TERMAL GROUPOIDS

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Abstract. We investigate the factor of the groupoid of terms through the largest congruence with a given set among its blocks. The set is supposed to be closed for overterms.

0. Introduction

There are two related constructions of universal algebras from a given set S of terms. Denote by U the set of all terms containing a subterm belonging to S. There exist both the least and the greatest congruence with U among its blocks. The first construction is taking the factor of T through the least such congruence, the second through the largest. The first construction has been investigated, e.g., in the papers [1],[3] and [4]. In the present paper we start to investigate the second construction. In order to simplify notation, we restrict our attention to the algebras with a single binary operation, i.e., groupoids. By a termal groupoid we mean the factor of T through the largest congruence with U among its blocks, for a subset U of T closed for overterms.

Of special interest are the termal groupoids corresponding to a set of terms U such that U is closed for both overterms and substitution instances. In this case there also exists the greatest equational theory with U among its blocks.

We will concentrate on the abstract characterization of termal groupoids, questions of finiteness and the description of the equational theory.

For basics of universal algebra, the reader is referred to [2].

1. General termal groupoids

Let X be a nonempty set of variables. We denote by $\mathbf{T}(X)$ the absolutely free groupoid over X, i.e., the groupoid of terms over X.

For a term t we define two mappings α_t and β_t of $\mathbf{T}(X)$ into itself by $\alpha_t(u) = tu$ and $\beta_t(u) = ut$ (for all $u \in \mathbf{T}(X)$). These mappings are called elementary lifts. By a lift we mean a composition of a finite number of elementary lifts. Every lift L can be uniquely expressed as $L = Q^n_{t_n} \dots Q^1_{t_1}$ for some $n \geq 0$, some terms $t_1, \dots, t_n$ and symbols $Q^i \in \{\alpha, \beta\}$; the terms

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$t_1, \dots, t_n$ will be called its basic terms; the number n will be called the height of the lift. For $n = 0$, L is the identity on $\mathbf{T}(X)$.

A term u is a subterm of a term t if and only if $L(u) = t$ for a lift L. In this case we write $u \subseteq t$; we also say that t is an overterm of u. The relation $u \subseteq t$ is a (partial) ordering on $\mathbf{T}(X)$ satisfying the minimal condition.

Two terms u, v are said to be similar (we write $u \sim v$) if $f(u) = v$ for an automorphism f of $\mathbf{T}(X)$. (We could also say that v results from u by renaming variables.) Two sets U, V of terms are said to be similar (we write $U \sim V$) if they contain the same terms up to similarity.

For a subset U of $\mathbf{T}(X)$ we denote by $\mathbf{M}(U)$ the set of the terms minimal in U, i.e., the set of the terms $u \in U$ such that $v \notin U$ for any proper subterm v of u. Any two terms in $\mathbf{M}(U)$ are either similar or subterm-incomparable. We have $\mathbf{M}(\mathbf{T}(X)) = X$.

For a set U of terms, we denote by $\mathbf{O}(U)$ the set of all overterms of terms from U. By an up-set (of terms) we mean a subset U of $\mathbf{T}(X)$ such that $\mathbf{O}(U) = U$. For an arbitrary subset U, $\mathbf{O}(U)$ is the up-set generated by U. If U is a set of pairwise subterm-incomparable terms, then $\mathbf{M}(\mathbf{O}(U)) = U$.

For every nonempty subset U of $\mathbf{T}(X)$ we define a binary relation $\sqsubseteq_U$ on $\mathbf{T}(X)$ by $u \sqsubseteq_U v$ if and only if $L(u) \in \mathbf{O}(U)$ implies $L(v) \in \mathbf{O}(U)$ for any lift L. It is evident that this relation is a quasiordering and $u \sqsubseteq_U v$ implies $L(u) \sqsubseteq_U L(v)$ for any lift L. We write $u \equiv_U v$ if both $u \sqsubseteq_U v$ and $v \sqsubseteq_U u$. Clearly, the relation $\equiv_U$ is a congruence of $\mathbf{T}(X)$.

For a nonempty subset U of $\mathbf{T}(X)$, any groupoid isomorphic to $\mathbf{T}(X)/\equiv_U$ will be called the termal groupoid of U. A groupoid is said to be termal if it is the termal groupoid of a nonempty subset of $\mathbf{T}(X)$, for a set of variables X. By a finitely determined termal groupoid we mean a groupoid isomorphic to $\mathbf{T}(X)/\equiv_U$, for a finite set X and a finite subset U of $\mathbf{T}(X)$.

Proposition 1.1. *Let U be a nonempty subset of $\mathbf{T}(X)$. Then $\equiv_U$ is just the largest congruence of $\mathbf{T}(X)$ such that $\mathbf{O}(U)$ is one of its blocks. The termal groupoid $\mathbf{T}(X)/\equiv_U$ is finite if and only if there exists a congruence s of $\mathbf{T}(X)$ such that $\mathbf{T}(X)/s$ is finite and $\mathbf{O}(U)$ is a block of s.*

Proof. It is easy. □

Proposition 1.2. *Let U be a nonempty set of pairwise subterm-incomparable terms and let u, v be two terms such that $u \notin \mathbf{O}(U) - U$. Then $u \sqsubseteq_U v$ if and only if $L(u) \in U$ implies $L(v) \in \mathbf{O}(U)$ for any lift L. Consequently, if $u, v \notin \mathbf{O}(U)$, then $u \equiv_U v$ if and only if $L(u) \in U$ is equivalent with $L(v) \in U$ for any lift L.*

Proof. The direct implication is clear. In order to prove the converse, let $L(u) \in \mathbf{O}(U)$; we need to show that $L(v) \in \mathbf{O}(U)$. This is evident in the case that at least one of the basic terms of L belongs to $\mathbf{O}(U)$. In the opposite case we have $L = L_2 L_1$ for two lifts L_1, L_2 such that $L_1(u) \in U$. Then $L_1(v) \in \mathbf{O}(U)$ by the assumption, and hence $L(v) = L_2 L_1(v) \in \mathbf{O}(U)$. □

Corrollary 1.3. *If X is at most countable and if U is finite, then the relations $\sqsubseteq_U$ and $\equiv_U$ are both recursive.*

Corrollary 1.4. *Let U be a nonempty set of pairwise subterm-incomparable terms. Denote by S the set of the terms that are subterm-incomparable with any term from U. If S is nonempty, then it is a block of $\equiv_U$ and $s \sqsubseteq_U t$ for any $s \in S$ and any term t.*

In the next proposition we describe an effective construction of termal groupoids.

Proposition 1.5. *Let U be a nonempty set of pairwise subterm-incomparable terms. Denote by P the set of proper subterms of terms from U. Define an equivalence r on P by $(u, v) \in r$ if and only if $L(u) \in U$ is equivalent with $L(v) \in U$ for any lift L. If every term is subterm-comparable with a term from U, put $G = (P/r) \cup \{0\}$; in the opposite case put $G = (P/r) \cup \{0, 1\}$. Define multiplication on G as follows: $0a = a0 = 0$ for all a; $1b = b1 = 1$ for all $b \neq 0$; if $u, v \in P$ and $uv \in P$, put $(u/r)(v/r) = (uv)/r$; if $uv \in U$, put $(u/r)(v/r) = 0$; in all other cases put $(u/r)(v/r) = 1$. Then G is the termal groupoid of U.*

Proof. It follows from 1.2 and 1.4. □

Corrollary 1.6. *If U is finite, then the termal groupoid of U is finite.*

A groupoid G is said to be 0-simple if it contains a zero element 0 and for every nontrivial congruence s of G there exists an element $a \in G - \{0\}$ such that $(a, 0) \in s$. For example, every simple groupoid with zero is 0-simple.

Proposition 1.7. *Let G be a 0-simple groupoid and let X be a generating subset of G. Denote by f the homomorphism of $\mathbf{T}(X)$ onto G extending the identity and put $U = f^{-1}(0)$. Then G is the termal groupoid of U.*

Proof. The kernel of f is a congruence of $\mathbf{T}(X)$ and the up-set U is one of its blocks. It follows by 1.1 that the kernel of f is contained in $\equiv_U$. But then, there exists a congruence s of G such that for any terms u and v, $u \equiv_U v$ if and only if $(f(u), f(v)) \in s$. Since $\{0\}$ is a block of s, it follows from the 0-simplicity of G that s is the identity, so that $\equiv_U$ is the kernel of f. Consequently, G is isomorphic to $\mathbf{T}(X)/\equiv_U$. □

Theorem 1.8. *A groupoid is a termal groupoid if and only if it is 0-simple. A groupoid is a termal groupoid with respect to a finite set of variables X (and an arbitrary subset U of $\mathbf{T}(X)$) if and only if it is 0-simple and finitely generated.*

Proof. It follows from 1.1, 1.7 and the definitions. □

By a divisibility cycle in a groupoid G we mean a finite sequence $a_1, \dots, a_n$ ($n \geq 1$) of elements of G such that $a_1 \rho a_2 \rho \dots \rho a_n \rho a_1$, where ρ is the divisibility relation defined as follows: $a \rho b$ if and only if either $b = ac$ or $b = ca$ for an element $c \in G$.

An element z of a groupoid G is said to be a zero among nonzeros if it is a nonzero element and $za = az = z$ for all nonzero elements a of G. Of course, a groupoid can have at most one zero among nonzeros.

Theorem 1.9. *The following are equivalent for a groupoid G:*

(1) *G is a finitely determined termal groupoid;*
(2) *G is a finite 0-simple groupoid in which all members of any divisibility cycle are equal to either 0 or a zero among nonzeros;*
(3) *G is a finite 0-simple groupoid and whenever G is isomorphic to the factor $\mathbf{T}(X)/\equiv_U$ for a finite set X and a set U of pairwise subterm-incomparable terms, then U is finite.*

Proof. (1) implies (2): G is finite by 1.6 and 0-simple by 1.8. Let X be finite and let U be a finite set of pairwise subterm-incomparable terms from $\mathbf{T}(X)$; let f be a homomorphism of $\mathbf{T}(X)$ onto G such that $\equiv_U$ is the kernel of f. Suppose that there exists a divisibility cycle $a_1, \dots, a_n$ in G containing neither 0 nor a zero among nonzeros. There is a term u with $a_1 = f(u)$. The existence of the cycle means that there is a lift L of height n such that $f(L(u)) = f(u) = a_1$. Then $f(L^i(u)) = a_1$ for all $i \geq 0$. Since a_1 is neither 0 nor a zero among nonzeros, each of the terms $L^i(u)$ is a proper subterm of a term from U. It follows that there are arbitrarily long terms in U, and U is infinite.

(2) implies (3): Let X be finite, U be a set of pairwise subterm-incomparable terms and f be a homomorphism of $\mathbf{T}(X)$ onto G with kernel $\equiv_U$. Suppose that U is infinite. Since X is finite, there exists a term $u \in U$ such that u can be expressed as $u = L(x)$ for a variable x and a lift L of height larger than the cardinality of G. We have $L = E_n \dots E_1$ for elementary lifts $E_1, \dots, E_n$. For every i put $L_i = E_i \dots E_1$. Since n is larger than the cardinality of G, there exist two numbers $0 \leq i < j \leq n$ with $f(L_i(u)) = f(L_j(u))$. But then $f(L_i(u)), \dots, f(L_{j-1}(u))$ is a divisibility cycle in G and $f(L_i(u))$ is neither 0 nor a zero among nonzeros.

(3) implies (1): This follows from 1.7. □

Proposition 1.10. *Let U be a set of pairwise subterm-incomparable terms containing at least one term not belonging to X. The termal groupoid of U is simple if and only if every term is either a subterm or an overterm of a term from U and for every pair u, v of proper subterms of terms from U there exists a lift L with $L(u) \equiv_U v$.*

Proof. Let G be the termal groupoid constructed as in 1.5; let G be simple. If some term is neither a subterm nor an overterm of a term from U, then the element 1 belongs to G and $\{0, 1\} \cup \mathrm{id}_G$ is a congruence, so that $G = \{0, 1\}$; this gives a contradiction. Let u be a proper subterm of a term from U. Define a binary relation s on $\mathbf{T}(X)$ by $(p, q) \in s$ if and only if either $p \equiv_U q$ or else $p \equiv_U L_1(u)$ and $q \equiv_U L_2(u)$ for two lifts L_1 and L_2. It is easy to verify that s is a congruence containing $\equiv_U$. Moreover, s properly contains $\equiv_U$, and thus s is the largest congruence of $\mathbf{T}(X)$. This means that for every pair u, v of proper subterms of terms from U there exists a lift L with $L(u) \equiv_U v$.

Now let the termal groupoid G be not simple. There is a nontrivial congruence s of $\mathbf{T}(X)$ properly containing $\equiv_U$. Since G is 0-simple, there is a term $u \notin \mathbf{O}(U)$ such that $(u, w) \in s$ for all terms $w \in U$. If every term is either a subterm or an overterm of a term from U, then u is a proper subterm of a term from U; there exists a term v with $(u, v) \notin s$, and v is also a proper subterm of a term from U. But then, it is easy to see that $L(u) \not\equiv_U v$ for any lift L. □

Proposition 1.11. *Every simple groupoid with* 0 *is termal. However, if a simple groupoid with* 0 *contains at least three elements, then it is never finitely determined.*

Proof. It follows from 1.7, 1.9 and 1.10. □

Example 1.12. Let $X = \{x\}$. For every $i \geq 2$ put $t_i = x(x \dots (xx))$, where x occurs i times. Put $U = \{t_i x : i \geq 2\}$. Then U is an infinite set of pairwise subterm-incomparable terms. The termal groupoid $\mathbf{T}(X)/\equiv_U$ is finite: it has four elements $0, x, a, b$ and multiplication table

	0	x	a	b
0	0	0	0	0
x	0	a	a	b
a	0	0	b	b
b	0	b	b	b

2. Fully termal groupoids

From now on let X be an infinitely countable set of variables.

For two terms u, v we write $u \leq v$ if there exists a substitution f (i.e., an endomorphism of $\mathbf{T}(X)$) such that $f(u)$ is a subterm of v. The relation $\leq$ is a quasiordering on $\mathbf{T}(X)$ satisfying the minimal condition. Two terms u, v are similar if and only if both $u \leq v$ and $v \leq u$. We write $u < v$ if $u \leq v$ but $v \nleq u$.

By a full set of terms we mean a nonempty subset U of $\mathbf{T}(X)$ such that $u \in U$ and $u \leq t$ imply $t \in U$. For a nonempty subset J of $\mathbf{T}(X)$ we denote by $\mathbf{F}(J)$ the set of the terms t such that $u \leq t$ for some $u \in J$; this is the full set generated by J. A nonempty subset J of $\mathbf{T}(X)$ is said to be irreducible if it consists of pairwise incomparable terms (incomparable with respect to the quasiordering $\leq$). Every full set is generated by an irreducible subset J, and J is determined uniquely up to the similarity of terms by U.

By a fully termal groupoid we mean a termal groupoid determined by a full set of terms over the infinite set of variables X, i.e., a groupoid isomorphic to $\mathbf{T}(X)/\equiv_U$ for and a full subset U of $\mathbf{T}(X)$.

It is easy to see that for a full set of terms U, the congruence $\equiv_U$ of $\mathbf{T}(X)$ is invariant, i.e., $u \equiv_U v$ implies $f(u) \equiv_U f(v)$ for any automorphism f of $\mathbf{T}(X)$. (However, the congruence is not necessarily fully invariant.)

A term t is said to be linear if no variable has more than one occurrence in t.

Lemma 2.1. *Let J be an irreducible subset of $\mathbf{T}(X)$; put $U = \mathbf{F}(J)$. The following are equivalent:*

(1) *$x \equiv_U y$ for all variables x, y;*
(2) *$x \equiv_U y$ for at least one pair x, y of distinct variables x, y;*
(3) *J contains only linear terms.*

Proof. The equivalence of the first two conditions is obvious.

(1) implies (3): Suppose that J contains a non-linear term u. There is a variable x with at least two occurrences in u. So, we have $u = L(x)$ for a lift L such that x is contained in at least one of the basic terms of L. Since X is infinite, there exists a variable y with no occurrence in u. Put $v = L(y)$. By (1) we have $x \equiv_U y$ and hence $u \equiv_U v$; since $u \in U$, we get $v \in U$. Since $v < u$, we get a contradiction with $u \in J$.

(3) implies (1): Let $L(x) \in U$ for a lift L. Then $L(x) \geq u$ for a term $u \in J$. Since u is linear, it is easy to see that also $L(y) \geq u$, so that $L(y) \in U$. This proves $x \sqsubseteq_U y$ for all variables x, y. Consequently, $x \equiv_U y$ for all variables x, y. □

Lemma 2.2. *Let S be a set of linear terms closed under taking subterms. Define a binary relation s on $\mathbf{T}(X)$ by $(u, v) \in s$ if and only if the following two conditions are satisfied for all $w \in S$:*

(1) *$w \leq u$ if and only if $w \leq v$;*
(2) *$f(w) = u$ for a substitution f if and only if $g(w) = v$ for a substitution g.*

Then s is a congruence of $\mathbf{T}(X)$.

Proof. It is evident that s is an equivalence relation. Let $(u, v) \in s$ and let t be a term. By symmetry, it is sufficient to prove $(ut, vt) \in s$.

Let $f(w) = ut$ for a substitution f and a term $w \in S$. If w is a variable, then it is evident that $g(w) = vt$ for a substitution g. So, let $w = w_1 w_2$. We have $f(w_1) = u$ and $f(w_2) = t$. Since $(u, v) \in s$ and $w_1 \in S$, we have $g(w_1) = v$ for a substitution g; since no variable occurring in w_2 occurs in w_1, it is possible to choose g in such a way that it coincides with f on all the variables occurring in w_2. But then $g(w_2) = t$ and $g(w) = vt$.

Let $w \leq ut$ for a term $w \in S$. If $w \leq u$, then $w \leq v$ by $(u, v) \in s$, and hence $w \leq vt$. If $w \leq t$, then $w \leq vt$ is obvious. It remains to consider the case when $f(w) = ut$ for a substitution f. But in this case we already know that $g(w) = vt$ for a substitution g, so that $w \leq vt$ again. □

Lemma 2.3. *Let J be an irreducible subset of $\mathbf{T}(X)$ such that the groupoid $\mathbf{T}(X)/\equiv_U$ with $U = \mathbf{F}(J)$ is finite. Then J contains only linear terms.*

Proof. It follows from 2.1. □

Theorem 2.4. *Let J be a finite irreducible subset of $\mathbf{T}(X)$. The fully termal groupoid $\mathbf{T}(X)/\equiv_U$ is finite if and only if J contains only linear terms.*

Proof. The direct implication follows from 2.3. Let J be a finite set of pairwise incomparable linear terms. Denote by S the set of subterms of terms from J. Clearly, S is a finite set of linear terms and S is closed under taking subterms. Define the congruence s of $\mathbf{T}(X)$ in the same way as in 2.2. Clearly, $\mathbf{T}(X)/s$ is a finite groupoid (it has at most n^2 elements, where n is the cardinality of S). It is evident that $(u,v) \in s$ and $u \in U$ imply $v \in U$. Define a binary relation r on $\mathbf{T}(X)$ by $(u,v) \in r$ if and only if either $(u,v) \in s$ or $\{u,v\} \subseteq U$. Then r is a congruence of $\mathbf{T}(X)$ and U is a block of r. Since $s \subseteq r$, the factor $\mathbf{T}(X)/r$ is finite. From this it follows by 1.1 that $\mathbf{T}(X)/\equiv_U$ is finite. □

3. The equational theory of a fully termal groupoid

Again, let X be a countably infinite set of variables.

For a full subset U of $\mathbf{T}(X)$, we denote by $\mathbf{E}(U)$ the equational theory of the fully termal groupoid $\mathbf{T}(X)/\equiv_U$, i.e., the set of all equations that are satisfied in the groupoid.

Lemma 3.1. *Let r be a congruence of $\mathbf{T}(X)$. An equation (u,v) is satisfied in $\mathbf{T}(X)/r$ if and only if $(f(u),f(v)) \in r$ for any substitution f.*

Proof. It is easy. □

Proposition 3.2. *Let U be a full set of terms. The following are equivalent for an equation (u,v):*

(1) *$(u,v) \in \mathbf{E}(U)$;*
(2) *$f(u) \equiv_U f(v)$ for any substitution f;*
(3) *for any lift L and any substitution f, $L(f(u)) \in U$ if and only if $L(f(v)) \in U$;*
(4) *every immediate consequence of (u,v) belongs to $(U \times U) \cup (-U \times -U)$;*
(5) *every consequence of (u,v) belongs to $(U \times U) \cup (-U \times -U)$.*

Proof. The equivalence of (1) with (2) follows from 3.1. Condition (3) is just a reformulation of (2). Immediate consequences of an equation (u,v) are just the equations $(L(f(u)), L(f(v)))$ for a lift L and a substitution f, so (4) is a reformulation of (3). The equivalence with (5) is obvious. □

Proposition 3.3. *Let U be a full set of terms. Then U is a block of $\mathbf{E}(U)$, and $\mathbf{E}(U)$ is just the largest equational theory E such that U is a block of E.*

Proof. It follows from 3.2. □

Proposition 3.4. *Let U be a full set of terms and let $(u,v) \in \mathbf{E}(U)$ be an equation such that $u,v \notin U$. Then the terms u,v contain the same variables.*

Proof. Suppose that there exists a variable x contained in u but not contained in v. Take an arbitrary term $t \in U$ and denote by f the substitution such that $f(x) = t$ and $f(y) = y$ for all $y \in X - \{x\}$. Then $(f(u), f(v)) \in \mathbf{E}(U)$. But $f(u) \in U$ and $f(v) = v \notin U$, a contradiction. □

Proposition 3.5. *Let U be a full set of terms containing a linear term not belonging to X. Let $(x, u) \in \mathbf{E}(U)$ for a variable x and a term u. Then $u = x$.*

Proof. It is sufficient to derive a contradiction in the case that u is a composed term, $u = u_1u_2$. Take a linear term $t \in U$ of minimal length; we can suppose that t does not contain x. We can express t as $t = L(y_1y_2)$ for a lift L and a pair of distinct variables y_1, y_2. Since $(x, u) \in \mathbf{E}(U)$, we have $(L(x), L(u)) \in \mathbf{E}(U)$. But $L(x) \notin U$ (since $L(x)$ is a linear term shorter than t), while $L(u) \in U$ (since $L(u) \geq t$), a contradiction. □

For every $n \geq 0$ define a set P_n of similar linear terms of length 2^n as follows: $P_0 = X$; $P_{n+1} = \{uv : u, v \in P_n \text{ and } uv \text{ is linear}\}$.

Proposition 3.6. *Let U be the full set generated by a single linear term t. Then $(xy, yx) \in \mathbf{E}(U)$ if and only if $t \in P_n$ for some n.*

Proof. Let $(xy, yx) \in \mathbf{E}(U)$. If uv is an arbitrary composed subterm of t, then $t = L(uv)$ for a lift L; we have $(t, L(vu)) \in \mathbf{E}(U)$ and thus $L(vu) \geq t$, from which we easily obtain $u \sim v$. Since $u \sim v$ for any composed subterm uv of t, it follows that $t \in P_n$ for some n. The converse is obvious. □

Proposition 3.7. *Let J be a nonempty set of linear terms and U be the full set generated by J. Let G be a 0-simple groupoid and let a be a generator of G. Denote by h the homomorphism of $\mathbf{T}(X)$ onto G such that $h(x) = a$ for all variables x and suppose that $h^{-1}(\{0\}) = U$. Then $\equiv_U$ is the kernel of h and G is the termal groupoid of U.*

Proof. It is easy. □

Example 3.8. Let

$$X = \{x, y, z, u, x_1, x_2, \dots\}$$

and

$$J = \{u(((((x(yz))x_1)x_2)\dots)x_k) : k \geq 0\}.$$

Applying 3.7, one can easily verify that the groupoid $\mathbf{T}(X)/\equiv_U$ with $U = \mathbf{F}(J)$ has four elements (denote them $0, a, b, c$) and the following multiplication table:

	0	a	b	c
0	0	0	0	0
a	0	b	c	0
b	0	b	c	0
c	0	c	c	0

Notice that, according to 1.9, this termal groupoid is not finitely determined.

We are going to describe the equational theory $\mathbf{E}(U)$ in the special case when U is the full set generated by a single linear term. To this purpose we need some notation.

For two terms u and v, write $u \to v$ if there is a substitution f with $f(u) = v$.

For a subterm u of t, occurrences of u in t can be identified with the lifts L such that $L(u) = t$. Let L be a lift, expressed as the composition of elementary lifts, $L = Q^n_{u_n} \dots Q^1_{u_1}$ ($Q^i \in \{\alpha, \beta\}$ for all i). A lift K is said to be a regular part of L if there is a $k \in \{0, \dots, n\}$ such that $K = Q^n_{u_n} \dots Q^{n-k+1}_{u_{n-k+1}}$, $Q^{n-k+1} = Q^1, \dots, Q^n = Q^k$ and $u_{n-k+1} \to u_1, \dots, u_n \to u_k$.

For a linear term t, define a binary relation R_t on $\mathbf{T}(X)$ by $(u, v) \in R_t$ if and only if whenever $L(p) = t$ for a lift L and a subterm p of t with $p \to u$, then $K(q) = t$ for a regular part K of L and a subterm q of t with $q \to v$.

Theorem 3.9. *Let t be a linear term and U be the full set $\{w : w \geq t\}$ generated by t. An equation (u, v) belongs to $\mathbf{E}(U)$ if and only if either $u, v \geq t$ or else $u, v \ngeq t$ and $(u, v) \in R_t \cap R_t^{-1}$.*

Proof. Let $(u, v) \in \mathbf{E}(U)$ and $u, v \ngeq t$. We are going to prove that $(u, v) \in R_t$. Let $L(p) = t$ for a lift $L = Q^n_{u_n} \dots Q^1_{u_1}$ and a subterm p of t with $p \to u$. Evidently, $t \leq L(u)$ and hence $t \leq L(v)$. There is a substitution f such that $f(t)$ is a subterm of $L(v)$. Since none of the terms $v, u_1, \dots, u_n$ belongs to U, we get $f(t) = Q^k_{u_k} \dots Q^1_{u_1}(v)$ for some $k \in \{1, \dots, n\}$.

Let us prove by induction on $i = 0, \dots, k$ that

$$Q^{n-i+1} = Q^{k-i+1}, \dots, Q^n = Q^k$$

and

$$f(Q^{n-i}_{u_{n-i}} \dots Q^1_{u_1}(p)) = Q^{k-i}_{u_{k-i}} \dots Q^1_{u_1}(v).$$

This is clear for $i = 0$. Suppose that the assertion is true for a number $i < k$, and let us prove it for the number $i + 1$. If $Q^{n-i} \neq Q^{k-i}$, then $f(Q^{n-i-1}_{u_{n-i-1}}(p)) = u_{k-i}$; this is possible only if $k = n$, but then we immediately obtain a contradiction. Hence $Q^{n-i} = Q^{k-i}$. But then,

$$f(Q^{n-i-1}_{u_{n-i-1}} \dots Q^1_{u_1}(p)) = Q^{k-i-1}_{u_{k-i-1}} \dots Q^1_{u_1}(v)$$

and we are through.

For $i = k$ we obtain

$$Q^{n-k+1} = Q^1, \dots, Q^n = Q^k \text{ and } f(Q^{n-k}_{u_{n-k}} \dots Q^1_{u_1}(p)) = v.$$

Put $q = Q^{n-k}_{u_{n-k}} \dots Q^1_{u_1}(p)$ and $K = Q^n_{u_n} \dots Q^{n-k+1}_{u_{n-k+1}}$, so that $f(q) = v$ and $K(q) = t$. Clearly, $f(u_{n-k+1}) = u_1, \dots, f(u_n) = u_k$. Hence K is a regular part of L.

Conversely, let $(u, v) \in R_t \cap R_t^{-1}$ and $u, v \ngeq t$. In order to prove $(u, v) \in \mathbf{E}(U)$, it is sufficient to show that if $f(t) = L(u)$ for a substitution f and a lift L, then $L(v) \geq t$. Let $L = Q^n_{u_n} \dots Q^1_{u_1}$. We have $n \geq 1$. Suppose $L(v) \ngeq t$.

Let us prove by induction on $i = 0, \dots, n-1$ that there is a subterm p_i of t with $t = Q^n_{v_n} \dots Q^{n-i}_{v_{n-i}}(p_i)$ for some terms $v_{n-i}, \dots, v_n$. This is clear for $i = 0$. Let the assertion be true for a number $i < n-1$. In order to prove that it is true for the number $i+1$, it is sufficient to show that p_i is not a variable. So, suppose that p_i is a variable. We have $f(p_i) = Q^{n-i-1}_{u_{n-i-1}} \dots Q^1_{u_1}(u)$. Define a substitution g by $g(p_i) = Q^{n-i-1}_{u_{n-i-1}} \dots Q^1_{u_1}(v)$ and $g(y) = f(y)$ for all variables $y \neq p_i$. Then evidently $g(t) = L(v)$, so that $L(v) \geq t$, a contradiction.

In particular, there is a subterm $p = p_{n-1}$ of t with $t = Q^n_{v_n} \dots Q^1_{v_1}(p)$ for some terms $v_1, \dots, v_n$. We have $f(p) = u$ and $f(v_1) = u_1, \dots, f(v_n) = u_n$. Since $(u, v) \in R_t$, there exist a regular part $K = Q^n_{v_n} \dots Q^{n-k+1}_{v_{n-k+1}}$ of $Q^n_{v_n} \dots Q^1_{u_1}$, a subterm q of t with $K(q) = t$ and a substitution g with $g(q) = v$. The term

$$L(v) = Q^n_{u_n} \dots Q^1_{u_1}(g(Q^{n-k}_{v_{n-k}} \dots Q^1_{v_1}(p)))$$

contains the subterm

$$Q^k_{u_k} \dots Q^1_{u_1}(g(Q^{n-k}_{v_{n-k}} \dots Q^1_{v_1}(p))) = Q^n_{u_k} \dots Q^{n-k+1}_{u_1}(g(Q^{n-k}_{v_{n-k}} \dots Q^1_{v_1}(p))).$$

Since $v_{n-k+1} \to v_1 \to u_1, \dots, v_n \to v_k \to u_k$, the subterm is of the form $h(t)$ for a substitution h. We get $L(v) \geq t$. $\square$

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