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SELFDISTRIBUTIVE GROUPOIDS

PART D1

LEFT DISTRIBUTIVE SEMIGROUPS

PRAGUE AND CAEN

1999 – 2000

LIST OF CONTENTS

ACKNOWLEDGEMENT

This research was partly supported by the Grant Agency of the Czech Republic, grant #201/99/0263.

CHAPTER I

GENERAL THEORY OF LEFT DISTRIBUTIVE SEMIGROUPS

I.1 BASIC PROPERTIES OF LEFT DISTRIBUTIVE SEMIGROUPS

1.1 Proposition. *Let S be an LD-semigroup. Then, for all $x, y, z \in S$:*

(i) $xyz = xyxz = xy^2z$.
(ii) $x^ny = x^2y$ *for every* $n \geq 2$.
(iii) $(xy)^n = xy^n = xy^2 = (xy)^2$ *for every* $n \geq 2$.
(iv) $x^n = x^3$ *for every* $n \geq 3$.

Proof. (i) $xyz = xyxz = xyxyz = xy^2z$ by repeated use of the left distributive law.
(ii) For $n \geq 3$, $x^ny = xx^{n-2}xy = xx^{n-2}y = x^{n-1}y$.
(iii) For $n \geq 3$, $(xy)^n = xy^n = xyxy^{n-1} = xyxyy^{n-2} = xyxy^{n-2} = xy^{n-1}$.
(iv) For $n \geq 4$, $x^n = xxxx^{n-3} = xxx^{n-3} = x^{n-1}$. □

1.2 Proposition. *Let S be an LD-semigroup. Then:*

(i) $\mathrm{Id}(S)$ *is a left ideal of S and* $x^3, xy^2, xyx \in \mathrm{Id}(S)$ *for all* $x, y \in S$.
(ii) *S is elastic.*
(iii) *For every* $n \geq 3$, $o_{n,S} = o_{3,S}$.

Proof. (i) First, $xy^2 \in \mathrm{Id}(S)$ by 1.1(iii) and $(xyx)^2 = xyx^2 = xyx$. Now, $\mathrm{Id}(S)$ is a left ideal of S (see also A1.II.1.5(i)).
(ii) Every semigroup is elastic.
(iii) This is an immediate consequence of 1.1(iv). □

1.3 Proposition. *The following conditions are equivalent for an LD-semigroup S:*

(i) $\mathrm{Id}(S)$ *is an ideal of S.*
(ii) $S^3 \subseteq \mathrm{Id}(S)$.
(iii) *S satisfies the (semigroup) identity* $x^2y \approx x^2y^2$.

If these conditions are satisfied, then $S/\mathrm{Id}(S)$ is an A-semigroup.

Proof. (i) implies (ii). $xyz = xy^2z$ by 1.1(i), and $xy^2 \in \mathrm{Id}(S)$ by 1.2(i).
(ii) implies (iii). Since $x^2y \in \mathrm{Id}(S)$, we have $x^2y = x^2y \cdot x^2y = x^2y^2$.
(iii) implies (i). By 1.2(i), $\mathrm{Id}(S)$ is a left ideal. Let $x \in S$ and $a \in \mathrm{Id}(S)$. Then $ax = a^2x = a^2x^2 = a^2x \cdot a^2x = (ax)^2$. Thus $\mathrm{Id}(S)$ is a right ideal. □

1.4 Definition. An LD-semigroup satisfying the equivalent conditions of 1.3 will be called an *LDR-semigroup*.

1.5 Proposition. *The following conditions are equivalent for an LD-semigroup S:*

(i) $S^2 \subseteq \mathrm{Id}(S)$.
(ii) $\mathrm{Id}(S)$ *is an ideal of* S *and* $S/\mathrm{Id}(S)$ *is a Z-semigroup.*
(iii) S *satisfies the identity* $xy \approx xy^2$.
(iv) S/q_S *is idempotent.*

If these conditions are satisfied, then S *is an LDR-semigroup.*

Proof. Easy. □

1.6 Definition. By an *LDR_1-semigroup* we mean a semigroup satisfying $xy \approx xyx$. (Clearly, every LDR_1-semigroup is left distributive.)

1.7 Proposition. *Every LDR_1-semigroup satisfies the equivalent conditions of 1.5 (hence it is an LDR-semigroup).*

Proof. Let S be an LDR_1-semigroup. By 1.2(i), $xy = xyx \in \mathrm{Id}(S)$ for all $x, y \in S$. Thus $S^2 \subseteq \mathrm{Id}(S)$. □

1.8 Proposition. *Let S be an LD-semigroup. Then:*

(i) p_S *is a congruence of* S.
(ii) S/p_S *is an LDR_1-semigroup.*

Proof. (i) This is true for every semigroup.
(ii) We have $xy \cdot z = xyx \cdot z$ for all $x, y, z \in S$. □

1.9 Proposition. *The following conditions are equivalent for an LD-semigroup S:*

(i) $o_{2,S}$ *is an endomorphism of* S.
(ii) $o_{3,S}$ *is an endomorphism of* S.
(iii) S *satisfies the identity* $xy^2 \approx x^2y^2$.
(iv) S *is left semimedial.*

Proof. By 1.1(ii) and 1.1(iii) we have $(xy)^3 = xy^3 = xy^2 = (xy)^2$ and $x^3y^3 = x^2y^2$ for all $x, y \in S$. Now it is clear that the first three conditions are equivalent.

If (iii) is satisfied, then $xx \cdot yz = x^2yz = x^2y^2z = xy^2z = xyz = xy \cdot xz$ (use 1.1). Conversely, if S is left semimedial, then $x^2y^2 = xyxy = xy^2$. □

1.10 Definition. Every LD-semigroup satisfying the equivalent conditions of 1.9 will be called an *LDT-semigroup.*

1.11 Proposition. *Let S be an LDT-semigroup. Then:*

(i) $o_{3,S}$ *is a homomorphism of* S *onto* $\mathrm{Id}(S)$.
(ii) *Every block of* $\ker(o_{3,S})$ *is an A-semigroup.*

Proof. Easy. □

1.12 Proposition. *The following conditions are equivalent for an LD-semigroup S:*

(i) S *satisfies the identity* $xy \approx x^2y$.
(ii) S/p_S *is idempotent.*

Proof. Easy. □

1.13 Definition. Every LD-semigroup satisfying the equivalent conditions of 1.12 will be called an *LDT_1-semigroup.*

1.14 Proposition. *Let S be an LDT_1-semigroup. Then:*

(i) *S is an LDT-semigroup.*
(ii) *o_S is a homomorphism of S onto* $\mathrm{Id}(S)$.
(iii) *Every block of* $\ker(o_S)$ *is a Z-semigroup.*

Proof. Easy. □

1.15 Proposition. *Let S be an LD-semigroup. Then S/q_S is an LDT_1-semigroup.*

Proof. We have $zxy = zx^2y$ for all $x, y, z \in S$. □

1.16 Proposition. *The following conditions are equivalent for an LD-semigroup S:*

(i) *S satisfies the identity $x^2y \approx xy^2$ (i.e., S is delightful).*
(ii) *S satisfies the identities $x^2y \approx xy^2$ and $xyz \approx x^2yz$ (i.e., S is strongly delightful).*
(iii) *S is an LDRT-semigroup. (I.e., both LDR and LDT.)*

Proof. (i) implies (ii). We have $x^2yz = xy^2z = xyz$ by 1.1(i).

(ii) implies (iii). We have $x^2y = x \cdot x^2y = x^2y^2$ by 1.1(ii), so that S is an LDR-semigroup. Similarly, $xy^2 = xy^2 \cdot y = x^2y^2$ by 1.1(iii), so that S is an LDT-semigroup.

(iii) implies (i). This follows immediately from the definitions. □

1.17 Proposition. *Let S be an LDRT-semigroup. Then:*

(i) $\mathrm{Id}(S)$ *is an ideal of S and $S/\mathrm{Id}(S)$ is an A-semigroup.*
(ii) *$o_{3,S}$ is a homomorphism of S onto* $\mathrm{Id}(S)$ *and every block of* $\ker(o_{3,S})$ *is an A-semigroup.*
(iii) $\ker(o_{3,S}) \cap \equiv_{\mathrm{Id}(S)} = \mathrm{id}_S$ *and S is a subdirect product of* $\mathrm{Id}(S)$ *and* $S/\mathrm{Id}(S)$.

Proof. For (i) see 1.3; for (ii) see 1.11; (iii) is clear. □

1.18 Proposition. *Let S be an LDR_1-semigroup. Then there exists a congruence r of S such that S/r is commutative and every block of r containing at least two elements is a subsemigroup of S and an LZ-semigroup.*

Proof. Define r by $(a, b) \in r$ iff either $a = b$ or $a = cb$ and $b = da$ for some $c, d \in S$. Clearly, r is an equivalence and $(a, b) \in r$ implies $(ax, bx) \in r$ for any $x \in S$. On the other hand, using the left distributive law, one can see that $(a, b) \in r$ also implies $(xa, xb) \in r$. So, r is a congruence of S. Since S is an LDR_1-semigroup, we have $ab = aba$, $ba = bab$ and $(ab, ba) \in r$ for all $a, b \in S$. Thus S/r is commutative.

Now, let A be a block of r and $a, b \in A$, $a \neq b$. We have $a = cb$ and $b = da$ for some elements c, d. Then $ab = ada = ad = cbd = cdad = cda = cb = a$. Further, $(a, b) \in r$ implies $(aa, ab) \in r$, so that $(aa, a) \in r$, and we get $aa \in A$. If $a \neq aa$, then $a = a^3$ according to the previous observation, so that $a \in \mathrm{Id}(S)$ by 1.2(i), a contradiction. □

1.19 Proposition. *The following conditions are equivalent for an LD-semigroup S:*

(i) *S is right semimedial.*
(ii) *S is middle semimedial.*
(iii) *S is medial.*

(iv) *S/p_S is right permutable.*
(v) *S/q_S is left permutable.*

Proof. (i) implies (iii). $xyuv = xyu^2v = xuyuv = xuyv$.
(ii) implies (iii). $xyuv = xyuxv = xuyxv = xuyv$. □

1.20 Proposition. *The following conditions are equivalent for a semigroup S:*
(i) *S is a medial LDR-semigroup.*
(ii) *S is a medial LDRT-semigroup.*
(iii) *S is a D-semigroup.*

Proof. (i) implies (iii). $xyz = xyxz = xxyz = x^2y^2z = x^2y^2z^2 = x^2yz^2 = x^2zyz = xzyz$.
(iii) implies (ii). $xyuv = xuyuv = xuyv$, $xxy = xyxy = x^2y^2$ and $xyy = xyxy = x^2y^2$. □

1.21 Proposition. *The following conditions are equivalent for a semigroup S:*
(i) *S is an LD-semigroup and* $\mathrm{card}(\mathrm{Id}(S)) = 1$.
(ii) *S is an A-semigroup.*

Proof. (i) implies (ii). Let $\mathrm{Id}(S) = \{0\}$. By 1.2(i), 0 is a right absorbing element of S and $xy^2 = 0 = xyx$ for all $x, y \in S$. Now, $0x = 0x0x = 0x^2 = 0$ and hence $xyz = xyxz = 0z = 0$ for all $x, y, z \in S$. □

1.22 Proposition. *Let S be an LD-semigroup, $C = \mathcal{C}_l(S)$ and $D = S - C$. Then:*
(i) *Every element of C is a left neutral element of S.*
(ii) *If C is nonempty, then $q_S = \mathrm{id}_S$, S is an LDT_1-semigroup and C is an RZ-semigroup.*
(iii) *If D is nonempty, then D is a prime ideal of S.*
(iv) *If C is nonempty and S is an LDR_1-semigroup, then $C = \{e\}$ is a singleton and e is a neutral element of S.*

Proof. (i) For $a \in C$ and $x \in S$, $aax = aaax$ implies $x = ax$.
(ii) $C \neq \emptyset$ implies immediately that $q_S = \mathrm{id}_S$, and then S is an LDT_1-semigroup by 1.15. Further, C is a subsemigroup of S (see also A1.II.4.1(i)) and C is an RZ-semigroup by (i).
(iii) Since S is a semigroup, D is a left ideal of S. Let $a \in D$ and $x \in S$. Then $au = av$ for some $u, v \in S$, $u \neq v$, and we have $axu = axau = axav = axv$. Hence $ax \in D$ and we see that D is an ideal. Finally, if $ab \in D$, then $abu = abv$, $u \neq v$, and therefore either $a \in D$ or $b \in D$.
(iv) We have $ax = axa$ and $x = xa$ for all $a \in C$ and $x \in S$. The rest is clear by (i). □

I.2 EXAMPLES OF LEFT DISTRIBUTIVE SEMIGROUPS

2.1 Example. There are (up to isomorphism) precisely four two-element LD-semigroups. They are:

$$D(1), D(2), D(3), D(4)$$

(see A1.IV.4). The first three of them are idempotent; the last one is not.

2.2 Example. There are (up to isomorphism) precisely sixteen three-element LD-semigroups. They are:

$$D(7),\ldots,D(14),D(20),D(24),\ldots,D(28),D(36),D(46)$$

(see A1.IV.10). All of them, except $D(20)$ and $D(28)$, are distributive. The idempotent ones are $D(7),\ldots,D(14)$ and $D(20)$.

2.3 Example. The following table shows the numbers of isomorphism types of at most five-element LD-semigroups and LDI-semigroups:

	1	2	3	4	5
LDS	1	4	16	93	682
LDIS	1	3	9	38	179

2.4 Example. Consider the following five-element groupoid S:

S	0	1	2	3	4
0	1	1	3	4	4
1	1	1	4	4	4
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4

This groupoid is an LDR_1-semigroup; it is not an LDT-semigroup and it does not satisfy the identity $xyx \approx x^2yx$.

2.5 Example. Consider the following four-element groupoid S:

S	0	1	2	3
0	2	3	2	2
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

This groupoid is an LDR_1-semigroup; it is not an LDT-semigroup; it is subdirectly irreducible and satisfies $x^2 \approx x^2y$.

2.6 Example. Consider the following two three-element LD-semigroups:

$D(20)$	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

$D(28)$	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

$D(20)$ is an idempotent LDR_1-semigroup; it is not medial. $D(28)$ is an LDT_1-semigroup; it is medial and satisfies $xy^2 \approx yx^2$. Moreover, $\mathrm{Id}(D(28))$ is not an ideal and $D(28)$ is not an LDR-semigroup.

2.7 Example. Let f be a transformation of a nonempty set S and define multuplication on S by $xy = f(y)$ for all $x, y \in S$. Then S becomes a D-semigroup.

2.8 Proposition. *Let S be an LD-semigroup and $e \notin S$. Then:*

(i) *$S[e]$ is an LD-semigroup.*
(ii) *$S\{e]$ is an LD-semigroup.*
(iii) *$S[e\}$ is an LD-semigroup iff S is an LZ-semigroup.*
(iv) *$S\{e\}$ is an LD-semigroup iff S is an idempotent LDR_1-semigroup.*

Proof. Easy (see A1.IV.1.9). □

2.9 Proposition. *Let S be a D-semigroup and $e \notin S$. Then:*

(i) *$S[e]$ is a D-semigroup.*
(ii) *$S\{e]$ (resp. $S[e\}$) is a D-semigroup iff S is an RZ-semigroup (resp. LZ-semigroup).*
(iii) *$S\{e\}$ is a D-semigroup iff S is a semilattice.*

Proof. Use 2.8. □

I.3 BASIC FACTS ON SUBDIRECTLY IRREDUCIBLE LEFT DISTRIBUTIVE SEMIGROUPS

3.1 Proposition. *Let S be a subdirectly irreducible LD-semigroup. Then just one of the following two cases takes place:*

(i) *$\mathcal{C}_l(S) \neq \emptyset$, $q_S = \mathrm{id}_S$ and S is an LDT_1-semigroup.*
(ii) *$\mathcal{C}_l(S) = \emptyset$ and $q_S \neq \mathrm{id}_S$.*

Proof. Suppose first $\mathcal{C}_l(S) = \emptyset$. Then, for every $x \in S$, L_x is not injective, so that $\omega_S \subseteq q_{x,S}$; but then $\omega_S \subseteq q_S$. On the other hand, if $\mathcal{C}_l(S) \neq \emptyset$, then (i) is true by 1.22(ii). □

3.2 Proposition. *Let S be a subdirectly irreducible LD-semigroup such that $C = \mathcal{C}_l(S) \neq \emptyset$; put $D = S - C$. Then just one of the following five cases takes place:*

(i) *$S \simeq D(1)$.*
(ii) *$S \simeq D(2)$.*
(iii) *$S \simeq D(10)$.*
(iv) *S is neither idempotent nor an LDR-semigroup and $\mathrm{card}(D) \geq 2$ (then $p_S \neq \mathrm{id}_S$.)*
(v) *S is an idempotent LDR_1-semigroup, $\mathrm{card}(D) \geq 2$, $p_S = \mathrm{id}_S$, $C = \{e\}$ for a neutral element e of S, D is subdirectly irreducible and $p_D = \mathrm{id}_D \neq q_D$.*

Proof. By 3.1, $q_S = \mathrm{id}_S$ and S is an LDT_1-semigroup. By 1.22, either $D = \emptyset$ or D is a prime ideal of S. Let $(a, b) \in \omega_S$, $a \neq b$. Obviously, $D = \{x \in S : xa = xb\}$. If $D = \emptyset$, then S is an RZ-semigroup by 1.22(ii) and one can readily see that $S \simeq D(2)$ in that case.

Next assume that $D = \{0\}$ is a singleton. Then 0 is an absorbing element of S, C is an RZ-semigroup and it is easy to see that $s \cup \mathrm{id}_S$ is a congruence of S for any congruence s of C. If $\mathrm{card}(C) = 1$, then $S \simeq D(1)$. If $\mathrm{card}(C) \geq 2$, then $a, b \in C$, $C \simeq D(2)$ and $S \simeq D(10)$.

Finally, assume that $\mathrm{card}(D) \geq 2$. Since D is an ideal, $\equiv_D$ is a congruence of S and thus a, b both belong to D. Then $aa = ab$ and $ba = bb$.

Let $p_S \neq \mathrm{id}_S$. Then $(a, b) \in p_S$, $ab = bb$, and therefore $aa = bb$. It follows that either $aa \neq a$ or $bb \neq b$ and we see that S is not idempotent. Suppose that S is an LDR-semigroup. Then $\mathrm{Id}(S)$ is an ideal and, since either $a \notin \mathrm{Id}(S)$ or $b \notin \mathrm{Id}(S)$, we must have $\mathrm{card}(\mathrm{Id}(S)) = 1$ by the subdirect irreducibility. Then, by 1.21, S is an A-semigroup and thus $C = \emptyset$, a contradiction.

Let $p_S = \mathrm{id}_S$. Then, by 1.8, S is an LDR_1-semigroup; S is idempotent by 1.22(ii) and 1.17(iii). The rest is clear from 1.22(iv). □

3.3 Proposition. *Let S be a subdirectly irreducible delightful LD-semigroup (see 1.16). Then just one of the following four cases takes place:*

(i) $S \simeq D(2)$.
(ii) $S \simeq D(10)$.
(iii) *S is an idempotent LDR_1-semigroup with $p_S = \mathrm{id}_S$.*
(iv) *S is an A-semigroup.*

Proof. With respect to 1.16(iii) and 1.17(iii), we can assume that S is idempotent. Further, with respect to 3.1 and 3.2, we can assume that $q_S \neq \mathrm{id}_S$. Let $(a, b) \in \omega_S$, $a \neq b$. We have $(a, b) \in q_S$, so that $a = aa = ab$ abd $b = bb = ba$. Thus $ab \neq ba$ and $(a, b) \notin p_S$. But then $p_S = \mathrm{id}_S$ and S is an LDR_1-semigroup by 1.8(ii). □

3.4 Proposition. *Let S be a subdirectly irreducible D-semigroup. Then just one of the following two cases takes place:*

(i) *S is idempotent and S is isomorphic to one of the five distributive semigroups $D(1)$, $D(2)$, $D(3)$, $D(9)$ and $D(10)$.*
(ii) *S is an A-semigroup.*

Proof. With respect to 3.3, we can assume that S is an idempotent LDR_1-semigroup, i.e., S satisfies $xy \approx xyx$. Dually, using the right hand form of 3.3, we can assume that S satisfies $xy \approx yxy$. However, then S is commutative, i.e., it is a semilattice. A subdirectly irreducible semilattice is isomorphic to $D(1)$. □

3.5 Remark. Let S be a subdirectly irreducible LD-semigroup. We have either $t_S \neq \mathrm{id}_S$ or $t_S = \mathrm{id}_S$.

If $t_S \neq \mathrm{id}_S$, then $t_S = \omega_S = \{(a, b), (b, a)\}$ for some $a, b \in S$, $a \neq b$. Then $a^2 = ab = ba = b^2$, and so either $a \notin \mathrm{Id}(S)$ or $b \notin \mathrm{Id}(S)$.

If $t = \mathrm{id}_S$, then either $p_S = \mathrm{id}_S$ and S is an LDR_1-semigroup, or else $q_S = \mathrm{id}_S$ and S is an LDT_1-semigroup. In the latter case, 3.2 applies.

3.6 Proposition. *The groupoids $D(1)$, $D(2)$, $D(3)$ and $D(4)$ are (up to isomorphism) the only (congruence) simple LD-semigroups.*

Proof. The result follows easily from A1.II.7.4. □

CHAPTER II

FREE LEFT DISTRIBUTIVE SEMIGROUPS

II.1 CONSTRUCTION OF FREE LEFT DISTRIBUTIVE SEMIGROUPS

1.1 Construction. Let X be a nonempty set. Denote by $\mathbf{F}$ the (absolutely) free semigroup over X. Denote by F the union of the following four pairwise disjoint subsets A, B, C, D of $\mathbf{F}$:

$$
\begin{aligned}
A =&\{x^i : x \in X,\ 1 \le i \le 3\}\\
B =&\{x^i y^j : x, y \in X,\ x \ne y,\ 1 \le i, j \le 2\}\\
C =&\{x_1^i x_2 \dots x_{n-1} x_n^j : x_1, \dots, x_n \in X \text{ pairwise different}, n \ge 3,\ 1 \le i, j \le 2\}\\
D =&\{x_1^i x_2 \dots x_{n-1} x_n x_k : x_1, \dots, x_n \in X \text{ pairwise different}, n \ge 2,\ 1 \le k < n,\\
&\quad 1 \le i \le 2\}
\end{aligned}
$$

For every element u of $\mathbf{F}$, (uniquely) expressed as $u = x_1^{k_1} \dots x_n^{k_n}$ where $n \ge 1$, $x_i \in X$, $k_i \ge 1$ and $x_1 \ne x_2 \ne x_3 \ne \dots \ne x_n$, we define an element $f(u)$ of F as follows:

(i) If $n = 1$, let $f(u) = x_1^k$ where $k = \min(3, k_1)$.

(ii) If $n = 2$, let $f(u) = x_1^k x_2^l$ where $k = \min(2, k_1)$ and $l = \min(2, k_2)$.

(iii) If $n \ge 3$ and $x_n \notin \{x_1, \dots, x_{n-1}\}$, let $f(u) = x_1^k y_1 \dots y_m x_n^l$ where $k = \min(2, k_1)$, $l = \min(2, k_n)$ and (by induction on i) y_i is the first member of $x_1, \dots, x_{n-1}$ not contained in $\{x_1, y_1, \dots, y_{i-1}\}$.

(iv) If $n \ge 3$ and $x_n \in \{x_1, \dots, x_{n-2}\}$, let $f(u) = x_1^k y_1 \dots y_m x_n$ where $k = \min(2, k_1)$ and (by induction on i) y_i is the first member of $x_1, \dots, x_{n-1}$ not contained in $\{x_1, y_1, \dots, y_{i-1}\}$.

It is easy to see that $f(u) \in F$ in any case. Also, it is easy to see that $f(u) = u$ for $u \in F$. Let us define a binary operation $*$ on F in this way: $u * v = f(uv)$ for any $u, v \in F$. We are going to prove that $F(*)$ is a free LD-semigroup over X.

1.2 Lemma. *Let $u \in \mathbf{F}$. The identity $u \approx f(u)$ is satisfied in any LD-semigroup.*

Proof. It is easy; use I.1.1, I.1.2 and, of course, the left distributive law. □

1.3 Lemma. *Let $u, v \in F$ and $u \ne v$. Then there is an LD-semigroup not satisfying $u \approx v$.*

Proof. Suppose that $u \approx v$ is satisfied in all LD-semigroups. Since every LZ-semigroup is left distributive, the words u, v have the same first letters. Similarly, every RZ-semigroup is left distributive and hence u, v have the same last letters. Furthermore, every semilattice is distributive and we conclude that the set of letters occurring in u coincides with the set of letters occurring in v. Now, we distinguish the following cases.

Case 1: $u = x^i$ and $v = x^j$. The LD-semigroup $D(28)$ (see I.2.6) satisfies neither $x \approx x^2$ nor $x \approx x^3$. The LD-semigroup $D(46)$ (see A1.IV.8.1) does not satisfy $x^2 \approx x^3$. Using these observations, we conclude that $i = j$. Hence $u = v$, a contradiction.

Case 2: $u = x^i y^j$ and $v = x^k y^l$. The LD-semigroup S from I.2.4 satisfies none of the identities $xy \approx x^2 y$, $xy \approx x^2 y^2$, $xy^2 \approx x^2 y^2$ and $xy^2 \approx x^2 y$. The LD-semigroup $D(28)$ satisfies neither $xy \approx xy^2$ nor $x^2 y \approx x^2 y^2$. Consequently, $i = k$, $j = l$ and $u = v$, a contradiction.

Case 3: $u = x_1^i x_2 \dots x_{n-1} x_n^j \in C$ and $v = x_{p(1)}^k x_{p(2)} \dots x_{p(n-1)} x_{p(n)}^l \in C$ for a permutation p of $\{1, \dots, n\}$ with $p(1) = 1$ and $p(n) = n$. If $n \geq 4$, then every idempotent LD-semigroup satisfying $u \approx v$ is medial. However, $D(20)$ (see I.2.6) is a non-medial LDI-semigroup. Consequently, $n = 3$. It is easy to see that either $xy^2 \approx x^2 y^2$ or $x^2 y \approx x^2 y^2$ is a consequence of $u \approx v$, and we get a contradiction by Case 2.

Case 4: $u = x_1^i x_2 \dots x_{n-1} x_n^j \in C$ and $v = x_{p(1)}^k x_{p(2)} \dots x_{p(n-1)} x_{p(n)} x_{p(k)} \in D$ for a permutation p of $\{1, \dots, n\}$ with $p(1) = 1$ and $p(k) = n$. One can easily check that every LDI-semigroup satisfying $u \approx v$ is distributive. However, $D(20)$ is not distributive, a contradiction.

Case 5: $u = x_1^i x_2 \dots x_{n-1} x_n x_k \in D$ and $v = x_{p(1)}^j x_{p(2)} \dots x_{p(n-1)} x_{p(n)} x_{p(l)} \in D$ for a permutation p of $\{1, \dots, n\}$ with $p(1) = 1$ and $p(l) = k$. Since $D(20)$ is not middle semimedial, we have $p(2) = 2, \dots, p(n) = n$. However, the LD-semigroup from I.2.4 does not satisfy $xyx \approx x^2 yx$. Thus $i = j$ and $u = v$, a contradiction. □

1.4 Theorem. *For a nonempty set X, the groupoid $F(*)$ constructed in 1.1 is a free LD-semigroup over X.*

Proof. Denote by $\sim$ the set of the ordered pairs (u, v) of elements of $\mathbf{F}$ such that the equation $u \approx v$ is satisfied in all LD-semigroups. So, $\sim$ is a (fully invariant) congruence of $\mathbf{F}$ and $\mathbf{F}/\sim$ is a free LD-semigroup over X. We know (by 1.2) that $f(u) \sim u$ for any $u \in \mathbf{F}$, so that (by 1.3) $u \sim v$ iff $f(u) = f(v)$ for any $u, v \in \mathbf{F}$ and $\sim$ is just the kernel of f. Now, f is a homomorphism of $\mathbf{F}$ onto $F(*)$: if $u, v \in \mathbf{F}$, then both $f(uv)$ and $f(u) * f(v)$ belong to F and are congruent modulo $\sim$ with uv. The result follows from the homomorphism theorem. (In particular, the operation $*$ is associative; this is not immediate from the definition.) □

1.5 Corollary. *Every finitely generated LD-semigroup is finite. The variety of LD-semigroups is locally finite.* □

1.6 Remark. Proceeding similarly, one can construct free LDI-semigroups. In that case we get words of two types only: words of the form $x_1 \dots x_n$ for $n \geq 1$ and words of the form $x_1 x_2 \dots x_n x_k$ for $n \geq 2$ and $1 \leq k < n$, where (in both cases) $x_1, \dots, x_n$ are pairwise distinct letters.

1.7 Remark. By I.1.20, every D-semigroup is a medial LDRT-semigroup. The words in a free D-semigroup are of the following types only: x, x^2, x^3, xy, $x^2 y$, xyx, $x_1 x_2 \dots x_m$ and $x_1 x_2 \dots x_m x_1$ $(m \geq 3)$. Of course,

$$x_1 \dots x_m \sim x_1 x_{p(2)} \dots x_{p(m-1)} x_m \text{ and } x_1 x_2 \dots x_m x_1 \sim x_1 x_{q(2)} \dots x_{q(m)} x_1$$

for any permutation p of $\{x_2, \dots, x_{m-1}\}$ and any permutation q of $\{x_2, \dots, x_m\}$.

II.2 AUXILIARY RESULTS ON NUMBER-THEORETIC FUNCTIONS

2.1 Definition. Put

(i) $a(n,m) = n(n-1)\dots(n-m)$,
(ii) $a(n) = \sum_{m=0}^{n} a(n,m)$,
(iii) $b(n) = \sum_{m=0}^{n} ma(n,m)$

for all nonnegative integers n, m.

2.2 Lemma. *Let $n, m \geq 0$. Then:*

(i) $a(n+1, m+1) = (n+1)a(n,m)$.
(ii) $a(n+1) = (n+1)(a(n)+1)$.
(iii) $b(n+1) = (n+1)(a(n)+b(n))$.
(iv) $b(n) = (n-2)a(n)+n$.

Proof. By induction on n. □

2.3 Lemma. *For every $n \geq 1$, $a(n)+c(n)+1 = n!\mathbf{e}$, where $(n+1)^{-1} < c(n) < n^{-1}$ and $\mathbf{e} = \sum_{k=0}^{\infty} 1/(k!)$.*

Proof. Indeed, $n!\mathbf{e} - 1 = 2n! + 3\cdot 4\cdot \ldots \cdot n + 4\cdot 5\cdot \ldots \cdot n + \cdots + (n-1)n + n + c(n) = a(n) + c(n)$, where $c(n) = 1/(n+1) + 1/(n+1)(n+2) + 1/(n+1)(n+2)(n+3) + \ldots$. Clearly, $1/(n+1) < c(n) < 1/n$. □

2.4 Lemma. *For every $n \geq 1$, $na(n) = [nn!\mathbf{e}] - n$ (here, for a positive real number r, $[r]$ means the entire part of r).*

Proof. By 2.3, $na(n) = [nn!\mathbf{e}] - n - nc(n) + u$, where $0 < u < 1$. Then $-1 < u - nc(n) < (n+1)^{-1}$ and, since $u - nc(n)$ is a whole number, we must have $u - nc(n) = 0$. □

II.3 THE NUMBER OF ELEMENTS OF A FREE LEFT DISTRIBUTIVE SEMIGROUP

3.1 Theorem. *The cardinality $f_1(n)$ of the free LD-semigroup of rank n and the cardinality $f_2(n)$ of the free LDI-semigroup of rank n are given by*

$$f_1(n) = 2[n!n\mathbf{e}] - n,$$
$$f_2(n) = [n!(n-1)\mathbf{e}] + 1.$$

Proof. By 1.4, 2.1 and 2.2 we have $f_1(n) = 4a(n) + 2b(n) - n = n + 2na(n)$. In order to compute $f_1(n)$, it remains to use 2.4. The other formula is clear from 1.6. □

3.2 Remark.

(i) $f_1(n) = \varepsilon(n)(n+1)!$, where $\varepsilon(n) \to 2\mathbf{e}$. Moreover, $f_1(n)/f_2(n) \to 2$.
(ii) Let S be a finitely generated LD-semigroup and $n = \sigma(S)$ (see A1.I.1.5). If $n = 0$, then $\mathrm{card}(S) = 1$. If $n \geq 1$, then

$$n \leq \mathrm{card}(S) \leq 2[n!n\mathbf{e}] - n.$$

3.3 Remark.

(i) The cardinality $f_3(n)$ of the free idempotent LDR_1-semigroup of rank n is given by

$$f_3(n) = [n!\mathbf{e}] - 1.$$

(ii) The cardinality $f_4(n)$ of the free DI-semigroup of rank n is given by

$$f_4(n) = n(n+1)2^{n-2}.$$

(iii) The cardinality $f_5(n)$ (resp. $f_6(n)$) of the free LDI-semigroup satisfying $xyz \approx xzy$ (resp. $xyz \approx yxz$) of rank n is given by

$$f_5(n) = f_6(n) = n2^{n-1}.$$

(iv) The cardinality $f_7(n)$ of the free semilattice of rank n is given by

$$f_7(n) = 2^n - 1.$$

(v) The cardinality $f_8(n)$ of the free idempotent semigroup satisfying $x \approx xyx$ of rank n is given by

$$f_8(n) = n^2.$$

(vi) The cardinality $f_9(n)$ (resp. $f_{10}(n)$) of the free LZ-semigroup (resp. RZ-semigroup) of rank n is given by

$$f_9(n) = f_{10}(n) = n.$$

3.4 Remark. Denote by $f_{11}(n)$ the cardinality of the free D-semigroup of rank n. According to 1.7, $f_{11}(n) = 3n + 2n(n-1) + n(n-1)(\binom{n-2}{1} + \cdots + \binom{n-2}{n-2}) + n(\binom{n-1}{1} + \cdots + \binom{n-1}{n-1})$. After easy calculation, we find that

$$f_{11}(n) = n(n+1)(1 + 2^{n-2}).$$

3.5 Remark. Denote by $f_{12}(n)$ (resp. $f_{13}(n)$, $f_{14}(n)$, $f_{15}(n)$, $f_{16}(n)$) the cardinality of the free A-semigroup (resp. free unipotent A-semigroup, free commutative A-semigroup, free unipotent commutative A-semigroup, free Z-semigroup) of rank n. Then

$$\begin{aligned} f_{12}(n) &= n^2 + n + 1, \\ f_{13}(n) &= n^2 + 1 \\ f_{14}(n) &= (n^2 + 3n + 2)/2, \\ f_{15}(n) &= (n^2 + n + 2)/2, \\ f_{16}(n) &= n + 1. \end{aligned}$$

3.6 Table.

	1	2	3	4	5	6	7	8
$f_1(n)$	3	18	93	516	3255	23478	191793	1753608
$f_2(n)$	1	6	33	196	1305	9786	82201	762208
$f_3(n)$	1	4	15	64	325	1956	13694	109600
$f_4(n)$	1	6	24	80	240	672	1792	4608
$f_{5,6}(n)$	1	4	12	32	80	192	448	1024
$f_7(n)$	1	3	7	15	31	63	127	255
$f_8(n)$	1	4	9	16	25	36	49	64
$f_{9,10}(n)$	1	2	3	4	5	6	7	8
$f_{11}(n)$	3	12	36	100	270	714	1848	4680
$f_{12}(n)$	3	7	13	21	31	43	57	73
$f_{13}(n)$	2	5	10	17	26	37	50	65
$f_{14}(n)$	3	6	10	15	21	28	36	45
$f_{15}(n)$	2	4	7	11	16	22	29	37
$f_{16}(n)$	2	3	4	5	6	7	8	9

CHAPTER III

A-SEMIGROUPS AND THEIR VARIETIES

III.1 BASIC PROPERTIES OF A-SEMIGROUPS

1.1. An A-semigroup is a groupoid satisfying $x \cdot yz \approx uv \cdot w$. It is apparent that A-semigroups are nothing else than semigroups nilpotent of class at most 3. Thus every A-semigroup S contains an absorbing element 0 $(= 0_S)$ such that $xyz = 0$ for all $x, y, z \in S$.

1.2 Proposition. *Let S be an A-semigroup and $Z(S) = \{a \in S : Sa = 0 = aS\}$. Then:*

(i) *0, S^2 and $Z(S)$ are ideals of S.*
(ii) $\mathrm{Id}(S) = \mathrm{Int}(S) = \{0\} = S^3 \subseteq S^2 \subseteq Z(S) \subseteq S$.
(iii) *S^2, $Z(S)$, S/S^2 and $S/Z(S)$ are Z-semigroups.*
(iv) $Z(S) \times Z(S) \subseteq t_S$.
(v) $\sigma(S) = \mathrm{card}(S - S^2)$.

Proof. Easy. □

III.2 VARIETIES OF A-SEMIGROUPS

2.1 Notation. Denote by $\mathcal{A}_0$ the variety of trivial groupoids, by $\mathcal{A}_1$ the variety of Z-semigroups, by $\mathcal{A}_2$ the variety of commutative unipotent A-semigroups, by $\mathcal{A}_3$ the variety of commutative A-semigroups, by $\mathcal{A}_4$ the variety of unipotent A-semigroups and by $\mathcal{A} = \mathcal{A}_5$ the variety of A-semigroups.

2.2 Theorem. *The varieties $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are pairwise different varieties of A-semigroups and there are no other varieties of A-semigroups. We have*

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \mathcal{A}_5, \qquad \mathcal{A}_2 \subset \mathcal{A}_4 \subset \mathcal{A}_5$$

and there are no other inclusions except those which follow by transitivity. The lattice of varieties of A-semigroups is given in Fig. 1.

Proof. Let V be a variety of A-semigroups determined by an identity $u \approx v$, where u, v are two semigroup words of lengths k and l, respectively. If $k \geq 3$ and $l \geq 3$, then $V = \mathcal{A}_5$. If $k \geq 3$ and $l = 2$, then V is either $\mathcal{A}_4$ or $\mathcal{A}_1$. If $k \geq 3$ and $l = 1$, then $V = \mathcal{A}_0$. If $k = l = 2$, then V is either $\mathcal{A}_5$ or $\mathcal{A}_4$ or $\mathcal{A}_3$ or $\mathcal{A}_1$. If $k = 2$ and $l = 1$, then $V = \mathcal{A}_0$. Finally, if $k = l = 1$, then V is either $\mathcal{A}_5$ or $\mathcal{A}_0$. Hence every one-based variety of A-semigroups can be found among $\mathcal{A}_0, \dots, \mathcal{A}_5$. Since

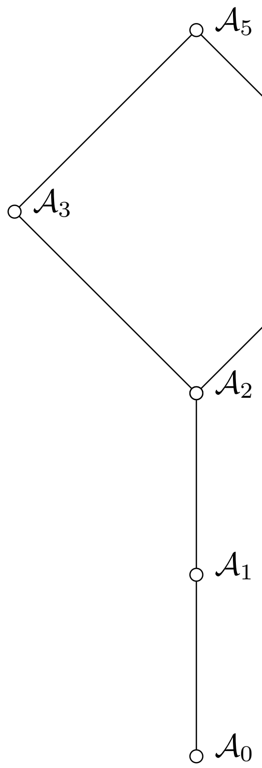


Fig. 1

this collection is closed under intersection (we have $\mathcal{A}_3 \cap \mathcal{A}_4 = \mathcal{A}_2$), it follows that there are no other subvarieties of $\mathcal{A}$.

All the inclusions are clear. The groupoid T given by

T	0	1	2	3
0	0	0	0	0
1	0	0	3	0
2	0	3	0	0
3	0	0	0	0

is in $\mathcal{A}_2$ but not in $\mathcal{A}_1$. The groupoid $D(46)$ (see A1.IV.8.1) is in $\mathcal{A}_3$ but not in $\mathcal{A}_4$, and the groupoid S given by

S	0	1	2	3	4
0	0	0	0	0	0
1	0	0	3	0	0
2	0	4	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0

is in $\mathcal{A}_4$ but not in $\mathcal{A}_3$. $\square$

III.3 FREE A-SEMIGROUPS

3.1 Construction. Let X be a nonempty set and let $f : X \times X \to Y$ be a bijective mapping, where $X \cap Y = \emptyset$. Let 0 be an element not belonging to $X \cup Y$. Define a multiplication on $F = X \cup Y \cup \{0\}$ by $xy = f(x, y)$ for $x, y \in X$ and $xy = 0$ otherwise. Then F becomes a free A-semigroup over the set X.

3.2 Proposition. *An A-semigroup S is a free A-semigroup if and only if it satisfies the following four conditions:*

(i) *S is nontrivial;*
(ii) *If $x, y, u, v \in S$ are such that $xy = uv \neq 0$, then $x = u$ and $y = v$;*
(iii) *If $x, y \in S - Z(S)$, then $xy \neq 0$;*
(iv) *$Z(S) = S^2$.*

Proof. Easy. □

3.3 Proposition. *An A-semigroup S is a subsemigroup of a free A-semigroup if and only if it satisfies the conditions 3.2(ii) and 3.2(iii).*

Proof. The direct implication is clear from 3.2 (if $S \subseteq F$, then $S - Z(S) \subseteq F - Z(F)$). Now, assume that S satisfies both 3.2(ii) and 3.2(iii) and put $A = S - Z(S)$ and $B = Z(S) - S^2$. It follows from 3.2(iii) that $S = A \cup B \cup A^2 \cup \{0\}$ is a disjoint union. Further, let C be a set such that $C \cap S = \emptyset$ and $\mathrm{card}(C) = \mathrm{card}(B)$, and let $g : B \to C$ be a bijection. Put $X = A \cup C$ and define a mapping $h : S \to F$ (where F is as in 3.1) as follows: $h(a) = a$ for every $a \in A$; $h(b) = g(b)^2$ for every $b \in B$; $h(xy) = xy$ for all $x, y \in A$; $h(0) = 0$. It follows from 3.2(ii) that h is well defined and, by 3.2(iii), h is an injective homomorphism of S onto the free A-semigroup F. □

3.4 Corollary. *Every Z-semigroup is a subsemigroup of a free A-semigroup.* □

3.5 Remark. The A-semigroup T from the proof of 2.2 is not a subsemigroup of any free A-semigroup.

3.6 Remark. The number of elements of a free semigroup in any subvariety of $\mathcal{A}$ has been computed in II.3.5.

III.4 SUBDIRECTLY IRREDUCIBLE A-SEMIGROUPS

4.1 Proposition. *Let S be an A-semigroup containing at least three elements. Then S is subdirectly irreducible if and only if the subsemigroup $T = S^2$ contains precisely two elements and $t_S = (T \times T) \cup \mathrm{id}_S$.*

Proof. Let S be subdirectly irreducible. As one can see easily, every subdirectly irreducible Z-semigroup contains only two elements. Consequently, S is not a Z-semigroup and $\mathrm{card}(T) \geq 2$. On the other hand, every nonempty subset M of T is an ideal of S, $(M \times M) \cup \mathrm{id}_S$ is a congruence, and it follows easily that $\mathrm{card}(T) = 2$ and $\omega_S = (T \times T) \cup \mathrm{id}_S$. Clearly, $\omega_S \subseteq t_S$. Conversely, if $(a, b) \in t_S$ and $a \neq b$, then $(\{a, b\} \times \{a, b\}) \cup \mathrm{id}_S$ is a congruence of S. Thus $\omega_S = t_S = (T \times T) \cup \mathrm{id}_S$.

Now assume that $T = \{0, a\}$ where $a \neq 0$, and that $t_S = (T \times T) \cup \mathrm{id}_S$. Let $r \neq \mathrm{id}_S$ be a congruence of S and let $(x, y) \in r$, $x \neq y$. If $xz \neq yz$ for some $z \in S$, then the elements xz and yz belong to T and we see that $(a, 0) \in r$. Similarly, $zx \neq zy$ implies $(a, 0) \in r$. If $xz = yz$ and $zx = zy$ for all $z \in S$, then $(x, y) \in t_S = (T \times T) \cup \mathrm{id}_S$. This proves $(a, 0) \in r$ in any case, so that S is subdirectly irreducible. □

4.2 Corollary. *Let S be a subdirectly irreducible A-semigroup containing at least three elements. Then $Z(S) = S^2$, $\omega_S = t_S$, $\sigma(S) = \mathrm{card}(S) - 2$ and every proper homomorphic image of S is a Z-semigroup.* □

4.3 Theorem. *An A-semigroup S is a subsemigroup of a subdirectly irreducible A-semigroup if and only if S^2 contains at most two elements.*

Proof. The direct implication follows from 4.1. Let S be an A-semigroup such that $S^2 \subseteq \{0, 1\}$, where 0 is the absorbing element of S (and 1 is some other element); let S be not subdirectly irreducible. Put $K = S - \{0, 1\}$. Let f be a bijection of K onto a set M with $S \cap M = \emptyset$. Put $G = S \cup M$ and define multiplication on G in the following way:

(i) S is a subsemigroup of G;
(ii) $x \cdot f(x) = f(x) \cdot x = 1$ and $f(x) \cdot f(x) = 0$ for all $x \in K$;
(iii) $f(x) \cdot y = y \cdot f(x) = 0$ and $f(x) \cdot f(y) = 1$ for all $x, y \in K$, $x \neq y$;
(iv) $z \cdot 0 = 0 \cdot z = z \cdot 1 = 1 \cdot z = 0$ for all $z \in G$.

It is easy to check that G is an A-semigroup. Of course, S is a subsemigroup of G. We have $G^2 = \{0, 1\}$, so that, according to 4.1, it remains to show that $t_G = (\{a, b\} \times \{a, b\}) \cup \mathrm{id}_G$.

Let $(a, b) \in t_G$, $a \neq b$. We are going to show that $a, b \in \{0, 1\}$. If $a, b \in M$, then $0 = aa = ab = 1$, a contradiction. Therefore, we can assume that $a \in S$.

Suppose $a \in K$. If $b \notin M$, then $1 = a \cdot f(a) = b \cdot f(a) = 0$, a contradiction. Thus $b \in M$ and we have $b = f(c)$ for some $c \in K$. If there exists an element d of K different from both a and c, then $0 = a \cdot f(d) = b \cdot f(d) = 1$, a contradiction. Thus $K = \{a, c\}$. If $a = c$, then $b = f(a)$ and $1 = a \cdot f(a) = b \cdot f(a) = 0$, a contradiction. If $ac = 0$, then $0 = ac = bc = 1$, which is not true; if $ca = 0$, we get a contradiction similarly. Thus $ac = 1 = ca$. Similarly $aa = 0$, and S is subdirectly irreducible by 4.1, a contradiction.

This proves that $a \in \{0, 1\}$. In this case, $xb = 0 = bx$ for every $x \in G$ and $b \in \{0, 1\}$. The rest is clear. □

4.4 Corollary. *Every Z-semigroup is a subsemigroup of a (commutative and unipotent) subdirectly irreducible A-semigroup.* □

4.5 Remark. The subdirectly irreducible A-semigroup G constructed in the proof of 4.3 is commutative (resp. unipotent), provided that S is commutative (resp. unipotent). Hence, the analogue of 4.3 remains true for commutative (resp. unipotent) A-semigroups.

CHAPTER IV

IDEMPOTENT LEFT DISTRIBUTIVE SEMIGROUPS AND THEIR VARIETIES

IV.1 BASIC PROPERTIES OF IDEMPOTENT LEFT DISTRIBUTIVE SEMIGROUPS

1.1 Proposition. *The following conditions are equivalent for an idempotent semigroup S:*

(i) *S is middle semimedial.*
(ii) *S is medial.*
(iii) *S is distributive.*

Proof. (i) implies (ii). We have $abcd = abcd \cdot abcd = a \cdot b \cdot cd \cdot a \cdot bcd = a \cdot cd \cdot b \cdot a \cdot bcd = a \cdot c \cdot d \cdot bab \cdot c \cdot d = a \cdot c \cdot bab \cdot d \cdot c \cdot d = a \cdot c \cdot ba \cdot bd \cdot c \cdot d = a \cdot c \cdot bd \cdot ba \cdot c \cdot d = acb \cdot d \cdot b \cdot ac \cdot d = acb \cdot d \cdot ac \cdot b \cdot d = acbd \cdot acbd = acbd$ for all $a, b, c, d \in$.

(ii) implies (iii). We have $abc = aabc = abac$ and $cba = cbaa = caba$ for all $a, b, c \in S$.

(iii) implies (i). We have $abca = abcba = acba$ for all $a, b, c \in S$. □

1.2 Proposition. *The pairwise nonisomorphic DI-semigroups $D(1)$, $D(2)$, $D(3)$, $D(9)$ and $D(10)$ are (up to isomorphism) the only subdirectly irreducible DI-semigroups. Moreover, $D(9)$ is right but not left permutable and $D(10)$ is left but not right permutable.*

Proof. See I.3.4. □

1.3 Proposition. *Let S be a rectangular band, i.e., an idempotent semigroup satisfying the identity $x \approx xyx$. Then:*

(i) *S is a DI-semigroup.*
(ii) *S/p_S is an LZ-semigroup and S/q_S is an RZ-semigroup.*
(iii) *$S \simeq S/p_S \times S/q_S$.*

Proof. (i) We have $abcd = aca \cdot bcd = a \cdot cabc \cdot d = acd = a \cdot cbc \cdot d = ac \cdot bdb \cdot cd = acb \cdot dbcd = acbd$ for all $a, b, c, d \in S$. Thus S is medial, and hence distributive by 1.1.

(ii) By (i), $xy = xzxy = xzy$ for all $x, y, z \in S$ and it follows that $(y, zy) \in q_S$ and S/q_S is an RZ-semigroup. Quite similarly, S/p_S is an LZ-semigroup.

(iii) Since S is idempotent, we have $t_S = p_S \cap q_S = \mathrm{id}_S$. On the other hand, by (ii), $a/p = ab/p$ and $b/q = ab/q$ for all $a, b \in S$. □

1.4 Proposition. *Let S be a subdirectly irreducible LDI-semigroup. Then either S is a DI-semigroup (and so S is isomorphic to one of $D(1)$, $D(2)$, $D(3)$, $D(9)$, $D(10)$) or S is an idempotent LDR_1-semigroup such that $p_S = \mathrm{id}_S$.*

Proof. See I.3.3 and 1.2. □

IV.2 VARIETIES OF IDEMPOTENT LD-SEMIGROUPS

2.1 Notation. Consider the following varieties of idempotent semigroups:

$\mathcal{I}_0$ ... trivial semigroups;
$\mathcal{I}_1$ ... semigroups satisfying $xy \approx x$;
$\mathcal{I}_2$ ... semilattices;
$\mathcal{I}_3$ ... semigroups satisfying $xy \approx y$;
$\mathcal{I}_4$ ... left permutable idempotent semigroups;
$\mathcal{I}_5$ ... rectangular bands (idempotent semigroups satisfying $x \approx xyx$);
$\mathcal{I}_6$ ... right permutable idempotent semigroups;
$\mathcal{I}_7$ ... normal bands (idempotent medial semigroups or DI-semigroups, see 1.1);
$\mathcal{I}_8$ ... idempotent LDR$_1$-semigroups (idempotent semigroups satisfying $xy \approx xyx$);
$\mathcal{I}_9 = \mathcal{I}$ ... LDI-semigroups.

2.2 Theorem. *The ten pairwise different varieties $\mathcal{I}_0, \dots, \mathcal{I}_9$ are just all subvarieties of the variety $\mathcal{I}$ of LDI-semigroups. We have*

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_4 \subset \mathcal{I}_8 \subset \mathcal{I}_9, \quad \mathcal{I}_1 \subset \mathcal{I}_5 \subset \mathcal{I}_7, \quad \mathcal{I}_2 \subset \mathcal{I}_6 \subset \mathcal{I}_7,$$
$$\mathcal{I}_0 \subset \mathcal{I}_2 \subset \mathcal{I}_4 \subset \mathcal{I}_7 \subset \mathcal{I}_9, \quad \mathcal{I}_0 \subset \mathcal{I}_3 \subset \mathcal{I}_5, \quad \mathcal{I}_3 \subset \mathcal{I}_6$$

and there are no other inclusions (except those that follow by transitivity). The lattice of subvarieties of $\mathcal{I}$ is given in Fig. 2.

Proof. All the non-sharp versions of the indicated inclusions are clear (use 1.1 and 1.3).

No nontrivial RZ-semigroup is in $\mathcal{I}_8$. Therefore, $\mathcal{I}_3 \not\subseteq \mathcal{I}_8$.

No nontrivial semilattice is in $\mathcal{I}_5$. Therefore, $\mathcal{I}_2 \not\subseteq \mathcal{I}_5$.

No nontrivial LZ-semigroup is in $\mathcal{I}_6$. Therefore, $\mathcal{I}_1 \not\subseteq \mathcal{I}_6$.

We have $D(20) \in \mathcal{I}_8 - \mathcal{I}_7$. This completes the inclusions part of the proof.

Now let V be a variety of LDI-semigroups determined (in $\mathcal{I}$) by a single identity $u \approx v$.

Assume first that $V \subseteq \mathcal{I}_7$. The variety V is generated by its subdirectly irreducible members. Using 1.2, we easily conclude that V is one of the varieties $\mathcal{I}_0$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{I}_5$, $\mathcal{I}_6$, $\mathcal{I}_7$.

Let $V \subseteq \mathcal{I}_8$. We can restrict ourselves to the case when $u = x_1 \dots x_n$ and $v = y_1 \dots y_m$ where $x_1, \dots, x_n$ are pairwise different and also $y_1, \dots, y_m$ are pairwise different. If $\mathrm{var}(u) \neq \mathrm{var}(v)$, then $V \subseteq \mathcal{I}_5$ and, in fact, V is either $\mathcal{I}_0$ or $\mathcal{I}_1$. So, assume that $\mathrm{var}(u) = \mathrm{var}(v)$. Then $n = m$ and there is a permutation p of $\{1, 2, \dots, n\}$ such that $y_i = x_{p(i)}$. If $p(1) \neq 1$, then V is either $\mathcal{I}_0$ or $\mathcal{I}_2$. Let $p(1) = 1$, $p \neq \mathrm{id}$, and let $2 \leq k \leq n-1$ be the smallest number with $p(k) \neq k$. Using the substitution $x_1, \dots, x_{k-1} \to x$, $x_k \to y$ and $x_{k+1}, \dots, x_n \to z$, we can show that the identity $xyz \approx xzy$ is satisfied in V, and so $V \subseteq \mathcal{I}_4$. Thus V is either $\mathcal{I}_0$ or $\mathcal{I}_1$ or $\mathcal{I}_2$ or $\mathcal{I}_4$.

Assume, finally, that $V \not\subseteq \mathcal{I}_7$ and $V \not\subseteq \mathcal{I}_8$. By 1.4, every subdirectly irreducible member of V is either in $\mathcal{I}_7$ or in $\mathcal{I}_8$. Consequently, $V = \mathcal{I}_9$. □

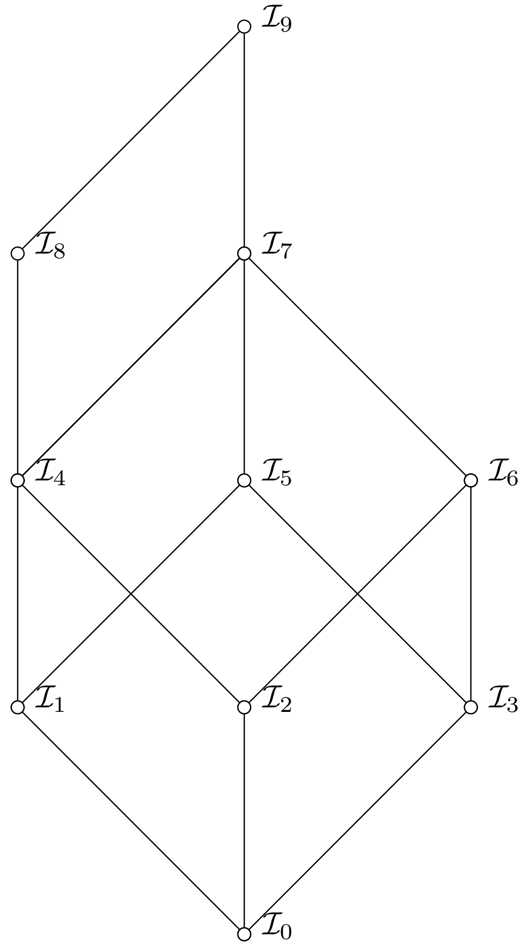


Fig. 2

IV.3 SUBDIRECTLY IRREDUCIBLE IDEMPOTENT LDR$_1$-SEMIGROUPS

3.1 Remark. According to 1.4, there exist (up to isomorphism) only two subdirectly irreducible LDI-semigroups that are not LDR$_1$-semigroups, namely, $D(2)$ and $D(10)$.

3.2 Proposition. *Let S be a subdirectly irreducible LDR$_1$I-semigroup such that $q_S = \mathrm{id}_S$. Then just one of the following two cases takes place:*

(i) $S \simeq D(1)$;

(ii) *S possesses at least three elements, among them a neutral element e, such that $T = S - \{e\}$ is a subsemigroup of S, $q_T \neq \mathrm{id}_T$ and T is a subdirectly irreducible LDR$_1$I-semigroup possessing no neutral element.*

Proof. See I.3.2. □

3.3 Proposition. *Let T be a nontrivial semigroup and e be an element not belonging to T. Then $T\{e\}$ is a subdirectly irreducible LDR$_1$I-semigroup if and only if T is a subdirectly irreducible LDR$_1$I-semigroup possessing no neutral element.*

Proof. See I.2.8(iv). □

3.4 Proposition. *Let T be a nontrivial semigroup and o be an element not belonging to T. Then $T[o]$ is a subdirectly irreducible LDR_1I-semigroup if and only if T is a subdirectly irreducible LDR_1I-semigroup possessing no absorbing element.*

Proof. Easy. □

3.5 Proposition. *Let S be a subdirectly irreducible LDR_1I-semigroup possessing an absorbing element o. Then just one of the following two cases takes place:*

(i) $S \simeq D(1)$;

(ii) *S contains at least three elements, $T = S - \{o\}$ is a subsemigroup of S, T is a subdirectly irreducible LDR_1I-semigroup and T contains no absorbing element.*

Proof. Assume that $\operatorname{card}(S) \geq 3$ and that $(a,b) \in \omega_S$, $a \neq b$, $a \neq o$. Let $u \in T$; put $I = \{x \in S : xu = o\}$ and $J = Su$. Then both I and J are ideals of S and $\operatorname{card}(J) \geq 2$; we have $o, u \in J$. Consequently, $\omega_S \subseteq (J \times J) \cup \mathrm{id}_S$ and $a = vu$ for some $v \in S$. We have $a = vu = vuu = au$, and so $a \notin I$. Thus $\omega_S \nsubseteq (I \times I) \cup \mathrm{id}_S$, $\operatorname{card}(I) = 1$ and $I = \{o\}$. We have proved that T is a subsemigroup of S and the rest is clear from 3.4. □

3.6 Definition. A subdirectly irreducible $\mathrm{LDR_1I}$-semigroup S will be called primary if S contains no neutral element and no absorbing element either.

3.7 Theorem. *Let S be a subdirectly irreducible LDR_1I-semigroup. Then just one of the following five cases takes place:*

(i) $S \simeq D(1)$.

(ii) *S is primary.*

(iii) *S contains at least three elements, among them a neutral element e, no absorbing element, $T = S - \{e\}$ is a subsemigroup of $S = T\{e\}$ and T is a primary subdirectly irreducible LDR_1I-semigroup.*

(iv) *S contains at least three elements, among them an absorbing element o, no neutral element, $T = S - \{o\}$ is a subsemigroup of $S = T[o]$ and T is a primary subdirectly irreducible LDR_1I-semigroup.*

(v) *S contains at least four elements, among them both a neutral element e and an absorbing element o, $T = S - \{e, o\}$ is a subsemigroup of $S = (T\{e\})[o] = (T[o])\{e\}$ and T is a primary subdirectly irreducible LDR_1I-semigroup.*

Proof. Combine 3.2, 3.3, 3.4 and 3.5 □

3.8 Notation. For a semigroup S, let $\mathrm{LA}(S)$ denote the set of left absorbing elements of S, i.e., $\mathrm{LA}(S) = \{a \in S : aS = \{a\}\}$. If $L = \mathrm{LA}(S)$ is nonempty, then L is an ideal of S and $L = \mathrm{Int}(S)$. Moreover, L is equal to the intersection of all left ideals of S and every nonempty subset of L is a right ideal of S.

3.9 Lemma. *Let S be an idempotent semigroup and I be a right ideal of S. Then $I \subseteq \mathrm{LA}(S)$ iff I is an LZ-semigroup.*

Proof. If I is an LZ-semigroup and if $a \in I$ and $x \in S$, then $ax \in I$ and $ax = a \cdot ax = a$. □

IV.4 SUBDIRECTLY IRREDUCIBLE SEMIGROUPS IN $\mathcal{I}_8$

4.1 Remark. Recall that $\mathcal{I}_8$ is the variety of $\mathrm{LDR_1I}$-semigroups, i.e., the variety of idempotent semigroups satisfying $xyx \approx xy$. The aim of this section is to prove that every semigroup from $\mathcal{I}_8$ can be embedded into a subdirectly irreducible semigroup from $\mathcal{I}_8$. This is a special case of a more general result by Goralčík and Koubek [GorK,?]. The proof contained in [GorK,?] contains several inaccuracies, making it almost unreadable.

4.2 Definition. We fix two distinct elements α, β. A semigroup $S \in \mathcal{I}_8$ will be called admissible if $\{\alpha, \beta\} \subseteq \mathrm{LA}(S)$ and $s\alpha = s\beta \in \{\alpha, \beta\}$ for all $s \in S - \mathrm{LA}(S)$.

An admissible semigroup $S \in \mathcal{I}_8$ will be called reductive if for every pair u, v of distinct elements of S there exists an element $s \in \mathrm{LA}(S)$ with $us \neq vs$.

4.3 Proposition. *Every semigroup $S \in \mathcal{I}_8$ containing neither α nor β can be extended to an admissible semigroup in $\mathcal{I}_8$.*

Proof. Put $T = S \cup \{\alpha, \beta\}$ and define multiplication on T as follows: S is a subsemigroup of T; $\alpha s = \alpha$ and $\beta s = \beta$ for all $s \in T$; $s\alpha = s\beta = \alpha$ for all $s \in S$. It is easy to see that $T \in \mathcal{I}_8$, $\mathrm{LA}(T) = \{\alpha, \beta\}$ and T is admissible. □

4.4 Proposition. *Every admissible semigroup $S \in \mathcal{I}_8$ can be extended to a reductive admissible semigroup in $\mathcal{I}_8$.*

Proof. Take an element $e \notin S$ and put $R = S\{e\}$. Let $x \to x'$ be a bijection of R onto a set R' with $R \cap R' = \{\alpha, \beta\}$, such that $\alpha' = \alpha$ and $\beta' = \beta$. Put $T = S \cup R'$ and define multiplication on T as follows:

(i) S is a subsemigroup of T;
(ii) $st' = (st)'$ for $s, t \in S$;
(iii) $se' = s'$ for $s \in S$;
(iv) $s'w = s'$ for $s \in S$, $w \in T$;
(v) $e'w = e'$ for $w \in T$.

It is easy to see that the multiplication is correctly defined, $T \in \mathcal{I}_8$, $\mathrm{LA}(T) = R'$, and T is admissible. It remains to prove that T is reductive. Let $s, t \in T$, $s \neq t$. If $s, t \in S$, then $se' = s' \neq t' = te'$. If $s, t \in R'$, then $ss = s \neq t = ts$. Finally, if $s \in S$ and $t \in R' - \{\alpha, \beta\}$, then $s\alpha \neq t = t\alpha$. □

4.5 Notation. In the next lemmas we suppose that $S \in \mathcal{I}_8$ is a given admissible reductive semigroup and c, d is a pair of distinct elements of $\mathrm{LA}(S)$ with $d \notin \{\alpha, \beta\}$.

Take two distinct elements x, y not belonging to S and denote by Z the LZ-semigroup with the underlying set $\{x, y\}$. Denote by F the free product of S and Z in $\mathcal{I}_8$, so that S and Z are disjoint subsemigroups of F, F is generated by $S \cup Z$ and for any $A \in \mathcal{I}_8$, any pair of homomorphisms $S \to A$, $Z \to A$ can be extended to a homomorphism $F \to A$.

By a canonical form of an element $u \in F$ we mean an expression $u = u_1 \dots u_n$, where

(i) $1 \leq n \leq 3$,
(ii) if $n = 2$, then either $u_1 \in Z$, $u_2 \in S$ or $u_1 \in S$, $u_2 \in Z$,
(iii) if $n = 3$, then $u_1 \in S$, $u_2 \in Z$, $u_3 \in S$ and $u_1 u_3 \neq u_1$.

Observe that for $n = 3$, $u_1 \in S - LA(S)$ (in particular, if $n = 3$, then $u_1 \notin \{\alpha, \beta\}$).

4.6 Lemma. *Every element of F can be expressed in a canonical form.*

Proof. As this is clear for the elements of $S \cup Z$, it is sufficient to show that the set of the elements expressible in a canonical form is a subsemigroup of F. For this sake, it is certainly sufficient to show that if $u = u_1 \dots u_n$ canonically, then each of the elements ux, uy and us (for $s \in S$) also has a canonical form. This can be done easily by considering the possible cases. For example, $xsy = xsxy = xsx = xs$. Also, if $st = s$, then $sxt = sxst = sxs = sx$. □

4.7 Lemma. *Let $u = u_1 \dots u_n$ and $u = v_1 \dots v_m$ be two canonical expressions of the same element $u \in F$. Then $n = m$ and either $u_1 = v_1, \dots, u_n = v_n$ or else $n = 3$, $u_1 = v_1$, $u_2 = v_2$ and $u_1u_3 = v_1v_3$.*

Proof. Denote by h_1 the homomorphism of F onto the two-element semilattice $\{0, 1\}$ (where $01 = 0$) such that $h_1(S) = \{1\}$ and $h_1(Z) = \{0\}$; define h_2 similarly, but setting $h_2(S) = \{0\}$ and $h_2(Z) = \{1\}$. Clearly, $h_1(u_1 \dots u_n) = 0$ iff $Z \cap \{u_1, \dots, u_n\} \neq \emptyset$; also, $h_2(u_1 \dots u_n) = 0$ iff $S \cap \{u_1, \dots, u_n\} \neq \emptyset$. From this it follows that it is sufficient to consider the case when $n \geq 2$ and $m \geq 2$.

For every $e \in \mathrm{LA}(S)$ denote by h_e the homomorphism of F into S extending the identity on S and the constant homomorphism of Z onto $\{e\}$. If $u_1 \in S$, then $h_e(u_1 \dots u_n) = u_1e$. If $v_1 \in Z$, then $h_e(v_1 \dots v_m) = e$. So, if $u_1 \in S$ and $v_1 \in Z$, then $u_1e = e$ for any $e \in \mathrm{LA}(S)$; in particular, $u_1\alpha = \alpha$ and $u_1\beta = \beta$, contradicting the admissibility of S. We conclude that u_1, v_1 either belong both to S or belong both to Z. In the case when $u_1, v_1 \in S$, we get $u_1e = v_1e$ for all $e \in \mathrm{LA}(S)$, so that $u_1 = v_1$ by the reductivity of S.

Denote by h_3 the homomorphism of F into $Z\{1\}$ extending the constant homomorphism of S onto $\{1\}$ and the identity on Z. If $u_1 = v_1 \in S$, then $h_3(u_1 \dots u_n) = u_2$ and $h_3(v_1 \dots v_m) = v_2$, so that $u_2 = v_2$. If $u_1, v_1 \in Z$, then $h_3(u_1 \dots u_n) = u_1$ and $h_3(v_1 \dots v_m) = v_1$, so that $u_1 = v_1$.

So far we have proved that $u_1 = v_1$ and if $u_1 = v_1 \in S$, then $u_2 = v_2$.

Denote by h_4 the homomorphism of F into $S\{1\}$ extending the identity on S and the constant homomorphism of Z onto $\{1\}$. If $u_1 = v_1 \in Z$, then $h_4(u_1 \dots u_n) = u_2$ and $h_4(v_1 \dots v_m) = v_2$. So, $u_2 = v_2$.

Let s, t, t' be elements of S. If $sx = sxt$, then $xsx = xsxt$, i.e., $xs = xst$ and hence $s = st$, so that sxt is not a canonical form. If $sxt = sxt'$, then (similarly) $st = st'$. □

4.8 Notation. We have seen that every element $u \in F$ can be expressed canonically, $u = u_1 \dots u_n$, and u_1 is uniquely determined by u; we say that u begins with u_1.

Denote by R the relation, containing the following pairs of elements of F:

$$(\alpha, xc), (\beta, yc), (x\alpha, x\beta), (y\alpha, y\beta), (\alpha, \alpha x), (\alpha, \alpha y), (\beta, \beta x), (\beta, \beta y), (xd, yd).$$

Denote by ρ the congruence of F generated by R.

Put $A_\alpha = \{s \in S : s\alpha = \alpha\}$ and $A_\beta = \{s \in S : s\beta = \beta\}$.

Put $B_\alpha = \{\alpha\} \cup \{xs : s \in S - \{d\}\} \cup A_\alpha ZS$ (notice that $A_\alpha Z \subseteq A_\alpha ZS$).

Put $B_\beta = \{\beta\} \cup \{ys : s \in S - \{d\}\} \cup A_\beta ZS$.

For $s \in \mathrm{LA}(S) - \{\alpha, \beta\}$ put $B_s = \{s, sx, sy\}$.

For $s \in S - \mathrm{LA}(S)$ put $B_s = \{s\}$.

4.9 Lemma. *Let $(v,w) \in R \cup R^{-1}$ and let p,q be two elements of $F\{1\}$ such that $pvq \in B_\alpha$ (or $pvq \in B_\beta$). Then $pwq \in B_\alpha$ (or $pwq \in B_\beta$, respectively).*

Proof. Let $pvq \in B_\alpha$ (the other case is similar). Consider first the case $pvq = \alpha$. Then clearly $p,q \in S\{1\}$, $v \in \{\alpha,\beta\}$, $w \in \{xc, yc, \alpha x, \alpha y\}$. If $p \neq 1$, then $\alpha = pv = p\alpha$, so that $p \in A_\alpha$ and $pwq \in A_\alpha ZS$. If $p = 1$, then $\alpha = vq = v$, so that $w \in \{xc, \alpha x\}$ and we have either $pwq = xcq = xc$ or $pwq = \alpha xq = \alpha x$; in both cases, $pwq \in B_\alpha$.

Let $pvq \in \{xs : s \in S - \{d\}\} \cup A_\alpha ZS$. If $p \notin S\{1\}$, it follows easily from 4.7 that p, and then also pwq belong to $\{xs : s \in S - \{d\}\} \cup A_\alpha ZS$. So, let $p \in S\{1\}$.

Let $p \in S$. Then $pvq \in A_\alpha ZS$; since v either begins with an element of Z or belongs to $\{\alpha, \beta, \alpha x, \alpha y, \beta x, \beta y\}$, we get $p \in A_\alpha$. If w either begins with an element of Z or is one of the elements $\alpha x, \alpha y, \beta x, \beta y$, we get $pwq \in A_\alpha ZS$. So, let $w \in \{\alpha, \beta\}$. Then $pw = \alpha$. If $q \in S\{1\}$, we get $pwq = \alpha \in B_\alpha$. Otherwise, $pwq = \alpha q \in A_\alpha ZS \subseteq B_\alpha$.

Finally, let $p = 1$. Then $pvq = vq$, so that v does not begin with y and $v \notin \{xd, \beta, \beta x, \beta y\}$. Hence both v and w belong to $\{\alpha, xc, x\alpha, x\beta, \alpha x, \alpha y\}$. But then $pwq = wq \in B_\alpha$. □

4.10 Lemma. *Let $(v,w) \in R \cup R^{-1}$ and let p,q be two elements of $F\{1\}$ such that $pvq \in B_s$, where $s \in S - \{\alpha,\beta\}$. Then $pwq \in B_s$.*

Proof. Consider first the case $pvq = s$. Then $p,v,q \in S\{1\}$, $v \in \{\alpha,\beta\}$, $s = pv \notin \{\alpha,\beta\}$, so by the admissibility of S we get $p = s \in \mathrm{LA}(S) - \{\alpha,\beta\}$. Hence $pwq = swq \in \{s, sx, sy\} = B_s$.

It remains to consider the case $s \in \mathrm{LA}(S) - \{\alpha,\beta\}$, $pvq \in \{sx, sy\}$.

Let $p \notin S\{1\}$. It follows easily from 4.7 and from $s \in \mathrm{LA}(S)$ that $p = pvq$. Then $pwq = pvq \in B_s$.

Let $p \in S\{1\}$. If v begins with either x or y, then from $pvq \in \{sx, sy\}$ we get $p = s$ and then $pwq = swq \in \{s, sx, sy\}$. So, let $v \in \{\alpha, \beta, \alpha x, \alpha y, \beta x, \beta y\}$. Then either $p\alpha$ or $p\beta$ does not belong to $\{\alpha,\beta\}$, so $p \in \mathrm{LA}(S)$ and we again obtain $p = s$ and $pwq = swq \in \{s, sx, sy\}$. □

4.11 Lemma. *Let $(s,t) \in \rho \cap (S \times S)$. Then $s = t$.*

Proof. Since $(s,t) \in \rho$, there is a finite sequence $s_0, \dots, s_n$ of elements of F such that $s_0 = s$, $s_n = t$ and for every $i = 1, \dots, n$ we have $s_{i-1} = pvq$, $s_i = pwq$ for some $p,q \in F\{1\}$ and $(v,w) \in R \cup R^{-1}$. It remains to use 4.9 and 4.10. □

4.12 Lemma. *Every congruence of F containing ρ and containing the pair (c,d) contains (α,β).*

Proof. Let $\sim$ be a congruence containing ρ and (c,d). We have $\alpha \sim xc \sim xd \sim yd \sim yc \sim \beta$. □

4.13 Proposition. *Let S be a reductive admissible semigroup from $\mathcal{I}_8$ and let $c,d \in S$, $c \neq d$. Then S can be extended to an admissible semigroup $T \in \mathcal{I}_8$ such that $(\alpha,\beta) \in \theta_{c,d}$, where $\theta_{c,d}$ is the congruence of T generated by (c,d).*

Proof. Since S is reductive, it is sufficient to consider the case $\{c,d\} \subseteq \mathrm{LA}(S)$. If $\{c,d\} = \{\alpha,\beta\}$, we can put $T = S$. So, we can assume that $d \notin \{\alpha,\beta\}$.

Let us keep the notation introduced in 4.5 and 4.8. Denote by T the semigroup F/ρ, in which we identify (or replace) every element s/ρ (for $s \in S$) with s (this is possible according to 4.11). So, T is an extension of S. We have $T \in \mathcal{I}_8$, since $F \in \mathcal{I}_8$.

We have $\{\alpha, \beta\} \subseteq \mathrm{LA}(T)$: this follows from $(\alpha x, \alpha) \in \rho$, $(\alpha y, \alpha) \in \rho$, $(\beta x, \beta) \in \rho$ and $(\beta y, \beta) \in \rho$.

Let $s \in \mathrm{LA}(S)$. Then $(\alpha, \alpha x) \in \rho$ implies $(s\alpha, s\alpha x) \in \rho$, i.e., $(s, sx) \in \rho$. Similarly, $(s, sy) \in \rho$. From this it follows that $(s, st) \in \rho$ for any $t \in F$, so that $s \in \mathrm{LA}(T)$. This proves $\mathrm{LA}(S) \subseteq \mathrm{LA}(T)$. Now it is easy to see that $\mathrm{LA}(T)$ also contains all the elements sx/ρ, sy/ρ, xs/ρ and ys/ρ with $s \in \mathrm{LA}(S)$.

Let $u = u_1 \dots u_n$ (canonically) be an element of F such that $u/\rho \in T - \mathrm{Ł}(T)$. We have $u_i \notin \mathrm{LA}(S)$ for all i.

We have $(\alpha, xc) \in \rho$, so that $(x\alpha, xxc) \in \rho$, i.e., $(x\alpha, xc) \in \rho$ and hence $(\alpha, x\alpha) \in \rho$. Hence also $(\alpha, x\beta) \in \rho$. Similarly, $(\beta, y\alpha) \in \rho$ and $(\beta, y\beta) \in \rho$. This shows that if $u_i \in \{x, y\}$, then $(u_i\alpha)/\rho = (u_i\beta)/\rho \in \{\alpha, \beta\}$. If $u_i \in S - \mathrm{LA}(S)$, then $u_i\alpha = u_i\beta \in \{\alpha, \beta\}$ by the admissibility of S. Now it is easy to see that $(u\alpha)/\rho = (u\beta)/\rho \in \{\alpha, \beta\}$.

We see that T is admissible. The rest follows from 4.12. □

4.14 Proposition. *Let S be an admissible semigroup from $\mathcal{I}_8$. Then S can be extended to an admissible semigroup $T \in \mathcal{I}_8$ such that for any $c, d \in S$ with $c \neq d$, the congruence of T generated by (c, d) contains (α, β).*

Proof. By 4.4 and 4.14, for every admissible semigroup $S \in \mathcal{I}_8$ and every $c, d \in S$ with $c \neq d$ there exists an admissible semigroup $T_{c,d} \in \mathcal{I}_8$ such that (α, β) belongs to the congruence of $T_{c,d}$ generated by (c, d). The result follows by a standard argument using transfinite construction; observe that the union of a chain of admissible semigroups from $\mathcal{I}_8$ is an admissible semigroup from $\mathcal{I}_8$. □

4.15 Theorem. *Every semigroup $S \in \mathcal{I}_8$ can be extended to a subdirectly irreducible semigroup from $\mathcal{I}_8$.*

Proof. By 4.3, it is enough to consider the case when S is admissible. Define a countable chain of admissible semigroups $S_0, S_1, \dots$ as follows: $S_0 = S$; S_{i+1} is an extension of S_i claimed by 4.14. The union of this chain is the desired semigroup. □

CHAPTER V

THE LATTICE OF VARIETIES OF LEFT DISTRIBUTIVE SEMIGROUPS

V.1 THE SUBVARIETIES OF $\mathcal{T} \cap \mathcal{R}$

1.1 Notation. We denote by $\mathcal{L}$ the variety of LD-semigroups, by $\mathcal{I}$ the variety of idempotent LD-semigroups (so that $\mathcal{I} = \mathcal{I}_9$), by $\mathcal{R}$ the variety of LDR-semigroups and by $\mathcal{T}$ the variety of LDT-semigroups.

1.2 Lemma. *$\mathcal{T} \cap \mathcal{R} = \mathcal{A} \vee \mathcal{I}$ and every subvariety of $\mathcal{T} \cap \mathcal{R}$ is equal to $\mathcal{A}_i \vee \mathcal{I}_j$ for some $0 \le i \le 5$ and $0 \le j \le 9$.*

Proof. By I.1.17, every semigroup in $\mathcal{T} \cap \mathcal{R}$ is a subdirect product of an A-semigroup and an idempotent LD-semigroup. Now, use Theorems III.2.2 and IV.2.2. □

1.3 Lemma. *For $j \notin \{0, 2\}$ we have $\mathcal{A}_2 \vee \mathcal{I}_j = \mathcal{A}_4 \vee \mathcal{I}_j$ and $\mathcal{A}_3 \vee \mathcal{I}_j = \mathcal{A}_5 \vee \mathcal{I}_j$.*

Proof. Let G be the free semigroup in $\mathcal{A}_3 \vee \mathcal{I}_j$ over two generators x and y. Clearly, $xy \ne yx$ in G and $xy, yx \notin \mathrm{Id}(G)$. From this it follows that $G/\mathrm{Id}(G) \notin \mathcal{A}_3$ and hence $(\mathcal{A}_3 \vee \mathcal{I}_j) \cap \mathcal{A}_5 \not\subseteq \mathcal{A}_3$. Consequently, $(\mathcal{A}_3 \vee \mathcal{I}_j) \cap \mathcal{A}_5 = \mathcal{A}_5$, which means that $\mathcal{A}_3 \vee \mathcal{I}_j = \mathcal{A}_5 \vee \mathcal{I}_j$. One can prove $\mathcal{A}_2 \vee \mathcal{I}_j = \mathcal{A}_4 \vee \mathcal{I}_j$ similarly. □

1.4 Lemma. *Let either $i \notin \{2, 3\}$ or $j \in \{0, 2\}$. Then a semigroup S belongs to $\mathcal{A}_i \vee \mathcal{I}_j$ if and only if $S \in \mathcal{T} \cap \mathcal{R}$, $\mathrm{Id}(S) \in \mathcal{I}_j$ and $S/\mathrm{Id}(S) \in \mathcal{A}_i$.*

Proof. Denote by V the class of all semigroups S with this property. It is easy to see that V is a variety, and hence $V = \mathcal{A}_i \vee \mathcal{I}_j$. □

1.5 Lemma. *Let (i, j) and (k, l) be two ordered pairs from $\{0, \dots, 5\} \times \{0, \dots, 9\}$. Then $\mathcal{A}_i \vee \mathcal{I}_j \subseteq \mathcal{A}_k \vee \mathcal{I}_l$ if and only if $\mathcal{I}_j \subseteq \mathcal{I}_l$ and one of the following three cases takes place: either $\mathcal{A}_i \subseteq \mathcal{A}_k$ or $l \notin \{0, 2\}$, $i = 4$, $k = 2$ or $l \notin \{0, 2\}$, $i = 5$, $k = 3$.*

Proof. Apply 1.2, 1.3 and 1.4. □

1.6 Lemma. *The variety $\mathcal{T} \cap \mathcal{R}$ has the following 44 subvarieties:*

$L_0 = \mathcal{A}_0 \vee \mathcal{I}_0 = \mathcal{A}_0 = \mathcal{I}_0$,
$L_1 = \mathcal{A}_0 \vee \mathcal{I}_1 = \mathcal{I}_1$,
...
$L_9 = \mathcal{A}_0 \vee \mathcal{I}_9 = \mathcal{I}_9$,
$L_{10} = \mathcal{A}_1 \vee \mathcal{I}_0 = \mathcal{A}_1$,
$L_{11} = \mathcal{A}_1 \vee \mathcal{I}_1$,
...
$L_{19} = \mathcal{A}_1 \vee \mathcal{I}_9$,
$L_{20} = \mathcal{A}_2 \vee \mathcal{I}_0$,
$L_{21} = \mathcal{A}_2 \vee \mathcal{I}_1 = \mathcal{A}_4 \vee \mathcal{I}_1$,

$L_{22} = \mathcal{A}_2 \vee \mathcal{I}_2$,
$L_{23} = \mathcal{A}_2 \vee \mathcal{I}_3 = \mathcal{A}_4 \vee \mathcal{I}_3$,
$\dots$
$L_{29} = \mathcal{A}_2 \vee \mathcal{I}_9 = \mathcal{A}_4 \vee \mathcal{I}_9$,
$L_{30} = \mathcal{A}_3 \vee \mathcal{I}_0$,
$L_{31} = \mathcal{A}_3 \vee \mathcal{I}_1 = \mathcal{A}_5 \vee \mathcal{I}_1$,
$L_{32} = \mathcal{A}_3 \vee \mathcal{I}_2$,
$L_{33} = \mathcal{A}_3 \vee \mathcal{I}_3 = \mathcal{A}_5 \vee \mathcal{I}_3$,
$\dots$
$L_{39} = \mathcal{A}_3 \vee \mathcal{I}_9 = \mathcal{A}_5 \vee \mathcal{I}_9 = \mathcal{T} \cap \mathcal{R}$,
$L_{40} = \mathcal{A}_4 \vee \mathcal{I}_0$,
$L_{41} = \mathcal{A}_4 \vee \mathcal{I}_2$,
$L_{42} = \mathcal{A}_5 \vee \mathcal{I}_0$,
$L_{43} = \mathcal{A}_5 \vee \mathcal{I}_2$.

Proof. It follows from 1.5. □

V.2 THE VARIETIES $S_{i,j}$, $R_{i,j}$ AND $T_{i,j}$

2.1 Notation. We denote by $\mathrm{M}(u_1 \approx v_1, \dots)$ the variety of LD-semigroups satisfying $u_1 \approx v_1, \dots$. Put

$S_1 = \mathrm{M}(x^2 \approx x^3, xy^2 \approx xyx)$,
$S_2 = \mathrm{M}(x^2 \approx x^3)$,
$S_3 = \mathrm{M}(xy^2 \approx xyx)$,
$S_4 = \mathcal{L}$ (the variety of all LD-semigroups),
$S_{i,j} = \{S \in S_i : \mathrm{Id}(S) \in \mathcal{I}_j\}$ for $1 \le i \le 4$ and $0 \le j \le 9$,
$R_1 = \mathrm{M}(xy \approx xyx)$,
$R_2 = \mathrm{M}(xy \approx xy^2)$,
$R_3 = \mathrm{M}(x^2 \approx x^3, xy^2 \approx xyx, x^2y \approx x^2y^2) = \mathcal{R} \cap S_1$,
$R_4 = \mathrm{M}(x^2 \approx x^3, x^2y \approx x^2y^2) = \mathcal{R} \cap S_2$,
$R_5 = \mathrm{M}(x^2y \approx x^2y^2, xy^2 \approx xyx) = \mathcal{R} \cap S_3$,
$R_6 = \mathrm{M}(x^2y \approx x^2y^2) = \mathcal{R}$,
$R_{i,j} = R_i \cap S_{4,j}$ for $1 \le i \le 6$ and $0 \le j \le 9$,
$T_1 = \mathrm{M}(xy \approx x^2y)$,
$T_2 = \mathrm{M}(x^2 \approx x^3, xy^2 \approx x^2y^2) = \mathcal{T} \cap S_2$,
$T_3 = \mathrm{M}(xy^2 \approx x^2y^2) = \mathcal{T}$,
$T_{i,j} = T_i \cap S_{4,j}$ for $1 \le i \le 3$ and $0 \le j \le 9$.

2.2 Lemma. *The following are true:*

(i) $S_{i,j}$ *is a subvariety of* $\mathcal{L}$ *and* $S_{i,j} \cap \mathcal{I} = \mathcal{I}_j$.
(ii) $S_1 = S_2 \cap S_3$ *and* $S_2 \vee S_3 \subseteq S_4$.
(iii) $\mathcal{A}_5 \subseteq S_{3,j} \subseteq S_{4,j}$, $\mathcal{A}_5 \not\subseteq S_{1,j}$ *and* $\mathcal{A}_5 \not\subseteq S_{2,j}$.
(iv) $S_{1,j} = S_{2,j} \cap S_{3,j}$, $S_{1,0} = S_{2,0} = \mathcal{A}_4$ *and* $S_{3,0} = S_{4,0} = \mathcal{A}_5$.
(v) $R_1 = R_2 \cap R_3$, $R_3 = R_4 \cap R_5$, $R_2 \subseteq R_4$ *and* $R_4 \vee R_5 \subseteq R_6$.
(vi) $T_1 \subseteq T_2 \subseteq T_3$.

Proof. It is easy. □

V.3 AUXILIARY RESULTS

3.1 Notation. Let X be a countably infinite set of variables. As before, we denote by $\mathbf{F}$ the free semigroup over X; the elements of $\mathbf{F}$ will be called words. Recall that F is a subset of $\mathbf{F}$, and every word is equivalent to a unique word from F with respect to the equational theory of LD-semigroups.

We denote by W_1 the set of the words t such that $f(t) \in \mathrm{Id}(S)$ for all LD-semigroups S and all homomorphisms f of $\mathbf{F}$ into S. Denote by W_2 the subsemigroup of $\mathbf{F}$ generated by $\{x^3 : x \in X\}$. Clearly, $W_2 \subseteq W_1$.

The first variable in a word t will be denoted by $o(t)$. Denote by $\mathrm{var}(t)$ the set of variables occurring in t.

3.2 Lemma. *Let r, s be two words with $o(r) \neq o(s)$ and let x be a variable such that $x \neq o(r)$. Then $\mathrm{M}(xr \approx xs) \subseteq \mathcal{T}$.*

Proof. Let y be a variable not occurring in xrs. Denote by y_1 the first variable in s. Consider the substitution f with $f(x) = f(y_1) = x$ and $f(z) = y$ for all variables $z \notin \{x, y_1\}$. Applying f to the equation $xry \approx xsy$ (which is a consequence of $xr \approx xs$), it is easy to see that either $xy^2 \approx x^2y$ or $xy^2 \approx x^2y^2$ is a consequence of $xr \approx xs$. However, $\mathrm{M}(xy^2 \approx x^2y) = \mathcal{T} \cap \mathcal{R}$ and $\mathrm{M}(xy^2 \approx x^2y^2) = \mathcal{T}$. □

3.3 Lemma. *Let r, s be two words.*

(i) *If $o(r) \neq o(s)$, then $\mathrm{M}(r \approx s) \subseteq \mathcal{T}$.*
(ii) *If $o(r) \neq o(s) = x$ and s starts with x^2 (i.e., either $s = x^2$ or $s = x^2t$ for some t), then $\mathrm{M}(xr \approx s) \subseteq \mathcal{T}$.*
(iii) *If x, y, z are variables and $y \neq z$, then $\mathrm{M}(xyr \approx xzs) \subseteq \mathcal{T}$.*

Proof. (i) Let x be a variable not occurring in rs. Then $\mathrm{M}(r \approx s) \subseteq \mathrm{M}(xr \approx xs) \subseteq \mathcal{T}$ by 3.2.

(ii) This follows from 3.2.

(iii) Let u be a variable not occurring in $xyzrs$. Consider the substitution f with $f(x) = f(z) = x$ and $f(v) = y$ for all variables $v \notin \{x, z\}$. Applying f to the equation $xyru \approx xzsu$, it is easy to see that either $xy^2 \approx x^2y$ or $xy^2 \approx x^2y^2$ is a consequence of $xyr \approx xzs$. □

3.4 Lemma. *Let r, s be two words.*

(i) *If x is a variable not occurring in r and if $s \notin \{x, x^2\}$ and $s \neq tx$ for any word t with $x \notin \mathrm{var}(t)$, then $\mathrm{M}(rx \approx s) \subseteq \mathcal{R}$.*
(ii) *If $\mathrm{var}(r) \neq \mathrm{var}(s)$, then $\mathrm{M}(r \approx s) \subseteq \mathcal{R}$.*

Proof. (i) Consider the substitution f with $f(x) = y$ and $f(v) = x$ for all variables $v \neq x$. Applying f to $rx \approx s$, we see that the equation $rx \approx s$ has a consequence $t \approx u$, where

$$t \in \{xy, x^2y\}$$

and

$$u \in \{x, x^2, x^3, y^3, xyx, x^2yx, xy^2, x^2y^2, yx, yx^2, y^2x, y^2x^2\}.$$

Every one of these 24 equations implies $x^2y = x^2y^2$.

(ii) By symmetry, we can assume that there is a variable $x \in \mathrm{var}(s) - \mathrm{var}(r)$. If $s = x$, then $\mathrm{M}(r \approx s)$ is the trivial variety. In the opposite case we have $sx \notin \{x, x^2\}$ and $\mathrm{M}(r \approx s) \subseteq \mathrm{M}(rx \approx sx) \subseteq \mathcal{R}$ by (i). □

3.5 Lemma. *Let V be a variety of LD-semigroups. If $V \cap \mathcal{I} \subseteq \mathcal{I}_6$, then $V \subseteq \mathcal{T}$. If $V \cap \mathcal{I} \subseteq \mathcal{I}_5$, then $V \subseteq \mathcal{R}$.*

Proof. First, let $V \cap \mathcal{I} \subseteq \mathcal{I}_6$. Then $abc = bac$ for all $a, b, c \in \mathrm{Id}(S)$, for any $S \in V$. Consequently, $V \subseteq \mathrm{M}(x^2yz^2 \approx y^2xz^2) \subseteq \mathcal{T}$ by 3.3(i).

Now, let $V \cap \mathcal{I} \subseteq \mathcal{I}_5$. Then $V \subseteq \mathrm{M}(x^3 \approx x^2yx^2) \subseteq \mathcal{R}$ by 3.4(ii). □

3.6 Lemma. *The following are true:*

(i) *Let r, s be two words such that $o(r) \neq o(s)$ and $\mathrm{var}(r) \neq \mathrm{var}(s)$. Then $\mathrm{M}(r \approx s) \subseteq \mathcal{T} \cap \mathcal{R}$.*

(ii) *Let V be a variety of LD-semigroups such that $V \cap \mathcal{I} \subseteq \mathcal{I}_3$. Then $V \subseteq \mathcal{T} \cap \mathcal{R}$.*

Proof. Use 3.3(i), 3.4(ii) and 3.5. □

3.7 Lemma. *Let r, s be two words.*

(i) *If $r, s \in W_2$, then $\mathrm{M}(r \approx s) = S_{4,j}$ for some j.*

(ii) *If $r, s \in W_1$, then $\mathrm{M}(r \approx s) \cap \mathcal{T} = T_{3,j}$ for some j.*

(iii) *If $r \in W_1$, then either $\mathrm{M}(r \approx s) \cap \mathcal{T} \subseteq \mathcal{R}$ or $\mathrm{M}(r \approx s) \cap \mathcal{T} = T_{3,j}$ or $\mathrm{M}(r \approx s) \cap \mathcal{T} = T_{2,j}$ for some j.*

Proof. Put $V = \mathrm{M}(r \approx s)$ and let $V \cap \mathcal{I} = \mathcal{I}_j$. Then $V \subseteq S_{4,j}$ and $V \cap \mathcal{T} \subseteq T_{3,j}$.

(i) Let $S \in S_{4,j}$ and let f be a homomorphism of $\mathbf{F}$ into S. Then $f(W_2) \subseteq \mathrm{Id}(S)$ and hence $f(r) = f(s)$. Thus $S \in V$ and $V = S_{4,j}$.

(ii) Let $S \in T_{3,j}$ and let f be a homomorphism of $\mathbf{F}$ into S. Denote by g the substitution with $g(x) = x^3$ for all variables x. Put $h(a) = a^3$ for all $a \in S$, so that h is an endomorphism of S. We have $g(\mathbf{F}) = W_2$ and $h(S) = \mathrm{Id}(S)$. Moreover, $\mathrm{Id}(S) \in \mathcal{I}_j \subseteq V \cap \mathcal{T}$ and $fg(\mathbf{F}) \subseteq \mathrm{Id}(S)$. Consequently, $fg(r) = fg(s)$. On the other hand, it is easy to see that $fg = hf$. Therefore $hf(r) = hf(s)$. But both $f(r)$ and $f(s)$ belong to $\mathrm{Id}(S)$, and so $f(r) = f(s)$.

(iii) By the construction of free LD-semigroups given in II.1.1 we can assume that $s = x_1^i x_2 \dots x_n$ where $n \geq 1$, $x_1, \dots, x_n$ are pairwise different variables and $i \leq 2$. Put $U = \mathrm{M}(s \approx s^3)$. Clearly, $V \cap \mathcal{T} = U \cap \mathcal{T} \cap \mathrm{M}(r \approx s^3)$. Since the words r and s^3 belong to W_1, we have $\mathrm{M}(r \approx s^3) \cap \mathcal{T} = T_{3,k}$ for some k. If $n = 1$ and $i = 1$, then $U = \mathcal{I}$ and $V \cap \mathcal{T} = \mathcal{I}_k$. If $n = 1$ and $i = 2$, then $U = S_2$ and $V \cap \mathcal{T} = T_{2,k}$. Let $n \geq 2$. Then

$$U = \mathrm{M}(x_1^i x_2 \dots x_n \approx x_1^i x_2 \dots x_{n-1} x_n^2) \subseteq \mathcal{R}$$

by 3.4(i). □

3.8 Lemma. *Let x, y be two variables and r, s be two words with $x \notin \mathrm{var}(rs)$. Let $V = \mathrm{M}(xyr \approx xys)$. If either $V \subseteq \mathcal{R}$ or $xyr, xys \in W_1$, then either $V = S_{4,j}$ or $V = R_{6,j}$ for some j.*

Proof. Put $r = u_1 \dots u_n$ and $s = v_1 \dots v_m$ ($u_i, v_i \in X$).

Let $V \subseteq \mathcal{R}$. It is enough to show that a semigroup $S \in \mathcal{R}$ satisfies $xyr \approx xys$ if and only if $\mathrm{Id}(S)$ satisfies $xyr \approx xys$. The direct implication is clear. Let $\mathrm{Id}(S)$ satisfy $xyr \approx xys$. In S we have

$$\begin{aligned} xyr &= xy^2r = (xy)^2r = (xy)^2r^2 = (xy)^3y^3r^3 = (xy)^3y^3u_1^3 \dots u_n^3 \\ &= (xy)^3y^3v_1^3 \dots v_m^3 = xys. \end{aligned}$$

Let $xyr, xys \in W_1$. Then $V = \mathrm{M}(xyu_1^3 \dots u_n^3 \approx xyv_1^3 \dots v_m^3)$. If $x = y$, then the result follows from 3.7(i). Hence suppose that $x \neq y$ and put $\mathcal{I}_j = V \cap \mathcal{I}$. Then $\mathcal{I}_j$ satisfies $yu_1 \dots u_n \approx yv_1 \dots v_m$ and $V \subseteq S_{4,j}$. Conversely, let $S \in S_{4,j}$. Then S satisfies $y^3u_1^3 \dots u_n^3 \approx y^3v_1^3 \dots v_m^3$ and hence $S \in V$. □

3.9 Lemma. *Let $i, j \leq 2 \leq n$, let $x_1, \dots, x_n$ be pairwise different variables and let p be a permutation of $\{1, \dots, n\}$ such that $p(1) \neq 1$. Put*

$$r = x_1^i x_2 \dots x_n, \qquad s = x_{p(1)}^j x_{p(2)} \dots x_{p(n)}$$

and $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathcal{T} \cap \mathcal{R}$ or $V = T_{3,6}$.

Proof. By 3.3(i), $V \subseteq \mathcal{T}$. If $p(n) \neq n$, then $V \subseteq \mathcal{R}$ by 3.4(i). So, we can assume that $p(n) = n$. Then $n \geq 3$, $\mathcal{I}_1 \not\subseteq V$, $V \cap \mathcal{I} = \mathcal{I}_6$ and we get $V \subseteq T_{3,6}$. Conversely, let $S \in T_{3,6}$ and $a_1, \dots, a_n \in S$. Then

$$a_1^3 \dots a_{n-1}^3 a_{n-1}^3 = a_{p(1)}^3 \dots a_{p(n-1)}^3 a_{n-1}^3$$

and

$$\begin{aligned} a_1 \dots a_n =& a_1^2 a_2 \dots a_n = a_1^3 a_2^3 \dots a_{n-1}^3 a_{n-1}^3 a_n = a_{p(1)}^3 \dots a_{p(n-1)}^3 a_{n-1}^3 a_n \\ =& a_{p(1)} \dots a_{p(n-1)} a_{n-1} a_n = a_{p(1)} \dots a_{p(n-1)} a_n. \quad \square \end{aligned}$$

3.10 Lemma. *Let r, s be two words such that $o(r) \neq o(s)$ and let $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathcal{T} \cap \mathcal{R}$ or $V = T_{2,j}$ or $V = T_{3,j}$ for some j.*

Proof. By 3.3(i) we have $V \subseteq \mathcal{T}$ and by 3.6(i) we can assume that $\mathrm{var}(r) = \mathrm{var}(s)$. Taking into account 3.7(iii), we may restrict ourselves to the case $r, s \in F - W_1$. Then $r = x_1^i x_2 \dots x_n$ and $s = y_1^k y_2 \dots y_m$. We have $n = m$ and there is a permutation p of $\{1, \dots, n\}$ with $p(1) \neq 1$, such that $y_1 = x_{p(1)}, \dots, y_n = x_{p(n)}$. The result now follows from 3.9. □

3.11 Lemma. *Let $i \leq 2$, $3 \leq n$, let $x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{2, \dots, n\}$ such that $p(2) \neq 2$. Put $r = x_1 x_2 \dots x_n$, $s = x_1^i x_{p(2)} \dots x_{p(n)}$ and $V = \mathrm{M}(r \approx s)$. Then:*

(i) *$V \subseteq \mathcal{T}$.*
(ii) *If $p(n) \neq n$, then $V \subseteq \mathcal{T} \cap \mathcal{R}$.*
(iii) *If $p(n) = n$, then $V = T_{3,7}$.*

Proof. (i) Use 3.3(iii).

(ii) Use (i) and 3.4(i).

(iii) It is easy to see that $V \cap \mathcal{I} = \mathcal{I}_7$ and $V \subseteq T_{3,7}$. Conversely, let $S \in T_{3,7}$ and let $a_1, \dots, a_n$ be elements of S. Then

$$\begin{aligned} a_1 \dots a_n &= a_1^3 \dots a_{n-1}^3 a_1^3 a_n = a_1^3 a_{p(2)}^3 \dots a_{p(n-1)}^3 a_1^3 a_n \\ &= a_1^2 a_{p(2)} \dots a_{p(n-1)} a_n. \quad \square \end{aligned}$$

3.12 Lemma. *Let $n \geq 3$, let $x_1, \ldots, x_n$ be pairwise different variables and let p be a non-identical permutation of $\{1, \ldots, n\}$ such that $p(1) = 1$. Put $V = \mathrm{M}(x_1^2 x_2 \ldots x_n \approx x_1^2 x_{p(2)} \ldots x_{p(n)})$. Then:*

(i) *If $p(n) \neq n$, then $V = R_{6,4}$.*
(ii) *If $p(n) = n$, then $V = S_{4,7}$.*

Proof. If $p(n) \neq n$, then $V \subseteq \mathcal{R}$ according to 3.4(i). The rest is similar to 3.11. □

3.13 Lemma. *Let $i, k, q, t \leq 2 \leq n$, let $x_1, \ldots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \ldots, n\}$. Put*

$$V = \mathrm{M}(x_1^i x_2 \ldots x_{n-1} x_n^k \approx x_{p(1)}^q x_{p(2)} \ldots x_{p(n-1)} x_{p(n)}^t).$$

Then either $V \subseteq \mathcal{T} \cap \mathcal{R}$ or $V = S_{4,j}$ or $V = T_{m,j}$ or $V = R_{6,j}$ for some m and j.

Proof. The result can be put together from the following nine cases.

(i) Let $p(1) \neq 1$. Then we can apply 3.10.

(ii) Let $p(1) = 1$, $k = t = 1$ and $i = q = 2$. This case is clear from 3.12.

(iii) Let $p(1) = 1$, $p(2) \neq 2$, $k = t = 1$ and $i + q \leq 3$. In this case we can use 3.11.

(iv) Let $p(1) = 1$, $p(2) = 2$, $k = t = 1$ and $i = q = 1$. If p is the identical permutation, then $V = \mathcal{L}$. Hence assume that p is non-identical. Then $n \geq 4$. If $p(n) \neq n$, then $V \subseteq \mathcal{R}$ by 3.4(i), $V \cap \mathcal{I} = \mathcal{I}_4$ and it is easy to see that $V = R_{6,4}$. Now, let $p(n) = n$. Then $V \cap \mathcal{I} = \mathcal{I}_7$ and $V \subseteq S_{4,7}$. Conversely, if $S \in S_{4,7}$ and if $a_1, \ldots, a_n$ are elements of S, then

$$a_1 \ldots a_n = a_1 a_2^3 \ldots a_{n-1}^3 a_2^3 a_n^3 = a_1 a_2^3 a_{p(3)}^3 \ldots a_{p(n-1)}^3 a_2^3 a_n$$
$$= a_1 a_2 a_{p(3)} \ldots a_{p(n-1)} a_n$$

and $S \in V$.

(v) Let $p(1) = 1$, $p(2) = 2$, $k = t = 1$, $i = 1$ and $q = 2$. We have $V \subseteq \mathcal{T}$ by 3.3(ii). If $p(n) \neq n$, then $V \subseteq \mathcal{T} \cap \mathcal{R}$ follows from 3.4(i). Let $p(n) = n$ and $n \geq 3$. Then it is easy to see that $V = \mathcal{T} \cap \mathrm{M}(x_1^2 x_2 \ldots x_n \approx x_1^2 x_2 x_{p(3)} \ldots x_{p(n)})$. If p is non-identical, then $V = T_{3,7}$ by 3.12; if p is the identity, then $V = T_{3,9}$.

(vi) Let $p(1) = 1$, $k = t = 2$, $i = 2$ and $q = 1$. Then $V \subseteq \mathcal{T}$ by 3.3(ii) and we can use 3.7(ii).

(vii) Let $p(1) = 1$, $k = t = 2$ and $i = q = 1$. If $p(2) = 2$, then the result follows from 3.8. If $p(2) \neq 2$, then $n \geq 3$, $V \subseteq \mathcal{T}$ by 3.3(iii) and the result follows from 3.7(ii).

(viii) Let $p(1) = 1$, $k = t = 2$ and $i = q = 2$. In this case, it is possible to use 3.7(i).

(ix) Let $p(1) = 1$, $k = 2$ and $t = 1$. If $p(n) \neq n$, then $V \subseteq \mathcal{R}$ by 3.4(i). If $p(n) = n$, then the inclusion $V \subseteq \mathcal{R}$ is obvious. Hence we have

$$V = \mathcal{R} \cap \mathrm{M}(x_1^i x_2 \ldots x_{n-1} x_n^2 \approx x_1^q x_{p(2)} \ldots x_{p(n-1)} x_{p(n)}^2).$$

The result is now clear from (vi), (vii) and (viii). □

3.14 Lemma. *Let r, s be two words and let $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathcal{T} \cap \mathcal{R}$ or $V = T_{i,j}$ for some i and j.*

Proof. According to 3.4(ii) and 3.7(iii), we can assume that $\mathrm{var}(r) = \mathrm{var}(s)$ and $r, s \in F - W_1$. However, then 3.13 can be applied. □

V.4 THE LATTICE OF SUBVARIETIES OF $\mathcal{T}$

4.1 Lemma. *The following are true:*

(i) $T_{1,j} \cap \mathcal{A} = \mathcal{A}_1$, $T_{2,j} \cap \mathcal{A} = \mathcal{A}_4$, $T_{3,j} \cap \mathcal{A} = \mathcal{A}_5$ *and* $T_{1,j} \cap \mathcal{I} = T_{2,j} \cap \mathcal{I} = T_{3,j} \cap \mathcal{I} = \mathcal{I}_j$ *for every* $0 \leq j \leq 9$.

(ii) $T_{1,j} = \mathcal{A}_1 \vee \mathcal{I}_j$, $T_{2,j} = \mathcal{A}_4 \vee \mathcal{I}_j$ *and* $T_{3,j} = \mathcal{A}_5 \vee \mathcal{I}_j$ *for* $j \in \{0,1,3,5\}$.

Proof. Use 1.5 and 3.5. □

4.2 Lemma. *Let* $1 \leq i,j \leq 3$ *and* $0 \leq p,q \leq 9$. *Then* $T_{i,p} \cap T_{j,q} = T_{r,s}$ *for some* r,s. *Moreover,* $T_{i,p} \subseteq T_{j,q}$ *if and only if* $i \leq j$ *and* $\mathcal{I}_p \subseteq \mathcal{I}_q$.

Proof. It is easy. □

4.3 Lemma. *The varieties* $T_{i,j}$ $(1 \leq i \leq 3,\ 0 \leq j \leq 9)$ *are pairwise distinct.*

Proof. Use 4.2. □

4.4 Lemma. *Let* V *be a subvariety of* $\mathcal{T}$. *Then either* V *is contained in* $\mathcal{T} \cap \mathcal{R}$ *or* $V = T_{i,j}$ *for some* i *and* j.

Proof. If $V \subseteq \mathcal{R}$, then $V \subseteq \mathcal{T} \cap \mathcal{R}$. So, let $V \not\subseteq \mathcal{R}$. Then, by 3.14, V is the intersection of some varieties $T_{i,j}$, so that $V = T_{i,j}$ for some i,j by 4.2. □

4.5 Proposition. *The variety* $\mathcal{T}$ *has the following 62 subvarieties:*

$L_0, \dots, L_{43}$,
$L_{44} = T_{1,2}$,
$L_{45} = T_{2,2}$,
$L_{46} = T_{3,2}$,
$L_{47} = T_{1,4}$,
$L_{48} = T_{2,4}$,
$L_{49} = T_{3,4}$,
$L_{50} = T_{1,6}$,
$L_{51} = T_{2,6}$,
$L_{52} = T_{3,6}$,
$L_{53} = T_{1,7}$,
$L_{54} = T_{2,7}$,
$L_{55} = T_{3,7}$,
$L_{56} = T_{1,8}$,
$L_{57} = T_{2,8}$,
$L_{58} = T_{3,8}$,
$L_{59} = T_{1,9}$,
$L_{60} = T_{2,9}$,
$L_{61} = T_{3,9} = \mathcal{T}$.

We have $L_{44}, \dots, L_{61} \not\subseteq L_{43} = \mathcal{T} \cap \mathcal{R}$. *We have* $T_{i,p} \subseteq T_{j,q}$ *if and only if* $i \leq j$ *and* $\mathcal{I}_p \subseteq \mathcal{I}_q$. *We have* $\mathcal{A}_m \vee \mathcal{I}_n \subseteq T_{r,s}$ *if and only if* $\mathcal{I}_n \subseteq \mathcal{I}_s$ *and either* $r = 3$ *or* $r = 2$, $m \in \{0,1,2,4\}$ *or* $r = 1$, $m \in \{0,1\}$.

Proof. Let V be a subvariety of $\mathcal{T}$ such that $V \not\subseteq \mathcal{R}$. By 4.4 and 4.1(ii), $V = T_{i,j}$ where $i \in \{1,2,3\}$ and $j \in \{2,4,6,7,8,9\}$. Conversely, if i and j are such numbers, then $T_{1,2} \subseteq T_{i,j}$ and hence $T_{i,j} \not\subseteq \mathcal{R}$. The rest is easy. □

V.5 AUXILIARY RESULTS

5.1 Lemma. *Let $i, j, k \leq 2$, $n \geq 0$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}(x^i x_1 \dots x_{n-1} x_n^j \approx x^k x_{p(1)} \dots x_{p(n)} x).$$

Then either $V \subseteq \mathcal{T}$ or $V = S_{r,s}$ or $V = R_{t,q}$ for some t and q.

Proof. We distinguish six cases.

(i) $n = 0$. Then either $S = \mathcal{L}$ or $V = S_{2,9}$ or $V = \mathcal{I}$.

(ii) $n \geq 1$ and $i = j = k = 2$. Then 3.7(i) can be applied.

(iii) $n \geq 1$, $i = k = 2$ and $j = 1$. By 3.4(i), $V \subseteq \mathcal{R}$ and then clearly $V = \mathcal{R} \cap U$ where

$$U = \mathrm{M}(x^i x_1 \dots x_{n-1} x_n^2 \approx x^2 x_{p(1)} \dots x_{p(n)} x).$$

But $U = S_{4,s}$ for some s and $V = R_{6,s}$.

(iv) $n \geq 1$ and $i + k = 3$. By 3.3(ii), $V \subseteq \mathcal{T}$.

(v) $n \geq 1$, $i = k = 1$ and $j = 2$. If $p(1) \neq 1$, then $V \subseteq \mathcal{T}$ due to 3.3(iii). Now we can assume that $p(1) = 1$. Consider first the case when p is the identity. Then it is easy to see that $V \subseteq S_{3,8}$. Conversely, if $S \in S_{3,8}$ and $a, b_1, \dots, b_n \in S$, then

$$ab_1 \dots b_n^2 = a(b_1 \dots b_n)^2 = ab_1 \dots b_n a$$

and $S \in V$. Now, let p be non-identical. Using similar arguments as in the last case, we see that $V = S_{3,4}$.

(vi) $n \geq 1$ and $i = j = k = 1$. Then $V \subseteq \mathcal{R}$,

$$V = \mathcal{R} \cap \mathrm{M}(x x_1 \dots x_{n-1} x_n^2 \approx x x_{p(1)} \dots x_{p(n)} x)$$

and either $V = R_{5,8}$ or $V = R_{5,4}$ by (v). □

5.2 Lemma. *Let $i, j \leq 2$, $n \geq 0$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}(x^i x_1 \dots x_n x \approx x^j x_{p(1)} \dots x_{p(n)} x).$$

Then either $V \subseteq \mathcal{T}$ or $V = S_{4,9}$ or $V = S_{4,7}$.

Proof. It is similar to the proof of 5.1. □

5.3 Lemma. *Let $i, j, k \leq 2 \leq n$, $1 \leq q < n$, $x, x_1, \dots, x_n$ be pairwise distinct variables and lep p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}(x^i x_1 \dots x_{n-1} x_n^j \approx x^k x_{p(1)} \dots x_{p(n)} x_{p(q)}).$$

Then either $V \subseteq \mathcal{T}$ or $V = S_{4,r}$ or $V = R_{6,r}$ for some r.

Proof. We distinguish five cases.

(i) $i = j = k = 2$. In this case we can use 3.7(i).

(ii) $i = k = 2$ and $j = 1$. Clearly, $V \subseteq \mathcal{R}$ and we can use 3.8.

(iii) $i + k = 3$. Then $V \subseteq \mathcal{T}$.

(iv) $i = k = 1$ and $p(1) \neq 1$. Then $V \subseteq \mathcal{T}$ by 3.2.

(v) $i = k = 1$ and $p(1) = 1$. If $j = 2$, then we can use 3.8. If $j = 1$, then $V \subseteq \mathcal{R}$ and we can again use 3.8. □

5.4 Lemma. *Let $i, j \leq 2 \leq n$, $1 \leq r, s < n$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}(x^i x_1 \dots x_n x_r \approx x^j x_{p(1)} \dots x_{p(n)} x_{p(s)}).$$

Then either $V \subseteq \mathcal{T}$ or $V = S_{4,q}$ or $V = S_{6,q}$ for some q.

Proof. It is similar to the proof of 5.3.

5.5 Lemma. *Let $i, j \leq 2 \leq n$, $1 \leq k < n$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}(x^i x_1 \dots x_n x \approx x^j x_{p(1)} \dots x_{p(n)} x_{p(k)}).$$

Then either $V \subseteq \mathcal{T}$ or $V = S_{r,s}$ for some r, s or $V = R_{t,s}$ for some t, s.

Proof. Clearly, $V \cap \mathcal{I} = \mathcal{I}_8$ and

$$V \subseteq \mathrm{M}(x^3_{p(k)} \dots x^3_{p(n)} x^3_{p(k)} \approx x^3_{p(k)} \dots x^3_{p(n)}).$$

Consequently, $V \subseteq U$ where

$$U = \mathrm{M}(x^i x_1 \dots x_n x \approx x^j x_{p(1)} \dots x_{p(n)})$$

and $V = U \cap S_{4,8}$. The result now follows from 5.1. □

5.6 Lemma. *Let r, s be two words such that* $\mathrm{var}(r) = \mathrm{var}(s)$ *and $o(r) = o(s)$. Put $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathcal{T} \cap \mathcal{R}$ or $V = T_{i,j}$ or$V = R_{p,q}$ or $V = S_{n,m}$ for some i, j, p, q, n, m.*

Proof. We can assume that $r, s \in F$. The result then follows from 3.13 and 5.1, . . . , 5.5. □

5.7 Lemma. *Let r, s be two words such that* $\mathrm{var}(r) \neq \mathrm{var}(s)$ *and let $V = \mathrm{M}(r \approx s)$. Then either $V = \mathcal{T} \cap \mathcal{R}$ or $V = R_{6,j}$ or $V = R_{4,j}$ for some j.*

Proof. By 3.4(ii), $V \subseteq \mathcal{R}$ and we can assume that $o(r) = o(s)$; denote this variable by x. Recall that $o(w)$ is the first variable in a word w. The last variable in w will be denoted by $\bar{o}(w)$. We distinguish nine cases.

(i) $r = x^2 p$ and $s = x^2 q$ where p, q are two words with $o(p) \neq x \neq o(q)$. Then $V = R_{6,j}$ by 3.7(i).

(ii) $r = x^i p$ and $s = x^2 q$ where p, q are two words with $o(p) \neq x \neq o(q)$ and $i + j = 3$. Then $V \subseteq \mathcal{T} \cap \mathcal{R}$ by 3.3(ii).

(iii) $r = xp$ and $s = xq$ where p, q are two words with $o(p) = o(q) \neq x$ and $\bar{o}(p) \neq x \neq \bar{o}(q)$. Then we can assume that $x \notin \mathrm{var}(pq)$ and the result follows from 3.8.

(iv) $r = xp$ and $s = xq$ where p, q are two words with $x \neq o(p) \neq o(q) \neq x$. Then $V \subseteq \mathcal{T} \cap \mathcal{R}$ by 3.3(iii).

(v) $r = xp$ and $s = xq$ where p, q are two words with $o(p) = o(q) \neq x$ and $\bar{o}(p) \neq x = \bar{o}(q)$. We can assume that $p = x_1 \dots x_n$, $x \notin \mathrm{var}(p)$, $q = y_1 \dots y_m x$, $x_1 = y_1$, $x \neq y_i$. Then $V \cap \mathcal{I} = \mathcal{I}_1$ and it is easy to see that $V = R_{6,1}$.

(vi) $r = xp$ and $s = xq$ where p, q are two words with $o(p) = o(q) \neq x = \bar{o}(p) = \bar{o}(q)$. We can assume that $p = x_1 \dots x_n x$, $q = y_1 \dots y_m x$, $x_1 = y_1$. Then $V \cap \mathcal{I} = \mathcal{I}_5$ and $V = R_{6,5}$.

(vii) $r = x$. Then $V \subseteq \mathcal{I}$.

(viii) $r = x^3$ and $s = x^i q$ where q is a word with $o(q) \neq x$. If $i = 1$, then $V \subseteq \mathcal{T} \cap \mathcal{R}$ by 3.3(ii). If $i = 2$, then 3.7(i) can be used.

(ix) $r = x^2$ and $s = x^i q$ where q is a word with $o(q) \neq x$. Then $V \subseteq S_2$ and $V = \mathrm{M}(x^3 \approx s) \cap S_2$. The result now follows from (viii). □

5.8 Proposition. *Let r, s be two words and let $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathcal{R} \cap \mathcal{T}$ or $V = R_{i,j}$ or $V = T_{i,j}$ or $V = S_{i,j}$ for some i, j.*

Proof. Apply 3.3, 5.6 and 5.7. □

V.6 THE LATTICE OF SUBVARIETIES OF $\mathcal{R}$

6.1 Lemma. *The following are true:*

(i) $R_{1,j} \cap \mathcal{A} = R_{2,j} \cap \mathcal{A} = \mathcal{A}_1$, $R_{3,j} \cap \mathcal{A} = R_{4,j} \cap \mathcal{A} = \mathcal{A}_4$, $R_{5,j} \cap \mathcal{A} = R_{6,j} \cap \mathcal{A} = \mathcal{A}_5$, $R_{1,j} \cap \mathcal{I} = R_{3,j} \cap \mathcal{I} = R_{5,j} \cap \mathcal{I} = \mathcal{I}_j \cap \mathcal{I}_8$ *and* $R_{2,j} \cap \mathcal{I} = R_{4,j} \cap \mathcal{I} = R_{6,j} \cap \mathcal{I} = \mathcal{I}_j$ *for every* $0 \leq j \leq 9$.

(ii) $R_{2,j} = \mathcal{A}_1 \vee \mathcal{I}_j$, $R_{4,j} = \mathcal{A}_4 \vee \mathcal{I}_j$, $R_{6,j} = \mathcal{A}_5 \vee \mathcal{I}_j$ *for every* $j \in \{0, 2, 3, 6\}$.

(iii) $R_{1,0} = R_{1,3} = \mathcal{A}_1 \vee \mathcal{I}_0$, $R_{1,2} = R_{1,6} = \mathcal{A}_1 \vee \mathcal{I}_2$, $R_{3,0} = R_{3,3} = \mathcal{A}_4 \vee \mathcal{I}_0$, $R_{3,2} = R_{3,6} = \mathcal{A}_4 \vee \mathcal{I}_2$, $R_{5,0} = R_{5,3} = \mathcal{A}_5 \vee \mathcal{I}_0$ *and* $R_{5,2} = R_{5,6} = \mathcal{A}_5 \vee \mathcal{I}_2$.

(iv) $R_{1,j} = R_{2,j}$, $R_{3,j} = R_{4,j}$ *and* $R_{5,j} = R_{6,j}$ *for every* $j \in \{1, 4, 8\}$.

(v) $R_{i,k} = R_{i,j}$ *for* $i \in \{1, 3, 5\}$ *and* $(k, j) \in \{(1, 5), (4, 7), (8, 9)\}$.

Proof. (i) is easy. In order to prove (ii), it is sufficient to show that $R_{6,6} \in \mathcal{T} \cap \mathcal{R}$. Let $S \in R_{6,6}$. We have $x^2y = x^2y^2$ and $efg = feg$ for all elements $x, y \in S$ and all idempotents $e, f, g \in S$. Hence $x^2y^2 = xx^3y^3y^3 = xy^3x^3y^3 = xy^2$.

(iii) follows from (ii). In order to prove (iv), it is sufficient to show that $R_{5,8} = R_{6,8}$. Let $S \in R_{6,8}$. We have $x^2y = x^2y^2$ and $efe = ef$ for all elements $x, y \in S$ and all idempotents $e, f \in S$. Hence $xyx = xy^3x^3 = xy^3x^3y^3 = xy^2$.

In order to prove (v), it is sufficient to show that $R_{5,8} = R_{5,9}$. Let $S \in R_{5,9}$. We have $x^2y = x^2y^2$ and $xy^2 = xyx$ for all elements $x, y \in S$. Then $efe = ef^2 = ef$ for all idempotents $e, f \in S$. □

6.2 Lemma. *Let $1 \leq i, j \leq 6$ and $0 \leq r, s \leq 9$. Then $R_{i,r} \cap R_{j,s} = R_{p,q}$ for some p and q.*

Proof. It is easy. □

6.3 Proposition. *We have the following inclusions between the varieties $R_{i,j}$:*

(i) $R_{i,j} \subseteq R_{p,q}$ *if* $R_i \subseteq R_p$ *and* $\mathcal{I}_j \subseteq \mathcal{I}_q$;

(ii) $R_{i,j} \subseteq R_{p,q}$ *if* $R_{i,j} = R_{p,q}$ *as described in 6.1.*

There are no other inclusions except those that follow by transitivity from these two cases.

Proof. The other inclusions would imply incorrect inclusions between subvarieties of $\mathcal{T} \cap \mathcal{R}$ (intersect both sides with $\mathcal{T}$). □

6.4 Proposition. *The variety $\mathcal{R}$ has the following 62 subvarieties:*

$L_0, \ldots, L_{43}$,
$L_{62} = R_{1,1} = R_{2,1} = R_{1,5}$,
$L_{63} = R_{3,1} = R_{4,1} = R_{3,5}$,
$L_{64} = R_{5,1} = R_{6,1} = R_{5,5}$,
$L_{65} = R_{1,4} = R_{2,4} = R_{1,7}$,
$L_{66} = R_{3,4} = R_{4,4} = R_{2,7}$,
$L_{67} = R_{5,4} = R_{6,4} = R_{5,7}$,

$L_{68} = R_{2,5}$,
$L_{69} = R_{4,5}$,
$L_{70} = R_{6,5}$,
$L_{71} = R_{2,7}$,
$L_{72} = R_{4,7}$,
$L_{73} = R_{6,7}$,
$L_{74} = R_{1,8} = R_{2,8} = R_{1,9}$,
$L_{75} = R_{3,8} = R_{4,8} = R_{3,9}$,
$L_{76} = R_{5,8} = R_{6,8} = R_{5,9}$,
$L_{77} = R_{2,9}$,
$L_{78} = R_{4,9}$,
$L_{79} = R_{6,9} = \mathcal{R}$.

Proof. Let V be a subvariety of $\mathcal{R}$ such that $V \not\subseteq \mathcal{T}$. It follows from 5.8 and 6.2 that $V = R_{i,j}$ for some $1 \le i \le 6$ and $0 \le j \le 9$. According to 6.1, V is one of the varieties $L_{62}, \dots, L_{79}$. Example I.2.5 shows that $L_{62} \not\subseteq \mathcal{T}$. □

V.7 THE LATTICE OF SUBVARIETIES OF $\mathcal{L}$

7.1 Lemma. *The following are true:*

(i) $S_{1,j} \cap \mathcal{A} = S_{2,j} \cap \mathcal{A} = \mathcal{A}_4$, $S_{3,j} \cap \mathcal{A} = S_{4,j} \cap \mathcal{A} = \mathcal{A}_5$, $S_{1,j} \cap \mathcal{I} = S_{3,j} \cap \mathcal{I} = \mathcal{I}_j \cap \mathcal{I}_8$, $S_{2,j} \cap \mathcal{I} = S_{4,j} \cap \mathcal{I} = \mathcal{I}_j$ *for every* $0 \le j \le 9$.
(ii) $S_{1,0} = S_{2,0} = S_{1,3} = \mathcal{A}_4 \vee \mathcal{I}_0$, $S_{3,0} = S_{4,0} = S_{3,3} = \mathcal{A}_5 \vee \mathcal{I}_0$, $S_{2,3} = \mathcal{A}_4 \vee \mathcal{I}_3$ *and* $S_{4,3} = \mathcal{A}_5 \vee \mathcal{I}_3$.
(iii) $S_3 \cap \mathcal{T} = T_{3,8}$.
(iv) $S_{1,2} = S_{2,2} = S_{1,6} = T_{2,2}$, $S_{3,2} = S_{4,2} = S_{3,6} = T_{3,2}$, $S_{2,6} = T_{2,6}$ *and* $S_{4,6} = T_{3,6}$.
(v) $S_{1,1} = S_{2,1} = R_{3,1}$, $S_{3,1} = S_{4,1} = R_{5,1}$, $S_{1,5} = R_{3,1}$, $S_{3,5} = R_{5,1}$, $S_{2,5} = R_{4,5}$ *and* $S_{4,5} = R_{6,5}$.

Proof. It is easy. □

7.2 Lemma. *Let* $0 \le i \le 9$ *and* $\mathcal{I}_j = \mathcal{I}_i \cap \mathcal{I}_8$. *Then* $S_{1,i} = S_{1,j}$ *and* $S_{3,i} = S_{3,j}$.

Proof. It is easy. □

7.3 Lemma. *Let* $i \in \{0, 1, 2, 4, 8\}$. *Then* $S_{1,i} = S_{2,i}$ *and* $S_{3,i} = S_{4,i}$.

Proof. It is easy. □

7.4 Lemma. *Let* $1 \le i, j \le 4$ *and* $0 \le r, s \le 9$. *Then* $S_{i,r} \cap S_{j,s} = S_{p,q}$ *for some* p *and* q.

Proof. It is easy. □

7.5 Proposition. *We have the following inclusions between the varieties* $S_{i,j}$*:*

(i) $S_{i,j} \subseteq S_{p,q}$ *if* $S_i \subseteq S_p$ *and* $\mathcal{I}_j \subseteq \mathcal{I}_q$*;*
(ii) $S_{i,j} \subseteq S_{p,q}$ *if* $S_{i,j} = S_{p,q}$ *according to 7.1, 7.2 or 7.3.*

There are no other inclusions except those that follow by transitivity from these two cases.

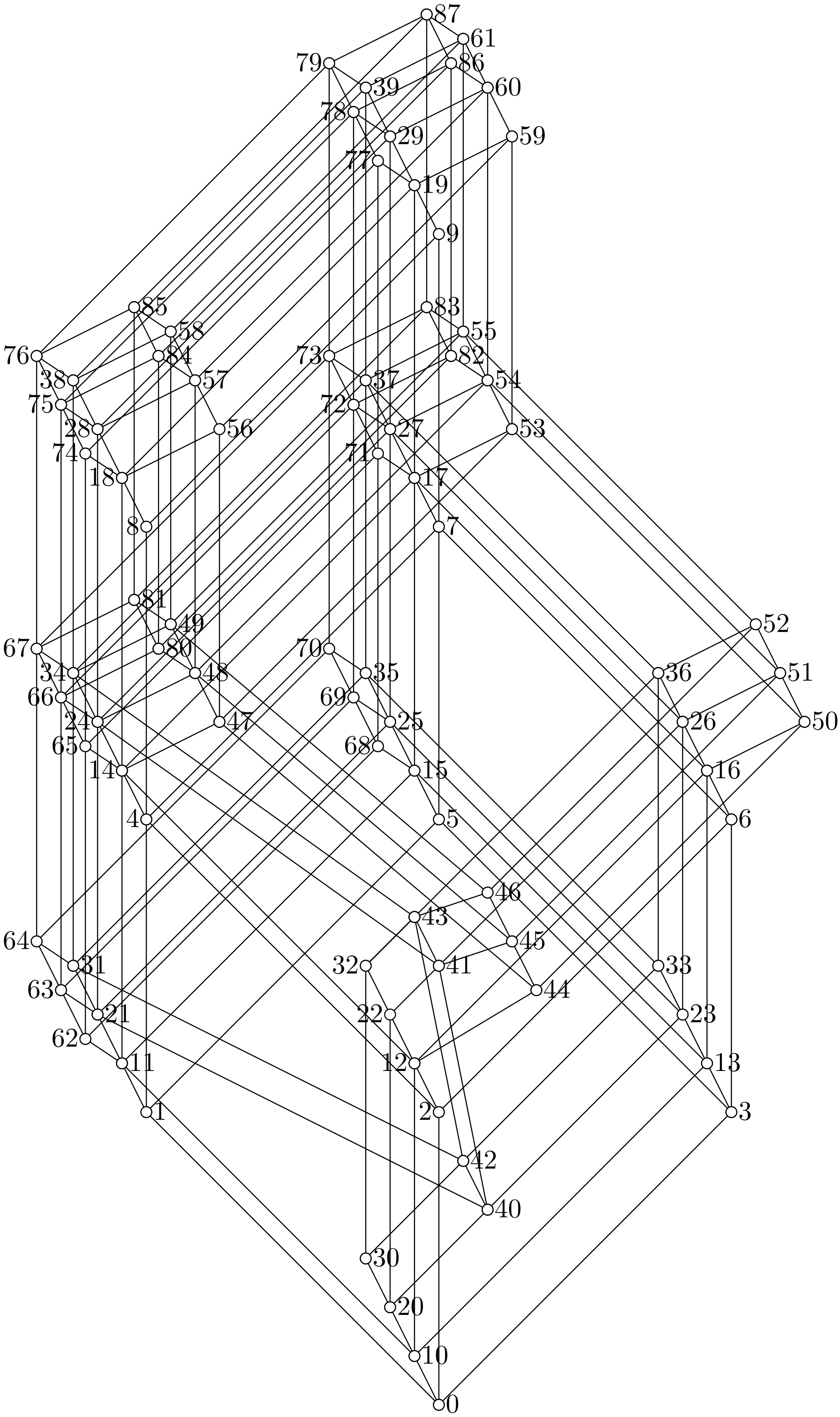
Proof. It is easy. □

7.6 Theorem. *The variety $\mathcal{L}$ has the following 88 subvarieties:*

$L_0, \ldots, L_{79}$,
$L_{80} = S_{1,4}$,
$L_{81} = S_{3,4}$,
$L_{82} = S_{2,7}$,
$L_{83} = S_{4,7}$,
$L_{84} = S_{1,8}$,
$L_{85} = S_{3,8}$,
$L_{86} = S_{2,9}$,
$L_{87} = S_{4,9} = \mathcal{L}$.

Proof. Apply 5.8 and 7.1,...,7.5. □

The lattice of varieties of LD-semigroups is pictured in Fig. 3. An element labeled i in the picture represents the variety L_i ($i = 0, \ldots, 87$).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11
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Fig. 3

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LIST OF SYMBOLS

INDEX