A DECIDABLE EQUATIONAL THEORY WITH UNDECIDABLE MEMBERSHIP PROBLEM FOR FINITE ALGEBRAS

Jaroslav Ježek

Recently, Hirsch and Hodkinson [1] proved that the variety of representable relation algebras has undecidable membership problem for finite algebras, i.e., there is no algorithm deciding for any finite algebra whether it belongs to the variety. Because the equational theory of representable relation algebras is undecidable, it is natural to ask if there is a variety with undecidable membership problem for finite algebras, the equational theory of which is decidable. The purpose of this paper is to give such an example.

For the basics of universal algebra and equational logic, the reader is referred to [5].

For a set S of positive integers, denote by S' the set of the positive integers n such that no multiple of n belongs to S.

Lemma 1. *There exists a recursive set S of positive integers such that the set S' is not recursive.*

Proof. As it is well known, there is a recursive binary relation R on the set of positive integers such that the range of R is not a recursive set. Let $i \leftrightarrow \langle c_1(i), c_2(i)\rangle$ be a recursive bijection of the set of positive integers onto the set of ordered pairs of positive integers. Define a set S of powers of prime numbers as follows: if p_m is the m-th prime number, let $p_m^i \in S$ if and only if $\langle c_1(i), c_2(i)\rangle \in R$ and $c_2(i) = m$. Clearly, S is a recursive set. For a prime number p_m, a multiple of p_m belongs to S if and only if m is in the range of R. Consequently, the set S' is not recursive. □

Theorem 2. *Consider the signature containing two unary operation symbols f, g and two constants $0, a$. Let S be a recursive set of positive integers such that the set S' is not recursive. Denote by U the set of the terms containing a subterm belonging to $\{0\} \cup \{fg^i fa : i \in S\}$, and let E be the equivalence on the set of terms such that U is a block of E and all the other blocks of E are singletons. Then E is a decidable equational theory with undecidable membership problem for finite algebras.*

Proof. It is easy to check that E is a fully invariant congruence of the term algebra, i.e., an equational theory. Since S is recursive, the set U is a recursive set of terms and E is a recursive set of ordered pairs of terms, so that E is a decidable

---

1991 *Mathematics Subject Classification.* 08B05,03C57,03C13.

While working on this paper the author was partially supported by the NSF grant No. DMS-9596043. Thanks are also to G. McNulty and R. McKenzie for their suggestions and help.

equational theory. For a positive integer n define an algebra $\mathbf{A}_n$ with the underlying set $\{a, 0, 1, \dots, n\}$ by

$$f(x) = \begin{cases} 1 & \text{for } x = a, \\ a & \text{for } x = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} x+1 & \text{for } x = 1, \dots, n-1, \\ 1 & \text{for } x = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{A}_n$ is a model of E if and only if $fg^i fa = 0$ in $\mathbf{A}_n$ for all $i \in S$. This condition is equivalent to $g^i 1 \neq 1$ in $\mathbf{A}_n$ for all $i \in S$; equivalently, no multiple of n belongs to S, i.e., $n \in S'$. Since S' is not recursive, there is no algorithm deciding if a finite algebra A_n is a model of E. □

The following three related problems are open.

Problem 1. *Let V be a finitely based variety of a finite signature. Denote by $c(V)$ the variety generated by the cancellative algebras in V. (An algebra A is called cancellative if for any fundamental operation F of arbitrary arity n and any elements $a_1, \dots, a_n, b_1, \dots, b_n \in A$ with $a_i = b_i$ for all but one i, $F(a_1, \dots, a_n) = F(b_1, \dots, b_n)$ implies $a_i = b_i$ for all i.) Is the membership problem for finite algebras in V always decidable?*

Problem 2. *Let V be a finitely based variety of a finite signature. Denote by $w(V)$ the variety generated by the V-algebras without irreducible elements. (An element is called irreducible if it is outside the ranges of all fundamental operations.) Is the membership problem for finite algebras in V always decidable?*

Problem 3. *For the variety M of medial groupoids (groupoids satisfying $(xy)(zu) = (xz)(yu)$), we have $c(M) = w(M)$. Is the membership problem for finite algebras in this variety decidable? (For the equational theory of medial groupoids, see [2], [3], [4] and [6].)*

References

1. R. Hirsch and I. Hodkinson, *Representability is not decidable for finite relation algebras*, to appear in Transactions of the AMS.
2. J. Ježek and T. Kepka, *Medial groupoids*, Rozpravy ČSAV, Řada mat. a přír. věd **93/2** (1983), 93pp.
3. J. Ježek and T. Kepka, *Equational theories of medial groupoids*, Algebra Universalis **17** (1983), 174–190.
4. J. Ježek and T. Kepka, *Free entropic groupoids*, Commentationes Math. Univ. Carolinae **22** (1981), 223–233.
5. R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties, Volume I*, Wadsworth & Brooks/Cole, Monterey, CA, 1987.
6. G. Pollák and Á. Szendrei, *Independent basis for the identities of entropic groupoids*, Commentationes Math. Univ. Carolinae **22** (1981), 71–85.

Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic