)$ for all i. Therefore, we can assume that $p_j \geq 1$ for some j. Then $p_{j+k} = p_j + k$ for all $k \geq 0$. Define β by

$$\beta(i) = \begin{cases} (p_i, q_i - 1) & \text{for } i > j, \\ g(\beta(i+1)) & \text{(by induction) for } i = j, j-1, \dots, 0. \end{cases}$$

It is easy to verify that $\beta \in \mathbf{B}_r$ and $\alpha = \bar{f}(\beta)$.

We see that the divisible commutative 2-unar (E, f, g) is right balanced, but not left balanced. Consequently, (E, g, f) is left balanced, but not right balanced. The cartesian product of these two 2-unars is neither left nor right balanced.

References

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