A NOTE ON FINITE SETS OF TERMS CLOSED UNDER SUBTERMS AND UNIFICATION

Jaroslav Ježek

Abstract. The paper contains two remarks on finite sets of groupoid terms closed under subterms and the application of unifying pairs.

Abstrakt. lnek obsahuje dv poznmky o konench mnoinch grupoidovch term, uzavench pro podtermy a pro aplikaci unifikujcch dvojic.

By a term we shall mean a groupoid term. Let us write $u+v \sim t$ if $t=f(u)=g(v)$ for a unifying pair f,g of the terms u and v, i.e., if t is a substitution instance of both u and v and any term that is a substitution instance of both u and v, is a substitution instance of t. (A survey of unification theory is contained, for example, in Dershowitz and Jouannaud [1].)

Let us call a set S of terms SU-closed if it is closed with respect to subterms and whenever $u+v \sim t$ for two terms $u, v \in S$, then $t \sim t' \in S$ for some t'.

Theorem 1. *There is no finite, SU-closed set of terms containing the following three terms:*

$$(xy \cdot z)x, \qquad x(yz \cdot u), \qquad x \cdot yx.$$

Proof. Let us define an infinite sequence $a_0, a_1, \dots$ of terms as follows: a_0 is a variable; $a_{i+1} = a_i x$ for a variable x not occurring in a_i. So, $a_i = (((x_0x_1)x_2)\dots)x_i$, where $x_0, \dots, x_i$ is a sequence of pairwise distinct variables. Also, put $b_i = ya_i$, where y is a variable not occurring in a_i. Hence $b_2 \sim x(yz \cdot u)$. It is easy to see that

$$(xy \cdot z)x + b_i \sim ((a_i x)y)a_i \supseteq a_{i+2}$$

for $i \geq 2$ (where x, y are two distinct variables not occurring in a_i, and

$$x \cdot yx + a_{i+1} \sim a_i \cdot xa_i \supseteq b_i$$

for $i \geq 3$. $\square$

The depth of a term is defined inductively as follows: the depth of a variable is 0; the depth of t_1t_2 is $1+\max(d_1,d_2)$, where d_1 is the depth of t_1 and d_2 is the depth of t_2. So, $xy \cdot zu$ is of depth 2.

Theorem 2. *There exists a finite, SU-closed set of terms containing (up to similarity) all terms of depth at most 2.*

Proof. The set consists of the terms of depth 2, plus the twelve terms

$$\begin{array}{lcl}
xx\cdot(xx\cdot xx) & \sim & x\cdot xx + xx\cdot y\\
xy\cdot(xy\cdot xy) & \sim & x\cdot xx + xy\cdot z\\
xx\cdot(xx\cdot y) & \sim & x\cdot xy + xx\cdot y\\
xy\cdot(xy\cdot z) & \sim & x\cdot xy + xy\cdot z\\
xx\cdot(xx\cdot x) & \sim & x\cdot xy + xx\cdot yx\\
xy\cdot(xy\cdot x) & \sim & x\cdot xy + xy\cdot zx\\
xy\cdot(xy\cdot y) & \sim & x\cdot xy + xy\cdot zy\\
xx\cdot(y\cdot xx) & \sim & x\cdot yx + xx\cdot y\\
xy\cdot(z\cdot xy) & \sim & x\cdot yx + xy\cdot z\\
xx\cdot(x\cdot xx) & \sim & x\cdot yx + xx\cdot xy\\
xy\cdot(x\cdot xy) & \sim & x\cdot yx + xy\cdot xz\\
xy\cdot(y\cdot xy) & \sim & x\cdot yx + xy\cdot yz
\end{array}$$

and their duals. $\square$

References

1. N. Dershowitz and J.-P. Jouannaud, *Rewrite systems*, Chapter 6, 243–320 in J. van Leeuwen, ed., Handbook of Theoretical Computer Science, B: Formal Methods and Semantics. North Holland, Amsterdam 1990.

Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic