CONSTRUCTIONS OVER TOURNAMENTS

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Abstract. We investigate tournaments that are projective in the variety that they generate, and free algebras over partial tournaments in that variety. We prove that the variety determined by three-variable equations of tournaments is not locally finite. We also construct infinitely many finite, pairwise incomparable simple tournaments.

1. Introduction

Let us denote by $\mathbf{T}$ the class of tournaments, i.e., directed graphs $(T, \to)$ such that for every pair a, b of distinct elements of T, precisely one of the two cases, either $a \to b$ or $b \to a$, takes place; and $a \to a$ for all $a \in T$. There is a natural one-to-one correspondence between tournaments and commutative groupoids satisfying $ab \in \{a, b\}$ for all a and b: set $ab = a$ if and only if $a \to b$. This makes it possible to identify tournaments with their corresponding groupoids and then to investigate tournaments by using algebraic methods (see [9]). In particular, we can investigate the variety generated by $\mathbf{T}$. We denote this variety by $\mathcal{T}$. In [4] we prove that the variety $\mathcal{T}$ is not finitely based. In [5] we prove some results in support of the following conjecture, which can be stated in two equivalent forms:

Conjecture.

(1) Every subdirectly irreducible algebra in $\mathcal{T}$ is a tournament.
(2) $\mathcal{T}$ is the quasivariety generated by tournaments.

In the present paper we investigate a construction of the $\mathcal{T}$-free algebra over a partial tournament, in hope that this also may be helpful in solving the problem. We also investigate tournaments that are projective in $\mathcal{T}$, and prove that the variety determined by three-variable equations of tournaments is not locally finite. In the last section we give a positive solution to a problem of E. Fried [3]. For basics of universal algebra, the reader is referred to [8].

For any $n \geq 1$, let $\mathcal{T}_n$ denote the variety generated by all n-element tournaments, and let $\mathcal{T}^n$ denote the variety determined by the at most n-variable equations of tournaments. So, $\mathcal{T}_n \subseteq \mathcal{T}_{n+1} \subseteq \mathcal{T} \subseteq \mathcal{T}^{n+1} \subseteq \mathcal{T}^n$

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for all n. For a variety V and a positive integer n, we denote by $\mathbf{F}_n(V)$ the free algebra in V on n generators. According to Theorem 3 of [5], $\mathbf{F}_n(\mathcal{T}) = \mathbf{F}_n(\mathcal{T}_n) = \mathbf{F}_n(\mathcal{T}^n)$ for all n, and the following four equations are a base for the equational theory of $\mathcal{T}^3$:

(e1) $xx = x$,
(e2) $xy = yx$,
(e3) $xy \cdot x = xy$,
(e4) $(xy \cdot xz)(xy \cdot yz) = xyz$

According to Lemma 5 of [5], for any three elements a, b, c of an algebra $A \in \mathcal{T}^3$ we have:

(p1) If $ab \to c$, then a, b, c generate a semilattice.
(p2) If $ab \to c \to a$, then $bc = ab$.
(p3) If $a \to c \to ab$, then $c \to b$.
(p4) If $a \to c$ and $b \to c$, then $ab \to c$.
(p5) If $a \to c \to b$ and a, b, c, ab are four distinct elements, then the subgroupoid generated by a, b, c either contains just these four elements and $c \to ab$, or else it contains precisely five elements $a, b, c, ab, ab \cdot c$ and $a \to ab \cdot c \to b$.

Our proof in [4] relied on an infinite sequence $\mathbf{M}_n$ ($n \geq 3$) of algebras with the following properties: $\mathbf{M}_n$ is subdirectly irreducible, $|\mathbf{M}_n| = n + 2$ and $\mathbf{M}_n \in \mathcal{T}^n - \mathcal{T}^{n+1}$. These algebras are defined as follows. $\mathbf{M}_n = \{a, c, c, d_1, \dots, d_{n-2}, e\}$;

$$ab = e,$$
$$e \to a \to c,$$
$$e \to b \to c,$$
$$e \to c,$$
$$a \to d_1 \to d_2 \to \dots \to d_{n-2} \to b,$$
$$d_i \to c \text{ for } i < n - 2,$$
$$c \to d_{n-2},$$
$$d_i \to e \text{ for all } i,$$
$$d_i \to a \text{ for } i > 1,$$
$$d_i \to b \text{ for all } i,$$
$$d_j \to d_i \text{ for } j > i + 1.$$

We have also introduced in [5] a five-element, subdirectly irreducible algebra $\mathbf{J}_3 \in \mathcal{T}^3 - \mathcal{T}^4$. This algebra is defined on $\{a, b, c, d, e\}$ by $a \to d \to b \to c \to a$, $c \to e$, $d \to c$, $d \to e$ and $ab = e$. The following is an even stronger formulation of the Conjecture: Is it true that every subdirectly irreducible algebra in $\mathcal{T} - \mathbf{T}$ contains a subalgebra isomorphic to either $\mathbf{J}_3$ or $\mathbf{M}_n$ for some $n \geq 3$?

2. Projective tournaments

Let V be a variety. An algebra $A \in V$ is said to be projective in V if for every $B, C \in V$, every homomorphism f of B onto C and every homomorphism h of A into C there exists a homomorphism g of A into B with $h = fg$.

The following are equivalent for an algebra $A \in V$:

(1) A is projective in V;
(2) A is a retract of a free algebra in V, i.e., there are an algebra F free in V, a homomorphism f of F onto A and a homomorphism g of A into F such that $fg = \mathrm{id}_A$;
(3) for any $B \in V$ and any homomorphism f of B onto A there is a homomorphism g of A into B with $fg = \mathrm{id}_A$.

The (easy) proof given in Theorem 5.1 of [1] for the variety of lattices can be extended to the case of an arbitrary variety without any difficulty.

2.1. **Theorem.** *A tournament A is projective in $\mathcal{T}$ if and only if for every $B \in \mathrm{SP}(\mathbf{T})$ and every homomorphism f of B onto A there is a homomorphism g of A into B with $fg = \mathrm{id}_A$.*

Proof. Let $C \in \mathcal{T} = \mathrm{HSP}(\mathbf{T})$, so that there is a homomorphism h of an algebra $B \in \mathrm{SP}(\mathbf{T})$ onto C. Let f be a homomorphism of C onto A. Then fh is a homomorphism of B onto A and hence there exists a homomorphism g_0 of A into B with $fhg_0 = \mathrm{id}_A$. Put $g = hg_0$. Then g is a homomorphism of A into C and $fg = fhg_0 = \mathrm{id}_A$. □

We denote by $\mathbf{C}_3$ and $\mathbf{C}_4$ the tournaments pictured in Fig. 1. Observe that $\mathbf{C}_4$ is, up to isomorphism, the only four-element tournament containing a four-cycle.

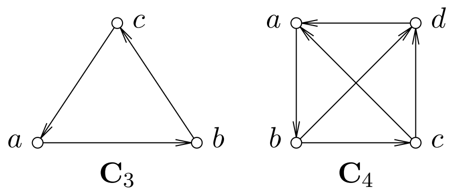

Fig. 1

2.2. **Theorem.** *$\mathbf{C}_3$ is projective in $\mathcal{T}^3$.*

Proof. Let f be a homomorphism of an algebra $B \in \mathcal{T}^3$ onto $\mathbf{C}_3$. Clearly, there is an element $c_0 \in B$ with $f(c_0) = c$, there is an element $b_0 \in B$ with $f(b_0) = b$ and $b_0 \to c_0$, and there is an element $a_0 \in B$ with $f(a_0) = a$ and

$a_0 \to b_0$. If $c_0 \to a_0$, then we can define g by $g(a) = a_0$, $g(b) = b_0$, $g(c) = c_0$ and we are through. Otherwise, the element $c_1 = a_0 c_0$ does not belong to $\{a_0, b_0, c_0\}$. If $b_0 \to c_1$, then we can put $g(a) = a_0$, $g(b) = b_0$, $g(c) = c_1$. According to (p5) applied to $a_0 \to b_0 \to c_0$, the only remaining possibility is that the element $b_1 = b_0 c_1$ does not belong to $\{a_0, b_0, c_0, c_1\}$ and $a_0 \to b_1$. But then we can put $g(a) = a_0$, $g(b) = b_1$, $g(c) = c_1$. □

2.3. **Lemma.** *Let $A, B \in \mathcal{T}^3$, let f be a homomorphism of B onto A and let $a, c, d \in A$ be three distinct elements such that $a \to c \to d$ and $a \to d$. Then for every $c_0, d_0 \in B$ with $f(c_0) = c$, $f(d_0) = d$, $c_0 \to d_0$ there is an element $a_0 \in B$ with $f(a_0) = a$, $a_0 \to c_0$, $a_0 \to d_0$.*

Proof. Of course, there exists an element $a_1 \in B$ with $f(a_1) = a$ and $a_1 \to c_0$. If $a_1 \to d_0$, then we can put $a_0 = a_1$. Otherwise, a_1 is incomparable with d_0. Put $a_2 = a_1 d_0$ and $a_0 = a_2 c_0$. Clearly, $a_0 \to c_0$. Since $a_1 \to d_0$ and $c_0 \to d_0$, we have $a_0 \to d_0$ by (p4). □

2.4. **Lemma.** *Let $A, B \in \mathcal{T}^3$, let f be a homomorphism of B onto A and let $a, b, c, d \in A$ be four distinct elements such that $a \to c \to d \to b \to c$, $a \to d$. Then there are elements $a_0, b_0, c_0, d_0 \in B$ with $f(a_0) = a$, $f(b_0) = b$, $f(c_0) = c$, $f(d_0) = d$, $a_0 \to c_0 \to d_0 \to b_0 \to c_0$, $a_0 \to d_0$.*

Proof. First use 2.2 to obtain b_0, c_0, d_0 and then use 2.3 to obtain a_0. □

2.5. **Theorem.** $\mathbf{C}_4$ *is projective in $\mathcal{T}^3$.*

Proof. (See Fig. 2.) Let f be a homomorphism of an algebra $B \in \mathcal{T}^3$ onto $\mathbf{C}_4$. By 2.4 there are elements $a_0, b_0, c_0, d_0 \in B$ with $f(a_0) = a$, $f(b_0) = b$, $f(c_0) = c$, $f(d_0) = d$, $a_0 \to b_0 \to d_0 \to a_0$, $c_0 \to a_0$, $c_0 \to d_0$. If $b_0 \to c_0$, we are through. Consider the opposite case. Put $b_1 = b_0 c_0$, so that b_1 is a new element (different from a_0, b_0, c_0, d_0). By (p4), $b_1 \to d_0$. If $a_0 \to b_1$, we are through. Otherwise, $a_1 = a_0 b_1$ is a new element. By (p5) applied to $c_0 \to a_0 \to b_0$, $c_0 \to a_1 \to b_0$. If $d_0 \to a_1$, we are through. Otherwise, $d_1 = a_1 d_0$ is a new element. By (p5) applied to $b_1 \to d_0 \to a_0$, $b_1 \to d_1 \to a_0$. Since $c_0 \to a_1$ and $c_0 \to d_0$, we have $c_0 \to d_1$. In total, $a_1 \to b_1 \to c_0 \to d_1 \to a_1$, $b_1 \to d_1$ and $c_0 \to a_1$. □

2.6. **Theorem.** *Let A be a tournament and A' be the tournament obtained from A by adding the unit element 1 (i.e., $x \to 1$ for all $x \in A$). If A is projective in $\mathcal{T}$ (or in $\mathcal{T}^3$), then A' is projective in $\mathcal{T}$ (or in $\mathcal{T}^3$, respectively).*

Proof. Let $B \in \mathcal{T}^3$ and let f be a homomorphism of B onto A. Take an element $e \in B$ with $f(e) = 1$. For every $a \in A$ choose an element $a' \in B$ such that $f(a') = a$ and $a \to e$ (the existence of a' is clear). Denote by S the subalgebra of B generated by $\{a' : a \in A\}$. It follows from (p4) that $x \to e$ for all $x \in S$. The restriction of f to S is a homomorphism of S onto A, so there is a homomorphism g_0 of A into S with $fg_0 = \mathrm{id}_A$. Define a mapping g of A' into B by $g \supseteq g_0$ and $g(1) = e$. Then g is a homomorphism and $fg = \mathrm{id}_{A'}$. □

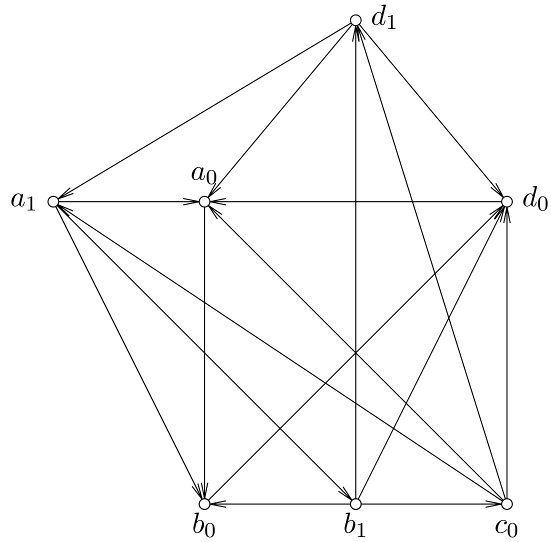

Fig. 2

2.7. **Corollary.** *Every finite chain is projective in* $\mathcal{T}^3$.

2.8. **Theorem.** *Let* A *be a tournament such that* A *is a disjoint union of two nonempty subsets* B, C *with the following properties*: $b \to c$ *for all* $b \in B$, $c \in C$; *the tournament* C *has no zero element. Then* A *is not projective in* $\mathcal{T}$.

Proof. Define a subset S of $A \times C$ by $(a, c) \in S$ if and only if either $a \in B$ or $a = c \in C$. Clearly, S is a subalgebra of $A \times C$, so $S \in \mathrm{SP}(\mathbf{T})$. The projection of $A \times C$ onto A, restricted to S, is a homomorphism of S onto A. So, if A is projective, then there is a homomorphism g of A into S such that whenever $g(a) = (a', c)$ then $a' = a$. Take an element $b \in B$. We have $g(b) = (b, c)$ for some $c \in C$. Since c is not a zero element of C, there exists an element $d \in C$ such that $d \neq c$ and $d \to c$. Then $(b, c) = g(b) = g(bd) = g(b)g(d) = (b, c)(d, d) = (b, d) \neq (b, c)$, a contradiction. □

2.9. **Corollary.** *Let* A *be a tournament with zero, such that* A *is not a chain. Then* A *is not projective in* $\mathcal{T}$.

3. Algebras, projective in $\mathcal{T}^3$

3.1. **Theorem.** $\mathbf{M}_3$ *is projective in* $\mathcal{T}^3$.

Proof. Let $B \in \mathcal{T}^3$ and let f be a homomorphism of B onto $\mathbf{M}_3$. By 2.4, there are elements $a_0, b_0, c_0, d_{10} \in B$ with $f(a_0) = a$, $f(b_0) = b$, $f(c_0) = c$, $f(d_{10}) = d_1$, $a_0 \to d_{10} \to b_0 \to c_0 \to d_{10}$ and $a_0 \to c_0$. Put $e_0 = a_0 b_0$, so that $f(e_0) = e$ and $e_0 \to c_0$. If $d_{10} \to e_0$, then these five elements are a subalgebra of B isomorphic to $\mathbf{M}_3$. Consider the opposite case. Put $d_{11} = d_{10}e_0$. By (p5) applied to $a_0 \to d_{10} \to b_0$ we have $a_0 \to d_{11} \to b_0$. So, if $c_0 \to d_{11}$, we are through. In the opposite case put $c_1 = c_0 d_{11}$. By (p5) applied to $e_0 \to c_0 \to d_{10}$ we have $e_0 \to c_1$. Since $a_0 \to c_0$ and $a_0 \to d_{11}$, we have $a_0 \to c_1$. So, if $b_0 \to c_1$, we are through. In the opposite case put $b_1 = b_0 c_1$. By (p5) applied to $d_{11} \to b_0 \to c_0$ we have $d_{11} \to b_1 \to c_1$. Since $e_0 \to b_0$ and $e_0 \to c_1$, we have $e_0 \to b_1$. Now $a_0 b_0 \to b_1 \to b_0$, so by (p2) we get $ab_1 = a_0 b_0 = e_0$ and we are through. □

3.2. **Corollary.** *The class of the algebras in* $\mathcal{T}^3$ *that do not contain a subalgebra isomorphic to* $\mathbf{M}_3$ *is a variety.*

3.3. **Theorem.** $\mathbf{M}_4$ *is not projective in* $\mathcal{T}^3$.

Proof. Define an algebra $B \in \mathcal{T}^3$ with the underlying set $\{a, b, c, d_{11}, d_{12}, d_{13}, d_{14}, d_2, e\}$ in such a way that the identity together with $d_{1i} \mapsto d_1$ is a homomorphism of B onto $\mathbf{M}_4$ and $d_{11}b = d_{14}$, $d_{11}e = d_{13}$, $d_{14}d_2 = d_{12}$, $d_{14}e = d_{13}$, $d_{12}c = d_{11}$, $d_{13}d_2 = d_{12}$. This homomorphism contradicts the assumption that $\mathbf{M}_4$ is projective in $\mathcal{T}^3$.

(The nine-element subdirectly irreducible algebra B is pictured in Fig. 3, with $d = d_2$, $f = d_{11}$, $g = d_{12}$, $h = d_{13}$, $i = d_{14}$. It contains a subalgebra isomorphic to $\mathbf{M}_3$, namely, $\{d_2, d_{13}, e, a, d_{12}\}$; it also contains a subalgebra isomorphic to $\mathbf{J}_3$, namely, $\{e, d_{11}, d_{12}, a, d_{13}\}$.) □

	a	b	c	d	e	f	g	h	i
a	a	e	a	d	e	a	a	a	a
b	e	b	b	d	e	i	g	h	i
c	a	b	c	c	e	f	f	h	i
d	d	d	c	d	d	f	g	g	g
e	e	e	e	d	e	h	g	h	h
f	a	i	f	f	h	f	f	h	i
g	a	g	f	g	g	f	g	g	g
h	a	h	h	g	h	h	g	h	h
i	a	i	i	g	h	i	g	h	i

Fig. 3

4. An infinite, 4-generated algebra in $\mathcal{T}^3$

We define an infinite groupoid $\mathbf{A}$ with underlying set $\{a_0, a_1, a_2, \dots\}$ as follows: the multiplication of $\mathbf{A}$ is both idempotent and commutative;

$\{a_0, a_1, a_2\}$ is a subtournament with $a_0 \to a_2 \to a_1 \to a_0$;

$$\text{for } i < 3 \leq j, \quad a_i a_j = \begin{cases} a_{j+1} & \text{if } j \equiv i+2 \mod 3, \\ a_j & \text{otherwise,} \end{cases}$$

$$\text{for } 3 \leq i < j, \quad a_i a_j = \begin{cases} a_{i+1} & \text{if } j \equiv i+1 \mod 3, \\ a_i & \text{otherwise.} \end{cases}$$

4.1. **Lemma.** *The mapping f defined by $f(a_i) = a_i$ for $i < 3$ and $f(a_i) = a_{i+3}$ for $i \geq 3$ is an isomorphism of $\mathbf{A}$ onto the subgroupoid $\mathbf{A} - \{a_3, a_4, a_5\}$.*

Proof. It is easy. □

4.2. **Theorem.** *The infinite groupoid $\mathbf{A}$ belongs to $\mathcal{T}^3$ and is generated by four elements a_0, a_1, a_2, a_3. Consequently, the variety $\mathcal{T}^3$ is not locally finite.*

Proof. We have $a_4 = a_1 a_3$, $a_5 = a_2 a_4$, $a_6 = a_0 a_5$, $a_7 = a_1 a_6$, etc. So, $\mathbf{A}$ is generated by a_0, a_1, a_2, a_3.

Suppose that an equation in three variables x, y, z is satisfied in all tournaments but not in $\mathbf{A}$. There is an interpretation sending the three variables to three elements a_i, a_j, a_k, under which the two sides evaluate to different elements. Since $\{a_0, a_1, a_2\}$ is a tournament, at least one of the elements a_i, a_j, a_k does not belong to $\{a_0, a_1, a_2\}$. If none of the three elements belongs to $\{a_3, a_4, a_5\}$, then it follows from 4.1 that the equation is also violated by the interpretation, sending the three variables to $a_{i'}, a_{j'}, a_{k'}$, where n' is defined by $n' = n$ for $n < 3$ and $n' = n - 3$ for $n \geq 3$. So, we can suppose that at least one of the three elements belongs to $\{a_3, a_4, a_5\}$. From the same reason we can suppose that either the three elements belong to $\{a_0, \dots, a_5\}$ or at least one of them belongs to $\{a_6, a_7, a_8\}$. And again, that either they all belong to $\{a_0, \dots, a_8\}$ or at least one of them belongs to $\{a_9, a_{10}, a_{11}\}$. In total, we can suppose $\{a_i, a_j, a_k\} \subseteq \{a_0, \dots, a_{11}\}$. However, one can easily check that the equations (e1),…,(e4) are satisfied under all the 12^3 interpretations sending x, y, z to $\{a_0, \dots, a_{11}\}$. □

For every $n \geq 3$ we define two groupoids $\mathbf{A}_n$ and $\mathbf{B}_n$ with the underlying set $\{a_0, \dots, a_{n-1}\}$ as follows. Let c be the only element of $\{0, 1, 2\}$ with $c \equiv n \mod 3$. Now all products in both $\mathbf{A}_n$ and $\mathbf{B}_n$ are the same as in $\mathbf{A}$, except that $a_c a_{n-1} = a_{n-1} a_c = a_{n-3}$ in $\mathbf{A}_n$, and $a_c a_{n-1} = a_{n-1} a_c = a_c$ in $\mathbf{B}_n$.

Clearly, $\mathbf{A}_3 = \mathbf{B}_3 \simeq \mathbf{C}_3$, $\mathbf{A}_4 = \mathbf{B}_4 \simeq \mathbf{C}_4$, $\mathbf{A}_5 = \mathbf{B}_5 \simeq \mathbf{M}_3$, and $\mathbf{A}_n \neq \mathbf{B}_n$ for $n \geq 6$.

4.3. **Theorem.** *The groupoids $\mathbf{A}_n$ and $\mathbf{B}_n$ all belong to $\mathcal{T}^3$. The groupoids $\mathbf{B}_n$ are all subdirectly irreducible, and $\mathbf{A}$ is isomorphic to a subdirect product of the groupoids $\mathbf{B}_n$ $(n \geq 3)$ and $\mathbf{C}_3 + 0$ (the groupoid $\mathbf{C}_3$ with zero element added). Although $\mathbf{A}$ is subdirectly reducible, id_A is not the intersection of a finite number of nontrivial congruences of $\mathbf{A}$.*

Proof. For every $n \geq 0$ define an equivalence μ_n on $\mathbf{A}$ as follows: $(a_i, a_j) \in \mu_n$ if and only if $i \equiv j \mod 3$ and either $i = j$ or $i, j \geq n$. While μ_1 and μ_2 are not congruences, it is easy to check that μ_n is a congruence of $\mathbf{A}$ for any $n \geq 3$. Since $\bigcap_{n \geq 3} \mu_n = \mathrm{id}_A$, it follows that $\mathbf{A}$ is subdirectly reducible: it is isomorphic to a subdirect product of the groupoids $\mathbf{A}/\mu_n$, $n \geq 3$. Now it is easy to see that $\mathbf{A}/\mu_n \simeq \mathbf{A}_{n+3}$ for $n \geq 3$. Consequently, $\mathbf{A}_n \in \mathcal{T}^3$ for $n \geq 6$. (But we have seen that this is also true for $n = 3, 4, 5$.) It is easy to see that every nontrivial congruence of $\mathbf{A}$ contains μ_n for some n, and so id_A is not the intersection of any finite number of nontrivial congruences of $\mathbf{A}$.

For $n \geq 6$, the identity on $\mathbf{A}_n$ is the intersection of two nontrivial congruences α and β of $\mathbf{A}_n$, where

$(a_i, a_j) \in \alpha$ iff $i \equiv j \mod 3$ and either $i, j \in \{0, 1, 2, a_{n-3}, a_{n-2}, a_{n-1}\}$ or $i = j$,

$(a_i, a_j) \in \beta$ iff either $i = j$ or $i, j \geq 3$.

Now $\mathbf{A}_n/\alpha \simeq \mathbf{B}_{n-3}$ and $\mathbf{A}_n/\beta \simeq \mathbf{C}_3 + 0$. Consequently, $\mathbf{B}_{n-3} \in \mathcal{T}^3$ for all $n \geq 6$ and $\mathbf{A}$ is isomorphic to a subdirect product of the groupoids $\mathbf{B}_3, \mathbf{B}_4, \dots$ and $\mathbf{C}_3 + 0$.

For $n \geq 3$, the groupoid $\mathbf{B}_n$ is subdirectly irreducible: for $n \geq 4$, its monolith is the congruence identifying a_{n-1} with a_c, where $c \in \{0, 1, 2\}$ and $c \equiv n - 1 \mod 3$. □

4.4. **Remark.** For $n \geq 5$, $\{a_0, a_1, a_2, a_{n-2}, a_{n-1}\}$ is a subgroupoid of $\mathbf{B}_n$ isomorphic to $\mathbf{M}_3$.

5. Free constructions over partial tournaments

By a partial tournament we mean a set A together with a reflexive, antisymmetric relation on A; the relation will be usually denoted by $\to$. By a homomorphism of a partial tournament A into a partial tournament B we mean a mapping f such that $a \to b$ implies $f(a) \to f(b)$.

By a $\mathcal{T}$-free algebra over a partial tournament A we mean an algebra $G \in \mathcal{T}$ together with a homomorphism g of A into G, such that G is generated by $g(A)$ and for any homomorphism h of A into any algebra $B \in \mathcal{T}$ there exists a homomorphism h' of G into B with $h = h'g$.

It is easy to see that the $\mathcal{T}$-free algebra G over A exists and is uniquely determined up to isomorphism; it will be denoted by $\mathbf{F}(A)$. For $A = \{a_1, \dots, a_n\}$, it can be constructed in the following way. Let F be the free algebra in $\mathcal{T}$ generated by a set of n variables $x_1, \dots, x_n$, and denote by r the congruence of F generated by the pairs $(x_i x_j, x_i)$ such that $a_i \to a_j$ in A. Clearly, $\mathbf{F}(A) = F/r$. In more detail, the factor F/r together with the mapping $a_i \mapsto x_i/r$ is the $\mathcal{T}$-free algebra over A.

However, this construction is very inefficient. It assumes that we are able to construct the free algebra F over $x_1, \dots, x_n$. For $n = 3$ we have $|F| = 15$, but for $n = 4$ we only know that F has more than (possibly much mure than) 500000 elements.

On the other hand, there is a candidate for G which can be constructed much more easily, at least in the case when the partial tournament is almost complete: Denote by $A_1, \dots, A_k$ all completions of A to tournaments (so that $k = 2^{\binom{n}{2}-m}$, where m is the number of the pairs $a_i \to a_j$ with $i \neq j$), for $a \in A$ put $\bar{a} = (a, a, \dots, a) \in A_1 \times \dots \times A_k$, and denote by S the subalgebra of $A_1 \times \dots \times A_k$ generated by $\bar{a}_1, \dots, \bar{a}_n$. The algebra S, together with the mapping $a_i \mapsto \bar{a}_i$, is a good candidate for a $\mathcal{T}$-free algebra over A. This algebra will be denoted by $\mathbf{F}_0(A)$.

One can easily see that $\mathbf{F}_0(A)$ is free over A in the quasivariety generated by tournaments. So, if the Conjecture is true, then $\mathbf{F}(A) = \mathbf{F}_0(A)$ for every partial tournament A. However, we do not know whether the Conjecture is true. So, we need to find at least a way how to prove $\mathbf{F}(A) = \mathbf{F}_0(A)$ in some particular cases.

Let $A = \{a_1, \dots, a_n\}$ be a finite partial tournament. Take a set of n variables $x_1, \dots, x_n$ and denote by T the groupoid of terms over the set $\{x_1, \dots, x_n\}$. We define a mapping ν of a subset of T into A as follows: $\nu(x_i) = a_i$; $\nu(t_1 t_2)$ is defined if and only if both $\nu(t_1)$ and $\nu(t_2)$ are defined and either $\nu(t_1) \to \nu(t_2)$ or $\nu(t_2) \to \nu(t_1)$; in the first case put $\nu(t_1 t_2) = \nu(t_1)$, and in the second case $\nu(t_1 t_2) = \nu(t_2)$. If defined, the element $\nu(t)$ is called the value of t in A (under the interpretation $x_i \mapsto a_i$).

By a correct term configuration for A we mean a mapping γ of A into T satisfying two conditions:

(1) for $a \in A$, the value of $\gamma(a)$ in A under the interpretation $x_i \mapsto a_i$ is equal to a;

(2) for $a \to b$ in A, the equation $\gamma(a)\gamma(b) \approx \gamma(a)$ is true in all tournaments.

5.1. **Theorem.** *Let A be a finite partial tournament for which there exists a correct term configuration. Then $\mathbf{F}(A) = \mathbf{F}_0(A)$.*

Proof. Let us keep the above notation, so that $\mathbf{F}(A) = F/r$ and $\mathbf{F}_0(A) = S$. Denote by h the extension of the identity to a homomorphism of T onto F, and by f the homomorphism of F onto S extending $x_i \mapsto \bar{a}_i$. Easily, $r \subseteq \ker(f)$ and all we need to prove is that $\ker(f) = r$.

It is easy to prove by induction on the length of t that if $t \in T$ is a term having a value a_i in A, then $(x_i, h(t)) \in r$. According to (1), it follows that $(h\gamma(a_i), x_i) \in r$ for all $a_i \in A$.

For every $s \in S$ take a term $\tau_s(x_1, \dots, x_n)$ such that $\tau_s(\bar{a}_1, \dots, \bar{a}_n) = s$ in S. This can be done in such a way that $\tau_{\bar{a}_i} = x_i$ for all i. If s_1, s_2 are elements of S, then for any tournament C and any n-tuple $c_1, \dots, c_n$ of elements of C such that $a_i \to a_j$ in A implies $c_i \to c_j$ in C we have $\tau_{s_1}(c_1, \dots, c_n)\tau_{s_2}(c_1, \dots, c_n) = \tau_{s_1 s_2}(c_1, \dots, c_n)$. (I.e., for every $s_1, s_2 \in S$ we obtain a certain quasiequation true in all tournaments.) Indeed, the assumption implies that there is an index $p \in \{1, \dots, k\}$ such that the mapping $\varphi : a_i \mapsto c_i$ is a homomorphism of A_p into C; where π_p is the projection of

$A_1 \times \dots \times A_k$ onto A_p, we have

$$\tau_s(c_1, \dots, c_n) = \tau_s(\varphi\pi_p(\bar{a}_1), \dots, \varphi\pi_p(\bar{a}_n)) = \varphi\pi_p t_s(\bar{a}_1, \dots, \bar{a}_n) = \varphi\pi_p(s)$$

for all $s \in S$, so that

$$\begin{aligned}\tau_{s_1}(c_1, \dots, c_n)\tau_{s_2}(c_1, \dots, c_n) &= \varphi\pi_p(s_1)\varphi\pi_p(s_2) = \varphi\pi_p(s_1 s_2) \\ &= \tau_{s_1 s_2}(c_1, \dots, c_n).\end{aligned}$$

Define an endomorphism ε of T by $\varepsilon(x_i) = \gamma(a_i)$. If $a_i \to a_j$ in A, then by (2) $\varepsilon(x_i) \to \varepsilon(x_j)$ is satisfied in all tournaments (under any interpretation). Consequently, for $s_1, s_2 \in S$, $\varepsilon(\tau_{s_1})\varepsilon(\tau_{s_2}) \approx \varepsilon(\tau_{s_1 s_2})$ is satisfied in all tournaments. This means $h\varepsilon(\tau_{s_1}) \cdot h\varepsilon(\tau_{s_2}) = h\varepsilon(\tau_{s_1 s_2})$ in F. So, the set $H = \{h\varepsilon(\tau_s) : s \in S\}$ is a subgroupoid of F. Since $(h\varepsilon(\tau_{\bar{a}_i}), x_i) = (h\gamma(a_i), x_i) \in r$, every element of F is congruent with an element of H modulo r. Consequently, $F/r \simeq H/r$. The rest is now clear. □

5.2. **Example.** Consider the partial tournament $A = \{x, y, z, u\}$ with $x \to z \to y \to u \to x$ and $z \to u$. The mapping

$\gamma(x) = x(xuy \cdot xuz)$,
$\gamma(y) = x(xuy \cdot xuz)(xu)(xuy)$,
$\gamma(z) = xuy \cdot xuz$,
$\gamma(u) = x(xuy \cdot xuz)(xu)$

is a correct term configuration for A. Condition (1) can be checked immediately. Clearly, $\gamma(y) \to \gamma(u) \to \gamma(x) \to \gamma(z)$ is true in all tournaments. $\gamma(z) \to \gamma(u)$ is easy to prove from the three-variable equations. It remains to prove $xuy \cdot xuz \to x(xuy \cdot xuz)(xu)(xuy)$, which is easy to do by considering several (not many) cases.

Denote by A_1 and A_2 the two completions of A, one by $x \to y$ and the other by $y \to x$. Easily, the subgroupoid of $A_1 \times A_2$ generated by $(x, x), (y, y), (z, z), (u, u)$ equals $A_1 \times A_2$. Consequently, $\mathbf{F}(A) = A_1 \times A_2$. With $(x, x) = a$, $(y, y) = f$, $(z, z) = k$, $(u, u) = p$, the multiplication table of this groupoid is given in Fig. 4.

From this table it is possible to read, for example, that if an algebra in $\mathcal{T}$ contains four elements x, y, z, u with $x \to z \to y \to u \to x$ and $z \to u$, then $xyuz = z$.

5.3. **Example.** Consider the partial tournament $A = \{x, y, z, u\}$ with $z \to y \to u \to z \to x$ and $y \to x$. Where $t = yxux$, the mapping

$\gamma(x) = x$,
$\gamma(y) = t(zxtx)ut$,
$\gamma(z) = t(zxtx)ut(t(zxtx))$,
$\gamma(u) = t(zxtx)u$

is a correct term configuration for A.

5.4. **Example.** Consider the partial tournament $A = \{x, y, z, u\}$ with $z \to y \to u \to z \to x$. The mapping

$\gamma(x) = x$,

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
a	a	b	a	d	a	b	a	d	a	b	a	d	m	n	m	p
b	b	b	c	b	b	b	c	b	b	b	c	b	n	n	o	n
c	a	c	c	c	a	c	c	c	a	c	c	c	m	o	o	o
d	d	b	c	d	d	b	c	d	d	b	c	d	p	n	o	p
e	a	b	a	d	e	f	e	h	i	j	i	l	e	f	e	h
f	b	b	c	b	f	f	g	f	j	j	k	j	f	f	g	f
g	a	c	c	c	e	g	g	g	i	k	k	k	e	g	g	g
h	d	b	c	d	h	f	g	h	l	j	k	l	h	f	g	h
i	a	b	a	d	i	j	i	l	i	j	i	l	i	j	i	l
j	b	b	c	b	j	j	k	j	j	j	k	j	j	j	k	j
k	a	c	c	c	i	k	k	k	i	k	k	k	i	k	k	k
l	d	b	c	d	l	j	k	l	l	j	k	l	l	j	k	l
m	m	n	m	p	e	f	e	h	i	j	i	l	m	n	m	p
n	n	n	o	n	f	f	g	f	j	j	k	j	n	n	o	n
o	m	o	o	o	e	g	g	g	i	k	k	k	m	o	o	o
p	p	n	o	p	h	f	g	h	l	j	k	l	p	n	o	p

Fig. 4

$\gamma(y) = zxuxuy,$
$\gamma(z) = zxuxuy(zxux),$
$\gamma(u) = zxuxuy(zxux)(zxuxu)$

is a correct term configuration for A. Thus $\mathbf{F}(A) = \mathbf{F}_0(A)$. This algebra has 61 elements. Observe that the 16-element free algebra from Example 5.2 could be also constructed as a factor of this 61-element algebra.

5.5. **Example.** Consider the partial tournament $A = \{x, y, z, u\}$ with $x \to z \to y \to x \to u \to y$. In this case it is easy to construct the free algebra directly: it has just five elements. Consequently, $\mathbf{F}(A) = \mathbf{F}_0(A)$ also in this case.

We can also find a correct term configuration γ for A:

$\gamma(x) = x(yz)(yu)(yz),$
$\gamma(y) = x(yz)(yu)(yz)y,$
$\gamma(z) = x(yz)(yu)(yz)y(yz),$
$\gamma(u) = x(yz)(yu)(yz)y(yu).$

5.6. **Theorem.** *Let $n \geq 3$. Then every tournament satisfies*

$$x_1x_2x_3 \dots x_n \to x_1x_2x_3 \dots x_nx_1(x_1x_2)(x_1x_2x_3)\dots(x_1x_2\dots x_{n-1}).$$

Consequently, there exists a correct term configuration for the partial tournament $A = \{x_1, \dots, x_n\}$ with $x_1 \to x_n \to x_{n-1} \to \dots \to x_2 \to x_1$, and $\mathbf{F}(A)$ is the subgroupoid of $A_1 \times \dots \times A_k$ generated by the constant k-tuples, where $A_1, \dots, A_k$ are all completions of A to tournaments.

Proof. Let a tournament B be given, and let us compute in B. For all $j = 1, \dots, n$ put $y_j = x_1 \dots x_j$, so that $y_j \in \{x_1, \dots, x_j\}$. Clearly, $y_{j+1} \to y_j$

for $j < n$. There is an index i with $y_n = x_i$, and we need to prove $x_i \to x_i y_1 \dots y_{n-1}$.

If $x_i y_1 \dots y_j = y_j$ for some $j < n$, then $x_i y_1 \dots y_{n-1} = y_j \dots y_{n-1} = y_{n-1}$, and we are through, since $x_i = y_n \to y_{n-1}$. So, we may assume that $x_i y_1 \dots y_j \neq y_j$ for all $j < n$. But then, by induction on j, $x_i y_1 \dots y_j = x_i$ for all $j < n$. In particular, $x_i y_1 \dots y_{n-1} = x_i$. □

5.7. **Remark.** For $n = 4$, an easy computation shows that the subalgebra of $P = A_1 \times \dots \times A_k$ generated by the constant k-tuples is equal to P. However, this is not true for $n \geq 5$. For $n = 4$, the algebra P has 256 elements. For $n = 5$, it has 5^{32} elements, so it is not easy to compute its subalgebra generated by the five constant 32-tuples. But if the subalgebra equals P, then also the product $P' = B_1 \times \dots \times B_8$, where $B_1, \dots, B_8$ are all the completions of A enriched by $x_2 \to x_0$ and $x_3 \to x_0$, is generated by the five constant 8-tuples; one can easily compute that the subalgebra of P' generated by the constant 8-tuples has 109375 elements, and this number is less than $5^8 = |P'|$.

5.8. **Theorem.** *Let A be a finite partial tournament and $a \in A$ be an element such that there is no element $b \neq a$ with $b \to a$, and there are at most two elements $c \neq a$ with $a \to c$. Denote by A' the partial tournament $A - \{a\}$. If there exists a correct term configuration for A', then there exists a correct term configuration for A.*

Proof. Let γ be a correct term configuration for A'. Let x be the variable corresponding to the element a. We can extend γ to a correct term configuration for A as follows: if there is no element $b \in A'$ with $a \to b$, put $\gamma(a) = x$; if there is precisely one such element b, and put $\gamma(a) = x\gamma(b)$; if there are two such elements b_1 and b_2, put $\gamma(a) = x\gamma(b_1)\gamma(b_2)\gamma(b_1)$. □

5.9. **Theorem.** *Let A be a finite partial tournament and let A' be obtained by adding a zero element z to A, i.e., $z \to a$ for all $a \in A$. If $\mathbf{F}(A) = \mathbf{F}_0(A)$, then also $\mathbf{F}(A') = \mathbf{F}_0(A')$.*

Proof. Clearly, $\mathbf{F}(A')$ is obtained by adding a zero element to $\mathbf{F}(A)$. □

5.10. **Remark.** It follows that $\mathbf{F}(A) = \mathbf{F}_0(A)$ for the partial tournament $A = \{x, y, z, u\}$ with $x \to y$, $x \to z$, $x \to u$; the algebra has 16 elements. On the other hand, it can be easily shown that there is no correct term configuration for this partial tournament. Suppose there is such a configuration γ. One can easily see that (modulo the idempotent law) $\gamma(y) = y$, $\gamma(z) = z$ and $\gamma(u) = u$. Put $t = \gamma(x)$. Then t is a term in four variables such that $t \to y$, $t \to z$ and $t \to u$ are satisfied in all tournaments. Substituting y for x in t we obtain a term in three variables with the same property. However, it is easy to check that in the 15-element $\mathcal{T}$-free algebra with three generators there is no element corresponding to such a term.

5.11. **Theorem.** $\mathbf{F}(A) = \mathbf{F}_0(A)$ *for every partial tournament A with at most four elements.*

Proof. If $|A| \le 3$, then it follows from 5.6 that there is a correct term configuration for A. Let $|A| = 4$. Of course, we can assume that A is not a tournament. By 5.9 we can assume that A has no zero element, and by 5.8 we can assume that for every $a \in A$ there exists a $b \neq a$ with $b \to a$. The cases when A contains a four-cycle are covered by 5.2 and 5.6. There are only three cases remaining, covered by 5.3, 5.4 and 5.5. □

5.12. **Remark.** The cardinality of $\mathbf{F}(A)$ for a partial tournament A with four elements x, y, z, u can be easily computed in some cases. For example:
For A given by $x \to y \to z \to u$, $|\mathbf{F}(A)| = 965$.
For A given by $x \to y \to z$, $|\mathbf{F}(A)| = 18010$.
For A given by $y \to x$ and $z \to x$, $|\mathbf{F}(A)| = 732$.
For A given by $x \to y$ and $x \to z$, $|\mathbf{F}(A)| = 736$.
For A given by $x \to y$ and $z \to u$, $|\mathbf{F}(A)| = 3611$.
For A given by $x \to y \to z \to x$, $|\mathbf{F}(A)| = 380$.

6. Infinitely many incomparable tournaments

Tournaments can be identified with algebras in two different ways. The approach to consider them as groupoids (algebras with one binary operation) was taken, for example, in [4], [5], [9] (and in the present paper). Alternatively, tournaments can be also identified with algebras with two binary operations xy and $x + y$, where xy is defined as above and $a + b = b + a = b$ for $a \to b$. This approach was taken, for example, in [2] and [3]. For tournaments themselves the difference is not significant, but if we want to consider the variety generated by tournaments, we get different results in both cases. In the case of two binary operations, the variety generated by tournaments is contained in the variety of weakly associative lattices, and hence is congruence distributive (see [2]).

In [3] E. Fried asks whether the variety generated by tournaments has uncountably many subvarieties, and remarks that this would be a consequence of a positive solution to the following problem: Does there exist an infinite set of finite subdirectly irreducible tournaments such that neither one is isomorphic to a subalgebra of some other one? In this section we are going to construct such an infinite set of tournaments; all of them will be simple.

The infinite sequence of tournaments A_n ($n \ge 8$) is defined in the following way: $A_n = \{a_{n,1}, \dots, a_{n,n}\}$,

$$a_{n,n} \to a_{n,1};$$
$$a_{n,i+2} \to a_{n,i} \text{ for } 1 \le i \le n-2;$$
$$a_{n,i} \to a_{n,j} \text{ for } 1 \le i < j \le n,\ j \neq i+2,\ (i,j) \neq (1,n).$$

6.1. **Lemma.** *Let $a_{n,i}, a_{n,j}$ be two distinct elements of A_n such that $a_{n,i} \to a_{n,j}$. Put $X = \{x \in A_n - \{a_{n,i}, a_{n,j}\} : a_{n,j} \to x \to a_{n,i}\}$. Then:*

(1) *For $(i,j) = (n,i)$, $X = \{a_{n,2}, a_{n,4}, a_{n,5}, \dots, a_{n,n-4}, a_{n,n-3}, a_{n,n-1}\}$ and $|X| \ge 4$.*

(2) *For* $1 \le j < i = j+2 \le n$, $X \subseteq \{a_{n,j-2}, a_{n,j+1}, a_{n,j+4}\}$.
(3) *For* $1 = i < j$, $X \subseteq \{a_{n,3}, a_{n,n}\}$.
(4) *For* $i < j = n$, $X \subseteq \{a_{n,1}, a_{n,n-2}\}$.
(5) *For* $2 \le i < i+1 = j \le n-1$, $X \subseteq \{a_{n,i-1}, a_{n,i+2}\}$.
(6) *For* $2 \le i < i+4 = j \le n-1$, $X = \{a_{n,i+2}\}$.
(7) *In all other cases,* $X = \emptyset$.

Proof. It is easy. □

6.2. **Lemma.** *Let* $n, m \ge 8$ *and let* α *be an embedding of* A_n *into* A_m. *Then* $\alpha(a_{n,1}) = a_{m,1}$ *and* $\alpha(a_{n,n}) = a_{m,m}$.

Proof. We have $\alpha(a_{n,n}) \to \alpha(a_{n,1})$ and, by Lemma 6.1(1), there are at least four elements $x \in A_m - \{\alpha(a_{n,1}), \alpha(a_{n,n})\}$ such that $\alpha(a_{n,1}) \to x \to \alpha(a_{n,n})$. By Lemma 6.1, it follows that $(\alpha(a_{n,n}), \alpha(a_{n,1})) = (a_{m,m}, a_{m,1})$. □

6.3. **Lemma.** *Let* $n, m \ge 8$ *and let* α *be an embedding of* A_n *into* A_m. *Then* $\alpha(a_{n,2}) = a_{m,2}$ *and* $\alpha(a_{n,3}) = a_{m,3}$.

Proof. Put $x = \alpha(a_{n,2})$, $y = \alpha(a_{n,3})$, $z = \alpha(a_{n,4})$ and $u = \alpha(a_{n,5})$. Then x, y, z, u are four distinct elements of $A_m - \{a_{m,1}, a_{m,m}\}$ such that $a_{m,1} \to x \to y \to z \to u$, $y \to a_{m,1}$, $z \to x$, $u \to y$, $x \to u$, $a_{m,1} \to u$. From $a_{m,1} \to x \to y \to a_{m,1}$ we get either $(x,y) = (a_{m,2}, a_{m,3})$ or $(x,y) = (a_{m,5}, a_{m,3})$. In the first case we are done, so suppose that $x = a_{m,5}$ and $y = a_{m,3}$. From $y \to z \to x$ (i.e., $a_{m,3} \to z \to a_{m,5}$) we get either $z = a_{m,4}$ or $z = a_{m,7}$.

Suppose $z = a_{m,4}$. From $z \to u \to y$ we get either $u = a_{m,2}$ or $u = a_{m,5}$. In the first case we get a contradiction with $x \to u$, and the second case contradicts $x \ne u$.

So, it remains to consider the case $z = a_{m,7}$. From $z \to u \to y$ we get $u = a_{m,5}$, a contradiction with $x \ne u$. □

6.4. **Lemma.** *Let* $n, m \ge 8$ *and let* α *be an embedding of* A_n *into* A_m. *Then* $\alpha(a_{n,i}) = a_{m,i}$ *for all* $i = 1, \dots, n$.

Proof. By Lemma 6.2 and Lemma 6.3, this is true for $i = 1, 2, 3$. Let $i \ge 4$ and suppose $\alpha(a_{n,j}) = a_{m,j}$ for all $j < i$. Put $x = \alpha(a_{n,i})$. We have $a_{n,i-1} \to a_{n,i} \to a_{n,i-2}$ in A_n, and thus $a_{m,i-1} \to x \to a_{m,i-2}$ in A_m. Moreover, $x \notin \{a_{m,1}, \dots, a_{m,i-1}\}$. But there is only one element x in A_m with these properties, namely, $x = a_{m,i}$. Hence $\alpha(a_{n,i}) = a_{m,i}$. □

6.5. **Lemma.** A_n *is a simple tournament for* $n \ge 8$.

Proof. Let $r \ne \mathrm{id}_{A_n}$ be a congruence of A_n. We need to prove that $r = A_n \times A_n$.

If $(a_{n,i}, a_{n,i+1}) \in r$ for some i, then in the case $i > 1$ we have $a_{n,i-1} \to a_{n,i} \to a_{n,i+1} \to a_{n,i-1}$, from which it follows that $(a_{n,i-1}, a_{n,i}) \in r$; and in the case $i+1 < n$ we have $(a_{n,i+1}, a_{n,i+2}) \in r$ from the same reason. Hence, if $(a_{n,i}, a_{n,i+1}) \in r$ for some i, then $r = A_n \times A_n$.

If $(a_{n,i}, a_{n,i+2}) \in r$ for some i, then

$$(a_{n,i}, a_{n,i+1}) = (a_{n,i}a_{n,i+1}, a_{n,i+2}a_{n,i+1}) \in r,$$

so that $r = A_n \times A_n$.

If $(a_{n,i}, a_{n,i+3}) \in r$ for some i, then one of the following two cases takes place. If $i \geq 3$, then

$$(a_{n,i}, a_{n,i-2}) = (a_{n,i}a_{n,i-2}, a_{n,i+3}a_{n,i-2}) \in r.$$

If $i \leq n - 5$, then

$$(a_{n,i}, a_{n,i+5}) = (a_{n,i}a_{n,i+5}, a_{n,i+3}a_{n,i+5}) \in r$$

and hence $(a_{n,i+3}, a_{n,i+5}) \in r$. But then, $r = A_n \times A_n$ in both cases.

Finally, if $(a_{n,i}, a_{n,j}) \in r$ where $j \geq i + 4$, then

$$(a_{n,i}, a_{n,i+1}) = (a_{n,i}a_{n,i+1}, a_{n,j}a_{n,i+1}) \in r,$$

so that $r = A_n \times A_n$. □

6.6. **Theorem.** *The tournaments A_n with $n \geq 8$ are all simple and pairwise incomparable in the sense that if $n \neq m$, then A_n cannot be embedded into A_m.*

Proof. It follows from the lemmas. □

As noted in [3], due to the ultraproduct theorem of Jónsson [7] and the fact that a homomorphic image of a tournament is isomorphic to a subtournament of that tournament, it follows from Theorem 6.6 that for any subset S of $\{A_8, A_9, \dots\}$, the variety (of algebras with two binary operations) generated by S does not contain any A_n with $n \geq 8$ and $n \notin S$.

6.7. **Corollary.** *The lattice of subvarieties of the variety (of algebras with two binary operations) generated by tournaments is uncountable. It contains a subset, order isomorphic to the lattice of all subsets of a countably infinite set.*

It is not clear, although it is likely, that the same is true for the variety generated by tournaments considered as algebras with one binary operation.

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