ORDINARY GRAPHS AND SUBPLANE PARTITIONS

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Abstract. We introduce a generalization of symmetric (v, k, λ) block designs, and show how these could potentially be used to construct projective planes of non-prime-power order.

If q is a prime power and $n^2+n+1 = N(q^2+q+1)$, then conceivably we could construct a projective plane of order n by gluing together N planes of order q. For example, $18^2+18+1 = 343 = 49 \cdot 7$. Can we make a projective plane of order 18 by gluing 49 planes of order 2?

With this in mind, we will discuss a class of directed graphs which arises when we attempt such a construction. The existence of a graph with the necessary parameters is the first problem we face. Then we will describe how the graphs may be used to build the putative plane of order n. The gluing maps are an even more serious obstacle. Nonetheless, it is an intriguing program, which just possibly could work.

1. Ordinary Graphs and their Associated Matrices

Let $\mathbf{G}$ be a loopless directed graph. For a vertex i of $\mathbf{G}$, let

$$\uparrow i = \{j : i \to j\}$$
$$\downarrow i = \{k : k \to i\}.$$

Define binary relations A, B, C on the vertices of $\mathbf{G}$ by

(1) $i\,A\,j$ if $i \neq j$ and there is no edge between i and j,
(2) $i\,B\,j$ if $i \to j$ or $j \to i$ but not both,
(3) $i\,C\,j$ if $i \to j$ and $j \to i$.

We say that $\mathbf{G}$ is an *ordinary graph* of type $\langle n, r, a, b, c\rangle$ if

(1) $\mathbf{G}$ has n vertices,
(2) $|\uparrow i| = |\downarrow i| = r$ for each vertex i of $\mathbf{G}$,
(3) if $i\,A\,j$ then $|\uparrow i \cap \uparrow j| = |\downarrow i \cap \downarrow j| = a$,
(4) if $i\,B\,j$ then $|\uparrow i \cap \uparrow j| = |\downarrow i \cap \downarrow j| = b$,
(5) if $i\,C\,j$ then $|\uparrow i \cap \uparrow j| = |\downarrow i \cap \downarrow j| = c$.

We allow a, b or c to be the special symbol $\times$ to indicate that in $\mathbf{G}$ the corresponding relation A, B or C is empty.

1991 *Mathematics Subject Classification.* 51A35, 51E05.

The research of the second author was supported in part by the grants GAČR 201/99/0263 and MSM 113200007.

This definition is easily interpreted in terms of the adjacency matrix $\mathbf{M}$ of $\mathbf{G}$. An $n \times n$ 0-1 matrix $\mathbf{M} = [m_{ij}]$ is the adjacency matrix of an ordinary graph of type $\langle n, r, a, b, c\rangle$ if it has a zero diagonal and

$$(\mathbf{M}\mathbf{M}^t)_{ij} = (\mathbf{M}^t\mathbf{M})_{ij} = \begin{cases} r & \text{if } i = j, \\ a & \text{if } i \neq j \text{ and } m_{ij} + m_{ji} = 0, \\ b & \text{if } m_{ij} + m_{ji} = 1, \\ c & \text{if } m_{ij} + m_{ji} = 2. \end{cases}$$

We will also refer to the adjacency matrix of an ordinary graph as an *ordinary matrix*.

Recall, for example, that the incidence matrix of a symmetric (v, k, λ) block design is a $v \times v$ 0-1 matrix satisfying $\mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M} = (k - \lambda)\mathbf{I} + \lambda\mathbf{J}$. Thus the corresponding graph is ordinary of type $\langle v, k, \lambda, \lambda, \lambda\rangle$, with possibly one or more of the λ's replaced by an $\times$.

Question. In the examples we have found so far, if a 0-1 matrix $\mathbf{M}$ has constant row and column sums, and satisfies the conditions above for $(\mathbf{M}\mathbf{M}^t)_{ij}$, then $\mathbf{M}^t\mathbf{M} = \mathbf{M}\mathbf{M}^t$ (so $\mathbf{M}$ is normal). Is this always the case? If so, it would generalize a well-known result of Ryser for symmetric block designs.

We will produce many examples in the next two sections. First, we have the basic counting lemma.

Lemma 1. *Let $\mathbf{G}$ be an ordinary graph of type $\langle n, r, a, b, c\rangle$. Let α be the number of (unordered) pairs in the A relation, and similarly β is the number of pairs in B, γ the number of pairs in C. Then*

(1) $\alpha + \beta + \gamma = \binom{n}{2}$,
(2) $\beta + 2\gamma = nr$,
(3) $a\alpha + b\beta + c\gamma = n\binom{r}{2}$.

Proof. (1) is clear, while (2) just counts the number of arrows in $\mathbf{G}$. For (3), we count the pairs $(i, \{j, k\})$ with $j \neq k$, $i \to j$ and $i \to k$. Each subset $\{j, k\}$ with $j\, A\, k$ will be counted a times, etc. □

An ordinary graph is *symmetric* if $i \to j$ implies $j \to i$ (equivalently, $\mathbf{M}^t = \mathbf{M}$ or $\beta = 0$). Likewise, a graph is *antisymmetric* if $i \to j$ implies $j \not\to i$ (equivalently, $\mathbf{M} + \mathbf{M}^t + \mathbf{I} \leq \mathbf{J}$ or $\gamma = 0$).

Note: the conflict of terminology is unfortunate, but a symmetric (v, k, λ) block design need not be symmetric in this sense.

Given an ordinary graph $\mathbf{G}$, we define the *complement* $\mathbf{G}^*$ to be the loopless graph such that, for vertices $i \neq j$, $i \to_{\mathbf{G}^*} j$ if and only if $i \not\to_{\mathbf{G}} j$. In matrix terms, if $\mathbf{M}$ is the adjacency matrix for $\mathbf{G}$, then $\mathbf{J} - \mathbf{I} - \mathbf{M}$ is the adjacency matrix for $\mathbf{G}^*$. The following calculation is easy.

Lemma 2. *Let $\mathbf{G}$ be an ordinary graph of type $\langle n, r, a, b, c\rangle$. If $A \neq \emptyset$, $B \neq \emptyset$ and $C \neq \emptyset$, then $\mathbf{G}^*$ is an ordinary graph of type $\langle n, n-1-r, n+c-2r, n+b-2r-1, n+a-2r-2\rangle$.*

The modifications for when $A = \emptyset$ or $B = \emptyset$ or $C = \emptyset$ are straightforward. For example, if $A = \emptyset$ and B, $C \neq \emptyset$, so that $\mathbf{G}$ has type $\langle n, r, \times, b, c\rangle$, then $\mathbf{G}^*$ has type $\langle n, n-1-r, n+c-2r, n+b-2r-1, \times\rangle$.

Corollary 3. *Let* $\mathbf{G}$ *be an ordinary graph of type* $\langle n, r, a, b, c\rangle$.

(1) *If* $A \neq \emptyset$, *then* $n + a \geq 2r + 2$.
(2) *If* $B \neq \emptyset$, *then* $n + b \geq 2r + 1$.
(3) *If* $C \neq \emptyset$, *then* $n + c \geq 2r$.

2. Examples

We have several methods for constructing ordinary graphs and matrices, sometimes overlapping.

Symmetric (v, k, λ) **block designs.** These include complete loopless graphs (with adjacency matrix $\mathbf{J} - \mathbf{I}$), graphs whose adjacency matrix is a permutation matrix, and graphs whose adjacency matrix is the incidence matrix of a finite projective plane. This class also includes ordinary tournaments, discussed in the next section.

Set intersection graphs. These are simple examples of symmetric ordinary graphs. Let X be an N-element set, where $N \geq 2$, and let $X_{(2)}$ be the collection of all 2-element subsets of X.

(1) For A, $B \in X_{(2)}$, define $A \to B$ if $|A \cap B| = 1$. Then $X_{(2)}$ forms an ordinary graph of type $\langle \binom{N}{2}, 2N-4, 4, \times, N-2\rangle$.

(2) If $N \geq 4$, for A, $B \in X_{(2)}$, define $A \to B$ if $|A \cap B| = 0$. Then $X_{(2)}$ forms an ordinary graph of type $\langle \binom{N}{2}, \binom{N-2}{2}, \binom{N-3}{2}, \times, \binom{N-4}{2}\rangle$.

(3) If $N = 7$ or $N = 10$, then the collection $X_{(3)}$ of 3-element subsets of X, with the relation $A \to B$ if $|A \cap B| = 1$, is an ordinary graph of type $\langle 35, 18, 9, \times, 9\rangle$ or $\langle 120, 63, 36, \times, 30\rangle$, respectively.

Circulant ordinary matrices. A matrix $\mathbf{M} = (m_{ij})$ for $0 \leq i, j < n$ is said to be *circulant* if there is a sequence $\mathbf{e} = \langle e_0, \ldots, e_{n-1}\rangle$ such that $m_{ij} = e_{(j-i) \bmod n}$. Given a sequence $\mathbf{e}$ of 0's and 1's, let us denote the corresponding circulant matrix by $\mathbf{C}(\mathbf{e})$. This familiar construction allows us to build many examples of ordinary matrices. Indeed, it is simple to write a program which will generate the circulant matrices of a given size, and check which ones are ordinary.

Note that circulant matrices are normal: $\mathbf{C}(\mathbf{e})^t\mathbf{C}(\mathbf{e}) = \mathbf{C}(\mathbf{e})\mathbf{C}(\mathbf{e})^t$.

Table 1 lists all types of circulant ordinary matrices with $n \leq 8$, except that circulant permutation matrices and complete matrices $\mathbf{J} - \mathbf{I}$ are not listed. A selection of others of the hundreds of ordinary matrices we have generated in this way is given in Table 2. Of crucial importance to us later is the following well-known theorem of J. Singer [9]; see [4], Theorem 11.3.1.

Theorem 4. *For each prime power* q, *there is a circulant ordinary matrix representing the incidence matrix of a desarguean projective plane of order* q.

$\langle \mathbf{e} \rangle$	type
0110	$\langle 4, 2, \times, 1, 0 \rangle$
0101	$\langle 4, 2, 2, \times, 0 \rangle$
01001	$\langle 5, 2, 1, \times, 0 \rangle$
01110	$\langle 5, 3, \times, 2, 1 \rangle$
010100	$\langle 6, 2, 1, 0, 0 \rangle$
010010	$\langle 6, 2, 2, 0, \times \rangle$
001010	$\langle 6, 2, 0, \times, 1 \rangle$
011010	$\langle 6, 3, 2, 1, 1 \rangle$
010110	$\langle 6, 3, \times, 1, 2 \rangle$
010101	$\langle 6, 3, 3, \times, 0 \rangle$
011110	$\langle 6, 4, \times, 3, 2 \rangle$
011011	$\langle 6, 4, 4, \times, 2 \rangle$
0101000	$\langle 7, 2, 1, 0, \times \rangle$
0110100	$\langle 7, 3, \times, 1, \times \rangle$
0110010	$\langle 7, 3, 2, 1, 0 \rangle$
0110001	$\langle 7, 3, 1, 1, 1 \rangle$
0111010	$\langle 7, 4, \times, 2, 2 \rangle$
0111110	$\langle 7, 5, \times, 4, 3 \rangle$
00101000	$\langle 8, 2, 0, 1, 0 \rangle$
01101000	$\langle 8, 3, 1, 1, 0 \rangle$
01010100	$\langle 8, 3, 2, 0, 0 \rangle$
01001100	$\langle 8, 3, 0, 1, 2 \rangle$
00101010	$\langle 8, 3, 0, \times, 2 \rangle$
01110100	$\langle 8, 4, 2, 2, 1 \rangle$
01101010	$\langle 8, 4, 2, 1, 2 \rangle$
01100110	$\langle 8, 4, 4, 2, 0 \rangle$
01010101	$\langle 8, 4, 4, \times, 0 \rangle$
01110101	$\langle 8, 5, 4, 4, 2 \rangle$
01111110	$\langle 8, 6, \times, 5, 4 \rangle$

TABLE 1. Circulant ordinary matrices with $n \leq 8$

Other ordinary graphs. It is also not hard to generate small ordinary graphs which are symmetric or antisymmetric. In this way, we find one additional type of ordinary graphs of size at most 8, namely $\langle 8, 6, 6, \times, 4 \rangle$. One can also write a program which attempts to construct an ordinary graph of a fixed type. For example, there is an ordinary graph of type $\langle 25, 8, 3, 2, \times \rangle$. On the other hand, the program shows that apparently plausible types may fail to exist.

Lemma 5. *There is no* $\langle 13, 7, a, b, c \rangle$ *ordinary graph with* $a = 7$ *or* $\times$, $b = 4$ *or* $\times$, *and* $c = 1$ *or* $\times$.

$\langle \mathbf{e} \rangle$	type
000100100	$\langle 9, 2, 0, \times, 1 \rangle$
011010010	$\langle 9, 4, 3, 1, 1 \rangle$
010110100	$\langle 9, 4, 2, 1, 2 \rangle$
0100011000	$\langle 10, 3, 0, 1, 2 \rangle$
0010101000	$\langle 10, 3, 0, 2, 1 \rangle$
01110100100	$\langle 11, 5, 2, 2, 2 \rangle$
01101101000	$\langle 11, 5, 3, 2, 1 \rangle$
01011100010	$\langle 11, 5, \times, 2, \times \rangle$
0101000001000	$\langle 13, 3, 1, 0, \times \rangle$
0110100000100	$\langle 13, 4, 1, 1, \times \rangle$
0110001000001	$\langle 13, 4, 1, 1, 1 \rangle$
0111010100010	$\langle 13, 6, 4, 2, 3 \rangle$
0101100001101	$\langle 13, 6, 3, \times, 2 \rangle$
011101100101000	$\langle 15, 7, 3, 3, 3 \rangle$
011101011001000	$\langle 15, 7, 4, 3, 2 \rangle$
010110010010110	$\langle 15, 7, 6, 2, 5 \rangle$
010011100101001	$\langle 15, 7, 3, 6, 2 \rangle$
01111000101000011	$\langle 17, 8, 4, 3, 4 \rangle$
01110101001100100	$\langle 17, 8, 5, 3, 3 \rangle$
01101000110001011	$\langle 17, 8, 4, \times, 3 \rangle$
0100101101010000000	$\langle 19, 6, 3, 1, \times \rangle$
0111101010000110010	$\langle 19, 9, 4, 4, 4 \rangle$
0111010100010010110	$\langle 19, 9, 5, 3, 4 \rangle$
0111010100001101100	$\langle 19, 9, 5, 4, 3 \rangle$
0100111101010000110	$\langle 19, 9, \times, 4, \times \rangle$
011001000000000101000	$\langle 21, 5, 1, 1, \times \rangle$
01100001010000000010	$\langle 21, 5, 1, 1, 1 \rangle$
01000001100000100000	$\langle 21, 5, 0, 2, 4 \rangle$
00010010010010000000	$\langle 21, 5, 0, 4, 3 \rangle$
011011011001000011001	$\langle 21, 10, 7, 3, 4 \rangle$
01101101011010010010	$\langle 21, 10, 8, 3, 3 \rangle$
01011011010011010100	$\langle 21, 10, 4, 3, 7 \rangle$
01011010010010111010	$\langle 21, 10, 5, 3, 6 \rangle$

TABLE 2. More examples of circulant ordinary matrices

ORDINARY TOURNAMENTS

A *tournament* is a loopless, directed graph such that for each pair i, j of distinct vertices, exactly one of the relations $i \to j$ or $j \to i$ holds. Thus a square 0-1 matrix $\mathbf{M}$ is the adjacency matrix of a tournament if and only if $\mathbf{M} + \mathbf{M}^t + \mathbf{I} = \mathbf{J}$.

An ordinary tournament has type $\langle n, r, \times, b, \times\rangle$. An easy application of Lemma 1 shows that $r = 2b+1$ and $n = 2r+1 = 4b+3$. Thus we have the following result.

Lemma 6. *An ordinary tournament has type* $\langle 4k+3, 2k+1, \times, k, \times\rangle$ *for some integer* $k \geq 0$. *Its adjacency matrix satisfies*

$$\mathbf{M} + \mathbf{M}^t + \mathbf{I} = \mathbf{J}$$
$$\mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M} = k\mathbf{J} + (k+1)\mathbf{I}.$$

In this case, we say that $\mathbf{M}$ is an ordinary tournament *of order* k. By a theorem of H. Ryser, if a square 0-1 matrix of size $4k+3$ satisfies $\mathbf{M} + \mathbf{M}^t + \mathbf{I} = \mathbf{J}$ and $\mathbf{M}\mathbf{M}^t = k\mathbf{J} + (k+1)\mathbf{I}$, then $\mathbf{M}^t\mathbf{M} = \mathbf{M}\mathbf{M}^t$, and hence $\mathbf{M}$ is the adjacency matrix of an ordinary tournament of order k (see e.g. [4], Theorem 10.2.3).

A fundamental question is: *Does there exist an ordinary tournament of order* k *for every* $k \geq 0$*?* Recall that a *Hadamard design* of order k is a square 0-1 matrix $\mathbf{H}$ of size $4k+3$ such that $\mathbf{H}\mathbf{H}^t = \mathbf{H}^t\mathbf{H} = k\mathbf{J} + (k+1)\mathbf{I}$. Since it is unknown whether Hadamard designs of order k exist for every k, we certainly don't know whether ordinary tournaments of every order exist. However, we raise the possibility that the problem may be easier for this more restricted class.

Quadratic residue tournaments. A classic construction due to R. Paley yields a nice infinite class of ordinary tournaments.

Theorem 7. *If* $4k+3$ *is a prime power, then there is an ordinary tournament of order* k.

Proof. Assume that $4k+3 = p^\alpha$. Define a sequence $\mathbf{e} = \langle e_i : i \in \mathrm{GF}(p^\alpha)\rangle$ by

$$e_i = \begin{cases} 1 & \text{if } i \text{ is a nonzero quadratic residue in } \mathrm{GF}(p^\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Then let $\mathbf{M}$ be the matrix such that $m_{ij} = e_{(j-i)}$. It is well known that $\mathbf{M}$ is the matrix of a Hadamard design, which is in fact a tournament since α is odd, so that -1 is not a quadratic residue in $\mathrm{GF}(p^\alpha)$. (If $\alpha = 1$ then $\mathbf{M}$ will be circulant, while if $\alpha > 1$ then $\mathbf{M}$ represents a difference set on a non-cyclic abelian group.) □

Thus, for example, we obtain ordinary tournaments for orders $k = 0$, 1, 2, 4, 5, 6, 7, 10, and 11. Our second main construction (below) will yield ordinary tournaments of orders 3 and 9. Thus far we have been unable to construct ordinary tournaments of orders 8 ($n = 35$) and 12 ($n = 51$).

The complement of a tournament is usually referred to as its *dual*. The dual is represented by the matrix $\mathbf{M}^t$, and a tournament is said to be *self-dual* if there is a permutation matrix $\mathbf{P}$ such that $\mathbf{M}^t = \mathbf{P}^t\mathbf{M}\mathbf{P}$. Ordinary tournaments obtained from quadratic residues are self-dual, with the map $i \to -i$ inducing the isomorphism.

One can also do the quadratic residue construction for the case when $n = 4k+1$ is prime, which yields an ordinary matrix of type $\langle 4k+1, 2k, k, \times, k-1\rangle$. Other residue systems modulo n (e.g., cubic residues) occasionally produce ordinary matrices.

Isomorphism types. It is reasonable to ask how many different ordinary tournaments of order k there are. For k small, this is amenable to computer searches.

Theorem 8. *The following list gives isomorphism types of ordinary tournaments of order k.*

0. *For $k = 0$ there is 1 isomorphism type.*
1. *For $k = 1$ there is 1 isomorphism type.*
2. *For $k = 2$ there is 1 isomorphism type.*
3. *For $k = 3$ there are 2 isomorphism types (dual to each other).*
4. *For $k = 4$ there are at least 2 isomorphism types.*
5. *For $k = 5$ there are at least 28 isomorphism types.*
6. *For $k = 6$ there are at least 20 isomorphism types.*

Constructing order $2k+1$ from order k. Another standard construction for Hadamard designs also works for ordinary tournaments. This method allows us to construct an ordinary tournament of order $2k + 1$ from one (or more) of order k. Let $\mathbf{0}$ (resp. $\mathbf{1}$) represent a column vector of 0's (resp. 1's) of length $4k + 3$.

Theorem 9. *Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be ordinary tournament matrices of order k. If $\mathbf{B}\mathbf{A}^t = \mathbf{C}\mathbf{B}$, then*

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}+\mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{1} \\ \mathbf{1}^t & \mathbf{0}^t & 0 \end{bmatrix}$$

is an ordinary tournament matrix of order $2k + 1$. In particular, we may take $\mathbf{B} = \mathbf{A}$ and $\mathbf{C} = \mathbf{A}^t$ to construct an ordinary tournament of order $2k + 1$ whenever there is one of order k.

The proof is a straightforward matrix calculation.

The structure of ordinary tournaments. In this section we discuss some of the detailed structure of ordinary tournaments, which is useful in constructions.

Let $\mathbf{T}$ be a tournament. For any $a, b \in \mathbf{T}$ we denote

$$\begin{aligned} \xi_{oo}^{ab} &= |\{x \in T - \{a,b\} : a \to x,\ b \to x\}|, \\ \xi_{oi}^{ab} &= |\{x \in T - \{a,b\} : a \to x,\ x \to b\}|, \\ \xi_{io}^{ab} &= |\{x \in T - \{a,b\} : x \to a,\ b \to x\}|, \\ \xi_{ii}^{ab} &= |\{x \in T - \{a,b\} : x \to a,\ x \to b\}|. \end{aligned}$$

Similarly, for $a, b, c \in \mathbf{T}$ we define

$$\xi_{oii}^{abc} = |\{x \in T - \{a,b,c\} : a \to x,\ x \to b,\ x \to c\}|,$$

etc. (The index i stands for "into", while o stands for "out of", and refers to the corresponding upper index.)

Lemma 10. *Let* $\mathbf{T}$ *be an ordinary tournament of order* k*, and let* $a, b \in \mathbf{T}$ *be such that* $a \to b$*. Then:*

$$\begin{aligned}
&\xi^{ab}_{ii} = k,\\
&\xi^{ab}_{oo} = k,\\
&\xi^{ab}_{oi} = k,\\
&\xi^{ab}_{io} = k+1.
\end{aligned}$$

Proof. The first two statements follow from definition. Denote the third and the fourth number by x and y, respectively. Then $k + x + 1 = 2k + 1$ and $k + y = 2k + 1$, from which we get $x = k$ and $y = k + 1$. □

Lemma 11. *Let* $\mathbf{T}$ *be an ordinary tournament of order* k*, and let* $a, b, c \in \mathbf{T}$ *be pairwise distinct. Then one of the following two cases takes place:*

(1) $\{a, b, c\}$ *is a* 3*-cycle, and* $\xi^{abc}_{ooo} + \xi^{abc}_{iii} = k$*;*
(2) $\{a, b, c\}$ *is a chain, and* $\xi^{abc}_{ooo} + \xi^{abc}_{iii} = k - 1$.

Proof. Clearly, a 3-element subset of a tournament is either a 3-cycle or a chain.

(1) Let $\{a, b, c\}$ be a 3-cycle. Without loss of generality, $a \to b \to c \to a$. Then $\xi^{abc}_{iii} + \xi^{abc}_{ioi} = k$ and $\xi^{abc}_{ioi} + \xi^{abc}_{ooi} = k$, so that $\xi^{abc}_{iii} = \xi^{abc}_{ooi}$. Now $\xi^{abc}_{ooi} + \xi^{abc}_{ooo} = k$, and hence $\xi^{abc}_{iii} + \xi^{abc}_{ooo} = k$.

(2) Let $\{a, b, c\}$ be a chain. Without loss of generality, $a \to b \to c$ and $a \to c$. Then $\xi^{abc}_{iii} + \xi^{abc}_{ioi} = k$ and $\xi^{abc}_{ioi} + \xi^{abc}_{ooi} = k$, so that $\xi^{abc}_{iii} = \xi^{abc}_{ooi}$. Now $\xi^{abc}_{ooi} + \xi^{abc}_{ooo} = k - 1$, and hence $\xi^{abc}_{iii} + \xi^{abc}_{ooo} = k - 1$. □

Lemma 12. *Let* $\mathbf{T}$ *be an ordinary tournament of order* k*. If* $k \geq 1$*, then* $\mathbf{T}$ *contains a* 3*-element subchain. If* $k \geq 2$*, then every* 3*-element subchain of* $\mathbf{T}$ *can be extended to a* 4*-element subchain; in particular,* $\mathbf{T}$ *contains a* 4*-element subchain.*

Proof. It follows from the previous two lemmas. □

Lemma 13. *Let* $\mathbf{T}$ *be an ordinary tournament of order* k *containing a* 4*-element subchain* $a \to b \to c \to d$ *(also,* $a \to c$*, etc.) which cannot be extended to a* 5*-element subchain. Then*

$$\begin{aligned}
&\xi^{abcd}_{iiio} = k-2,\ \xi^{abcd}_{iioi} = 1, && \xi^{abcd}_{iioo} = 1, && \xi^{abcd}_{ioii} = k-2,\\
&\xi^{abcd}_{ioio} = 4-k,\ \xi^{abcd}_{iooi} = 1, && \xi^{abcd}_{iooo} = k-2,\ \xi^{abcd}_{oiio} = 1,\\
&\xi^{abcd}_{oioi} = k-2,\ \xi^{abcd}_{oioo} = 1, && \xi^{abcd}_{ooio} = k-2.
\end{aligned}$$

Proof. The fact that the 4-element subchain cannot be extended to a 5-element one means that $\xi^{abcd}_{iiii} = \xi^{abcd}_{oiii} = \xi^{abcd}_{ooii} = \xi^{abcd}_{oooi} = \xi^{abcd}_{oooo} = 0$. There are 11 remaining numbers ξ^{abcd}. We can compute the numbers ξ^{uv}_{ii} and ξ^{uv}_{oo} for any 2-element subset $\{u, v\}$ of $\{a, b, c, d\}$ from these 11 unknowns, and also compute ξ^{a}_{i}, obtaining 13 equations in the 11 unknowns. It is easy to solve this system of linear equations; it has precisely one solution. □

Lemma 14. *Let* $\mathbf{T}$ *be an ordinary tournament of order* $k \geq 4$*. Then* $\mathbf{T}$ *contains a* 5*-element subchain.*

Proof. Suppose that $\mathbf{T}$ contains no 5-element subchain. By Lemma 12 there exists a 4-element subchain $a \to b \to c \to d$. It follows from Lemma 13 that $k \leq 4$, and hence $k = 4$. Then, by Lemma 13, $\xi_{iiio}^{abcd} = 2$. This means that there are two different elements e, f such that $e \to a \to b \to c$ and $f \to a \to b \to c$ are subchains. Without loss of generality, $e \to f$. But then $e \to f \to a \to b \to c$ is a 5-element subchain. □

Lemma 15. *Let* $\mathbf{T}$ *be an ordinary tournament of order* $k \geq 3$*. Then* $\mathbf{T}$ *contains a* 5*-element subchain.*

Proof. It remains to consider the case $k = 3$. There are two isomorphism types in this case; both were checked to contain 5-element subchains by a computer. □

Denote by M the maximal cardinality of a subchain of an ordinary tournament of order k. Using a computer, we were able to find the number M in these cases:

For $k = 0$, $M = 2$.
For $k = 1$, $M = 3$.
For $k = 2$, $M = 4$.
For $k = 3$, $M = 5$ (for both isomorphism types).
For $k = 4$, $M = 5$ (for both known isomorphism types).
For $k = 5$, $M = 6$ for all the 28 known isomorphism types.
For $k = 6$, for 3 isomorphism types we have $M = 7$ while for the remaining 17 known isomorphism types we have $M = 6$.

Theorem 16. *Every ordinary tournament is a cycle.*

Proof. Let $\mathbf{T}$ be an ordinary tournament of order k, and let C be a subcycle of $\mathbf{T}$ of a maximal possible cardinality. Put $p = |C|$; clearly, $p \geq 3$. We have $C = \{c_0, \ldots, c_{p-1}\}$ where $c_0 \to c_1 \to \cdots \to c_{p-1} \to c_0$. Let $e \in T - C$. If there are two indexes i and j with $c_i \to e$ and $e \to c_j$, then there are two such indexes with $j \equiv i + 1 \mod p$, and we obtain a longer subcycle if e is inserted between c_i and c_j, a contradiction. This means that the set complement $T - C$ is the disjoint union $T - C = A \cup B$, where $c \to a$ and $b \to c$ for all $a \in A$, $b \in B$, $c \in C$. If $a \to b$ for some $a \in A$ and $b \in B$, then we obtain a longer subcycle if we insert $c_0 \to a \to b \to c_1$ between c_0 and c_1, a contradiction. Hence $b \to a$ for all $a \in A$ and $b \in B$. For $b \in B$ we have $C \cup A \subseteq \uparrow b$; consequently, if B is nonempty then $|C| + |A| \leq 2k + 1$. Similarly, if A is nonempty then $|C| + |B| \leq 2k + 1$. If both A and B are nonempty, we conclude $2|C| + |A| + |B| \leq 4k + 2 < 4k + 3 = |C| + |A| + |B|$, a contradiction. So, either $A = \emptyset$ or $B = \emptyset$. If A is nonempty, we get $|C| \leq 2k + 1$, so that $|A| \geq 2k + 2$, so that for $c \in C$ we have $|\uparrow c| \geq 2k + 2$, a contradiction. Hence $A = \emptyset$. Similarly, $B = \emptyset$. □

Every tournament can be considered as a groupoid, with respect to the multiplication defined as follows: $aa = a$ for all a; if $a \to b$ then $ab = ba = a$.

Theorem 17. *Every ordinary tournament is a simple groupoid.*

Proof. Let $\mathbf{T}$ be an ordinary tournament of order k, and let r be a non-identical congruence of $\mathbf{T}$. There is at least one non-singleton block B of r. Take two different elements $a, b \in B$. Without loss of generality, $a \to b$. Clearly, if x is an element such that either $b \to x \to a$ or $a \to x \to b$, then $x \in B$. There are $(k+1)+k = 2k+1$ such elements x, and hence $|B| \geq 2k+3$. So, every non-singleton block of r contains more than a half of the elements of T. It follows that for any $c \in T - B$, $\{c\}$ is a block of r. But for any such c, either $c \to x$ for all $x \in B$ or $x \to c$ for all $x \in B$, and either the out- or the in-degree of c is too large. Hence $B = T$. □

3. Subplane partitions and ordinary graphs

Let $\Pi = \langle P, L, \leq \rangle$ be a projective plane of order n (so that $|P| = |L| = n^2+n+1$, each line contains $n+1$ points and each point lies on $n+1$ lines). Suppose there is a partition of Π into equal sized subplanes $\Pi_i = \langle P_i, L_i, \leq_i \rangle$ $(0 \leq i < N)$ of order q. Since $n^2+n+1 = N(q^2+q+1)$, it is necessary that n^2+n+1 is divisible by q^2+q+1. For example, we could have $n = 18$ and $q = 2$; in this case, $N = 49$ as $343 = 49 \cdot 7$. Such a partition is called a *subplane partition* of Π.

We will write $i \to j$ if $i \neq j$ and there exist $p \in P_i$ and $\ell \in L_j$ with $p \leq \ell$. A subplane partition of Π will be called *ordinary* if $i \to j$ implies that for every $p \in P_i$ there exists an $\ell \in L_j$ with $p \leq \ell$. In that case, for each pair (i, j) with $i \to j$, we obtain a bijection λ_{ij} of P_i onto L_j by defining $\lambda_{ij}(p) = \ell$ where $\ell \in L_j$ and $p \leq \ell$. In other words, $\lambda_{ij} = \leq |_{P_i \times L_j}$.

We want to show that $\langle N, \to \rangle$, with the arrow relation defined in the preceding paragraph, is an ordinary graph whenever $\Pi = \bigcup_{0 \leq i < N} \Pi_i$ is an ordinary partition.

Theorem 18. *Let $\Pi_i = \langle P_i, L_i, \leq_i \rangle$ $(0 \leq i < N)$ be an ordinary partition of a projective plane Π of order n into subplanes of order q. The following are true:*

(1) *For each $i = 0, \ldots, N-1$, $|\uparrow i| = |\downarrow i| = n - q$.*
(2) *If $i \, A \, j$, then $|\uparrow i \cap \uparrow j| = |\downarrow i \cap \downarrow j| = q^2 + q + 1$.*
(3) *If $i \, B \, j$, then $|\uparrow i \cap \uparrow j| = |\downarrow i \cap \downarrow j| = q^2$.*
(4) *If $i \, C \, j$, then $|\uparrow i \cap \uparrow j| = |\downarrow i \cap \downarrow j| = q^2 - q - 1$.*

Thus $\langle N, \to \rangle$ is an ordinary graph of type $\langle N, n-q, a, b, c \rangle$ with $a = q^2+q+1$ or $\times$, $b = q^2$ or $\times$, and $c = q^2 - q - 1$ or $\times$.

Proof. By duality, it is sufficient to find the cardinality of the first set in each case.

(1) Every point of P_i lies on $n+1$ lines, where $k+1$ of them belong to L_i; clearly, the remaining $n-k$ ones belong to pairwise different L_j's.

(2) Denote by $p_1, \ldots, p_{q^2+q+1}$ the points of P_i and by $r_1, \ldots, r_{q^2+q+1}$ the points of P_j. For every $u \in \{1, \ldots, q^2+q+1\}$ define k_u by $p_1 \vee r_u \in L_{k_u}$. It follows from the assumptions that these numbers k_u are pairwise distinct, and different from i and j; their number is q^2+q+1, and each has the property $i \to k_u \& j \to k_u$. Now every line from $L_{k_1} \cup \cdots \cup L_{k_{q^2+q+1}}$ (there are $(q^2+q+1)^2$ such lines) contains a point from P_i and a point from P_j, and hence can be expressed as $p_u \vee r_v$ for some $u \in P_i$ and $v \in P_j$. The number of lines that can be expressed in this form is, of course, at most $(q^2+q+1)^2$. So, every line $p_u \vee r_v$ with $u \in P_i$ and $v \in P_j$ belongs to $L_{k_1} \cup \cdots \cup L_{k_{q^2+q+1}}$. Consequently, if k is such that $i \to k \& j \to k$, then $k \in \{k_1, \ldots, k_{q^2+q+1}\}$.

(3) Let $i \to j$ and $j \not\to i$. Take a point $p \in P_i$. There is precisely one line in L_j containing p; this line contains precisely $q+1$ points of P_j; denote the remaining points of P_j by $r_1, \ldots, r_{q^2}$. Define $k_1, \ldots, k_{q^2}$ by $p \vee r_u \in L_{k_u}$. We can proceed similarly as in the previous case to show that these numbers $k_1, \ldots, k_{q^2}$ are the only numbers k with the property that $i \to k$ and $j \to k$.

(4) Let $i \to j$ and $j \to i$. Take a point $p \in P_i$. There is precisely one line $A \in L_j$ containing p. This line contains precisely $q+1$ points of P_j. Also, p belongs to $q+1$ lines of L_i, each of which contains precisely one point of P_j, and this point does not belong to A. So, there are precisely $q^2+q+1-(q+1)-(q+1) = q^2-q-1$ points r of P_j such that the line $p \vee r$ does not belong to either L_i or L_j. Denote these points by $r_1, \ldots, r_{q^2-q-1}$ and define $k_1, \ldots, k_{q^2-q-1}$ by $p \vee r_u \in L_{k_u}$. As in the proof of property (2), one can show that these are the only numbers k with $i \to k$ and $j \to k$. □

We also need to describe a basic property of the maps λ_{ij}.

Lemma 19. *Let $\Pi_i = \langle P_i, L_i, \leq_i \rangle$ for $0 \leq i < N$ be an ordinary partition of a projective plane Π of order n into subplanes of order q. If $i \neq j$ and $p \in P_i$ and $r \in P_j$, then exactly one of the following holds:*

(1) *$r \leq_j \lambda_{ij}(p)$, or*
(2) *$p \leq_i \lambda_{ji}(r)$, or*
(3) *there exists k such that $\lambda_{ik}(p) = \lambda_{jk}(r)$.*

The condition of this lemma just makes each pair of points join to a unique line. It follows from general principles that this also makes each pair of lines meet in a unique point; see e.g. [3], Theorem 20.8.1.

Conversely, if the conditions of Theorem 18 and Lemma 19 are satisfied, we can construct a projective plane of order n. We formulate this as follows.

Theorem 20. *Assume that $n^2+n+1 = N(q^2+q+1)$, and that there exists a projective plane Ψ of order q. Let $\Pi_i = \langle P_i, L_i, \leq_i \rangle$ for $0 \leq i < N$ be disjoint copies of Ψ. Suppose*

(1) *there exists an ordinary graph $\langle N, \to \rangle$ of type $\langle N, n-q, a, b, c \rangle$ with $a = q^2+q+1$ or $\times$, $b = q^2$ or $\times$, and $c = q^2-q-1$ or $\times$, and*

(2) *there exist bijections* $\lambda_{ij} : P_i \to L_j$ *for each pair with* $i \to j$ *such that if* $i \neq j$ *and* $p \in P_i$ *and* $r \in P_j$*, then exactly one of the following holds:*
   (a) $r \leq_j \lambda_{ij}(p)$*, or*
   (b) $p \leq_i \lambda_{ji}(r)$*, or*
   (c) *there exists* k *such that* $\lambda_{ik}(p) = \lambda_{jk}(r)$*.*

Then there exists a projective plane of order n *with an ordinary subplane partition into subplanes of order* q*.*

The construction and proof are entirely straightforward in view of the previous discussion, *viz.*, if $p \in P_i$ and $\ell \in L_j$, then $p \leq \ell$ if and only if $i = j$ and $p \leq_i \ell$, or $i \to j$ and $\ell = \lambda_{ij}(p)$. In the next section we will give a matrix version of the construction.

Now we have several tasks to address. The first is to identify pairs n, q which are candidates. If $n \equiv q \mod q^2 + q + 1$ or $n \equiv q^2 \mod q^2 + q + 1$ then $q^2 + q + 1$ divides $n^2 + n + 1$. In the first case it is evident, and in the second case $n^2 + n + 1 \equiv q^4 + q^2 + 1 = (q^2 + q + 1)(q^2 - q + 1) \equiv 0 \mod (q^2 + q + 1)$. If $q^2 + q + 1$ is prime, then the converse is true: $q^2 + q + 1$ divides $n^2 + n + 1$ only if $n \equiv q$ or $q^2 \mod q^2 + q + 1$.

Table 3 gives a list of candidates for $n \leq 60$. We have omitted those n for which no plane of order n exists by the Bruck-Ryser theorem.

The second task, given n and q, is to find an ordinary graph of the required type. Two natural cases immediately present themselves.

The first natural case is when q is a prime power and $n = q^2$. R. H. Bruck proved that the desarguean plane of order q^2 always has a subplane partition into subplanes of order q, based on J. Singer's result that any finite desarguean plane may be derived from a difference set [1]. Peter Yff showed that these planes may have other partitions into subplanes of order q; see [10], [11]. In each case, $N = q^2 - q + 1$ and the corresponding graph is complete of type $\langle N, N - 1, \times, \times, N - 2\rangle$. This still leaves open the possibility that nondesarguean planes of order q^2 may have a subplane partition.

The second natural case is when q is a prime power and $n = 2q^2 + q + 1$. In this case we could use an ordinary tournament of order q^2, which has type $\langle 4q^2 + 3, 2q^2 + 1, \times, q^2, \times\rangle$. For $q = 2$ we have $n = 11$, an intriguing possibility. However, any attempt to construct a nondesarguean plane of order 11 must bear in mind the known constraints; see, e.g., [5] or [6], and Theorem 25 below. For $q = 3$ we have $n = 22$, which is eliminated by the Bruck-Ryser theorem. The cases $q = 4$, $n = 37$ and $q = 5$, $n = 56$ seem *a priori* promising. The methods we have used to attempt these constructions will be described in the next section.

The third, and most difficult task, given candidates n, q and an appropriate ordinary graph, is to find bijections λ_{ij} satisfying the required property. One method for attempting this will also be given in the next section.

n	n^2+n+1	q	N	comments
4	$21 = 7 \cdot 3$	2	3	works
9	$91 = 13 \cdot 7$	3	7	works
9	$91 = 7 \cdot 13$	2	13	no, type $\langle 13, 7, 7, 4, 1\rangle$ DNE
11	$133 = 7 \cdot 19$	2	19	need type $\langle 19, 9, 7, 4, 1\rangle$
16	$273 = 21 \cdot 13$	4	13	works
16	$273 = 13 \cdot 21$	3	21	need type $\langle 21, 13, 13, 9, 5\rangle$
18	$343 = 7 \cdot 49$	2	49	need type $\langle 49, 16, 7, 4, 1\rangle$
23	$553 = 7 \cdot 79$	2	79	need type $\langle 79, 21, 7, 4, 1\rangle$
25	$651 = 31 \cdot 21$	5	21	works
25	$651 = 21 \cdot 31$	4	31	need type $\langle 31, 21, 21, 16, 11\rangle$
25	$651 = 7 \cdot 93$	2	93	need type $\langle 93, 23, 7, 4, 1\rangle$
29	$871 = 13 \cdot 67$	3	67	need type $\langle 67, 26, 13, 9, 5\rangle$
32	$1057 = 7 \cdot 151$	2	151	need type $\langle 151, 30, 7, 4, 1\rangle$
35	$1261 = 13 \cdot 97$	3	97	need type $\langle 97, 32, 13, 9, 5\rangle$
36	$1333 = 31 \cdot 43$	5	43	need type $\langle 43, 31, 31, 25, 19\rangle$
37	$1407 = 21 \cdot 67$	4	67	need type $\langle 67, 33, 21, 16, 11\rangle$
37	$1407 = 7 \cdot 201$	2	201	need type $\langle 201, 35, 7, 4, 1\rangle$
39	$1561 = 7 \cdot 223$	2	223	need type $\langle 223, 37, 7, 4, 1\rangle$
42	$1807 = 13 \cdot 139$	3	139	need type $\langle 139, 39, 13, 9, 5\rangle$
48	$2353 = 13 \cdot 181$	3	181	need type $\langle 181, 45, 13, 9, 5\rangle$
49	$2451 = 57 \cdot 43$	7	43	works
51	$2653 = 7 \cdot 379$	2	379	need type $\langle 379, 49, 7, 4, 1\rangle$
53	$2863 = 7 \cdot 409$	2	409	need type $\langle 409, 51, 7, 4, 1\rangle$
55	$3081 = 13 \cdot 237$	3	237	need type $\langle 237, 52, 13, 9, 5\rangle$
56	$3193 = 31 \cdot 103$	5	103	need type $\langle 103, 51, 31, 25, 19\rangle$
58	$3423 = 21 \cdot 163$	4	163	need type $\langle 163, 54, 21, 16, 11\rangle$
58	$3423 = 7 \cdot 489$	2	489	need type $\langle 489, 56, 7, 4, 1\rangle$
60	$3661 = 7 \cdot 523$	2	523	need type $\langle 523, 58, 7, 4, 1\rangle$

TABLE 3. Possible combinations for special subplane partitions

4. Matrix interpretation

Now suppose that n and q are given, and that we can find an ordinary matrix $\mathbf{M}$ of type $\langle N, n-q, a, b, c\rangle$, where as usual $N = \frac{n^2+n+1}{q^2+q+1}$, $a = q^2+q+1$ or $\times$, $b = q^2$ or $\times$, and $c = q^2-q-1$ or $\times$. Thus $\mathbf{M}$ is an $N \times N$ 0-1 matrix satisfying certain conditions on $\mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M}$ involving $n-q$, a, b and c. Let $\mathbf{P}$ be the incidence matrix of a projective plane of order q, i.e., $\mathbf{P}$ is a square 0-1 matrix of size q^2+q+1 satisfying $\mathbf{P}\mathbf{P}^t = \mathbf{P}^t\mathbf{P} = q\mathbf{I} + \mathbf{J}$.

Our goal is to construct the incidence matrix of a projective plane of order n, i.e., a square 0-1 matrix $\mathbf{H}$ of size n^2+n+1 satisfying $\mathbf{H}\mathbf{H}^t = n\mathbf{I} + \mathbf{J}$. (This implies that $\mathbf{H}^t\mathbf{H} = \mathbf{H}\mathbf{H}^t$.) The construction requires that we find suitable permutation matrices $\mathbf{E}_{ij}$ for each pair (i, j) with $i \to j$, which must

satisfy some matrix equations given below. The matrices $\mathbf{E}_{ij}$ will be square of size q^2+q+1.

Define $\mathbf{H}$ to be a block matrix $\mathbf{H}=(\mathbf{H}_{ij})$ for $0\le i,j<N$ where

$$\mathbf{H}_{ij}=\begin{cases}\mathbf{P} & \text{if } i=j,\\ \mathbf{E}_{ij} & \text{if } m_{ij}=1,\\ \mathbf{O} & \text{if } i\ne j \text{ and } m_{ij}=0.\end{cases}$$

In terms of this block construction, $(\mathbf{HH}^t)_{ij}=\sum_k \mathbf{H}_{ik}\mathbf{H}_{jk}^t$. Our objective is to have $\mathbf{HH}^t=n\mathbf{I}+\mathbf{J}$, and hence

$$\sum_{0\le k<N}\mathbf{H}_{ik}\mathbf{H}_{jk}^t=\begin{cases}n\mathbf{I}+\mathbf{J} & \text{if } i=j,\\ \mathbf{J} & \text{if } i\ne j.\end{cases}$$

Since the matrices $\mathbf{E}_{ik}$ are chosen to be permutation matrices, $\mathbf{E}_{ik}\mathbf{E}_{ik}^t=\mathbf{I}$. There are $n-q$ permutation matrices in each "row" of $\mathbf{H}$, while $\mathbf{PP}^t=q\mathbf{I}+\mathbf{J}$. Thus any choice of the matrices $\mathbf{E}_{ij}$ will satisfy the equations for $i=j$, and the difficulty lies with the off-diagonal block positions. We record this as follows.

Theorem 21. *The construction above will yield the incidence matrix of a projective plane of order n precisely when we can find permutation matrices $\mathbf{E}_{st}$ for $s\to t$ such that, for all pairs $(i,j)\in N^2$ with $i\ne j$, we have $\sum_k \mathbf{H}_{ik}\mathbf{H}_{jk}^t=\mathbf{J}$.*

Now the equations $\sum_k \mathbf{H}_{ik}\mathbf{H}_{jk}^t=\mathbf{J}$, with each $\mathbf{H}_{ik}$ either $\mathbf{P}$, $\mathbf{O}$, or an unknown permutation matrix $\mathbf{E}_{ik}$, are too unwieldy to handle. Therefore we will consider a sequence of simplifying assumptions, in hopes that we do not lose all the solutions in the process.

Assumption A. Assume that $\mathbf{P}$ is a circulant matrix, $\mathbf{P}=\mathbf{C}(\mathbf{p})$. As long as our plane of order q is desarguean, there is no loss of generality in this. Moreover, it allows us to handle $\mathbf{P}^t$ effectively.

Lemma 22. *If $\mathbf{P}=\mathbf{C}(p_0,\dots,p_{m-1})$, then $\mathbf{P}^t=\mathbf{C}(p_0,p_{m-1},p_{m-2},\dots,p_1)$, where $m=q^2+q+1$.*

Assumption B. Assume that $\mathbf{M}$ is a circulant matrix, $\mathbf{M}=\mathbf{C}(\mathbf{f})$. This is of course a loss of generality, because not every ordinary matrix is equivalent to a circulant one. Nonetheless, this class provides enough examples to keep us busy.

Assumption C. If $\mathbf{M}$ is circulant, then we may take $\mathbf{H}$ to be block-circulant:

$$\mathbf{H}_{ij}=\begin{cases}\mathbf{P} & \text{if } i=j,\\ \mathbf{F}_k & \text{if } j-i\equiv k \mod N \text{ and } f_k=1,\\ \mathbf{O} & \text{if } j-i\equiv k \mod N \text{ and } f_k=0,\end{cases}$$

where the matrices $\mathbf{F}_k$ are unknown permutation matrices.

This tactic greatly reduces the number of equations to be considered: we may fix $i = 0$ and just solve the equations $\sum_k \mathbf{H}_{0k}\mathbf{H}_{jk}^t = \mathbf{J}$ for $1 \le j < N$. But there is an added benefit: the equations for j and $N - j$ are transposes of each other. (This is a straightforward calculation.) Thus, if we take $\mathbf{H}$ to be block circulant as above, it suffices to solve the $\lceil \frac{N-1}{2} \rceil = \lfloor \frac{N}{2} \rfloor$ equations $\sum_k \mathbf{H}_{0k}\mathbf{H}_{jk}^t = \mathbf{J}$ for $1 \le j \le \lfloor \frac{N}{2} \rfloor$.

The cost is that we have introduced a lot of symmetry: the map $\varphi : h_{ij} \mapsto h_{i+t,j+t}$ where $t = q^2 + q + 1$ is an automorphism of the matrix $\mathbf{H}$, and hence of the corresponding projective plane. An automorphism of order N is not necessarily fatal to our attempt to construct a nondesarguean plane of order n, but it is certainly a significant restriction.

If we want to construct a projective plane of non-prime-power order, then we should stop with (at most) the first three assumptions, and allow the permutation matrices $\mathbf{F}_k$ to be arbitrary. However, in order to illustrate the general method, let us proceed and consider systems with an additional property.

Assumption D. Let $\mathbf{S} = \mathbf{C}(0100\ldots0)$, corresponding to the cyclic permutation on $t = q^2 + q + 1$. Assume that $\mathbf{F}_k = \mathbf{S}^{x_k}$ when $f_k = 1$, where each x_k is an unknown with $0 \le x_k < t$, and of course $\mathbf{F}_k = \mathbf{O}$ when $f_k = 0$.

Since desarguean planes are also cyclic, with Assumption A this introduces additional symmetries, *viz.,* a cyclic automorphism ψ of order t which permutes the indices within the blocks of $\mathbf{H}$ simultaneously. Moreover, ψ commutes with φ. The restrictions imposed by this assumption are significant, and will be discussed in the final section.

With Assumption D, the problem reduces to solving at least one member of a set of (possibly inconsistent) linear equations in $\mathbb{Z}_{q^2+q+1}$. The following lemma is crucial to our calculations.

Lemma 23. *Let* $\mathbf{S} = \mathbf{C}(0100\ldots0)$.

(1) $(\mathbf{S}^x)^t = \mathbf{S}^{-x}$.
(2) $\mathbf{S}$ *commutes with circulant matrices:*

$$\mathbf{SC}(\mathbf{e}) = \mathbf{C}(\mathbf{e})\mathbf{S} = \mathbf{C}(e_{m-1}, e_0, e_1, \ldots, e_{m-2}).$$

Now let us illustrate how these constructions work with several examples. We begin by reversing Bruck's decomposition of a desarguean plane of order q^2 into subplanes of order q. For small values of q, at least, it is feasible to build a plane of order q^2 as a union of planes of order q.

Example 1: Constructing a plane of order 4. In this case $\mathbf{M} = \mathbf{C}(011)$, we may take $\mathbf{P} = \mathbf{C}(0110100)$, and $\mathbf{S} = \mathbf{C}(0100000)$. The matrix $\mathbf{H}$ then has the form

$$\mathbf{H} = \begin{bmatrix} \mathbf{P} & \mathbf{S}^{x_1} & \mathbf{S}^{x_2} \\ \mathbf{S}^{x_2} & \mathbf{P} & \mathbf{S}^{x_1} \\ \mathbf{S}^{x_1} & \mathbf{S}^{x_2} & \mathbf{P} \end{bmatrix}$$

with x_1 and x_2 unknown integers modulo 7. Then $\mathbf{HH}^t =$

$$\begin{bmatrix} \mathbf{PP}^t + 2\mathbf{I} & \mathbf{PS}^{-x_2} + \mathbf{S}^{x_1}\mathbf{P} + \mathbf{S}^{x_2-x_1} & \mathbf{PS}^{-x_1} + \mathbf{S}^{x_1-x_2} + \mathbf{S}^{x_2}\mathbf{P}^t \\ \mathbf{S}^{x_2}\mathbf{P}^t + \mathbf{PS}^{-x_1} + \mathbf{S}^{x_1-x_2} & \mathbf{PP}^t + 2\mathbf{I} & \mathbf{S}^{x_2-x_1} + \mathbf{PS}^{-x_2} + \mathbf{S}^{x_1}\mathbf{P}^t \\ \mathbf{S}^{x_1}\mathbf{P}^t + \mathbf{S}^{x_2-x_1} + \mathbf{PS}^{-x_2} & \mathbf{S}^{x_1-x_2} + \mathbf{S}^{x_2}\mathbf{P}^t + \mathbf{PS}^{-x_1} & \mathbf{PP}^t + 2\mathbf{I} \end{bmatrix}$$

where $\mathbf{PP}^t = 2\mathbf{I} + \mathbf{J}$. The equations to be satisfied for $\mathbf{HH}^t = \mathbf{J}$ are all equivalent to

$$\mathbf{PS}^{-x_2} + \mathbf{S}^{x_1}\mathbf{P}^t + \mathbf{S}^{x_2-x_1} = \mathbf{J}$$

or, equivalently,

$$\mathbf{P} + \mathbf{P}^t\mathbf{S}^{x_1+x_2} + \mathbf{S}^{2x_2-x_1} = \mathbf{J}.$$

Now the first row of $\mathbf{P}$ is the vector 0110100, and the first row of $\mathbf{P}^t$ is 0001011. Therefore, in order to get all 1's, it suffices to have

$$\begin{aligned} x_1 + x_2 &\equiv 0 \mod 7 \\ -x_1 + 2x_2 &\equiv 0 \mod 7 \end{aligned}$$

which has the solution $x_1 = x_2 = 0$. Thus we obtain the an incidence matrix for a plane of order 4 by taking

$$\mathbf{H} = \begin{bmatrix} \mathbf{P} & \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{P} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} & \mathbf{P} \end{bmatrix}.$$

Example 2: Constructing planes of order 9 and 16. The first example is too small to illustrate the general procedure for constructing a desarguean plane of order q^2 from a plane of order q. The construction of a plane of order 9 from one of order 3 is more typical.

For $q = 3$, we have $N = q^2 - q + 1 = 7$. Thus $\mathbf{M} = \mathbf{C}(0111111)$, we may take $\mathbf{P} = \mathbf{C}(0110100000100)$, and $\mathbf{S} = \mathbf{C}(0100000000000)$. The matrix $\mathbf{H}$ then has the block form $\mathbf{H} = \mathbf{C}(\mathbf{P}, \mathbf{S}^{x_1}, \mathbf{S}^{x_2}, \mathbf{S}^{x_3}, \mathbf{S}^{x_4}, \mathbf{S}^{x_5}, \mathbf{S}^{x_6})$. The equations we obtain from $\mathbf{HH}^t = \mathbf{J}$ simplify to

$$\mathbf{P} + \mathbf{P}^t\mathbf{S}^{x_1+x_6} + \mathbf{S}^{2x_1-x_2} + \mathbf{S}^{x_1+x_2-x_3} + \mathbf{S}^{x_1+x_3-x_4} + \mathbf{S}^{x_1+x_4-x_5} + \mathbf{S}^{x_1+x_5-x_6} = \mathbf{J}$$

$$\mathbf{P} + \mathbf{P}^t\mathbf{S}^{x_2+x_5} + \mathbf{S}^{x_1+x_2-x_3} + \mathbf{S}^{2x_2-x_4} + \mathbf{S}^{x_2+x_3-x_5} + \mathbf{S}^{x_2+x_4-x_6} + \mathbf{S}^{x_2+x_6-x_1} = \mathbf{J}$$

$$\mathbf{P} + \mathbf{P}^t\mathbf{S}^{x_3+x_4} + \mathbf{S}^{x_3+x_1-x_4} + \mathbf{S}^{x_3+x_2-x_5} + \mathbf{S}^{2x_3-x_6} + \mathbf{S}^{x_3+x_5-x_1} + \mathbf{S}^{x_3+x_6-x_2} = \mathbf{J}.$$

Now, for $\mathbf{P} = \mathbf{C}(\mathbf{f})$, let $F = \{i : \mathbf{f}_i = 1\} = \{1, 2, 4, 10\}$. Likewise, for $\mathbf{P}^t = \mathbf{C}(\mathbf{g})$, let $G = \{i : \mathbf{g}_i = 1\} = \{3, 9, 11, 12\}$. Note that $F \cap G = \emptyset$, and let $R_0 = \mathbb{Z}_{13} - (F \cup G) = \{0, 5, 6, 7, 8\}$. Moreover, two translates of $\mathbf{P}^t$ share this property: $\mathbf{P}^t\mathbf{S}^9$ and $\mathbf{P}^t\mathbf{S}^{10}$ have 1's in places disjoint from $\mathbf{P}$. If $\mathbf{P}^t\mathbf{S}^9 = \mathbf{C}(\mathbf{h})$, let $H = \{i : \mathbf{h}_i = 1\} = \{5, 7, 8, 12\}$, so that $R_9 = \mathbb{Z}_{13} - (F \cup H) = \{0, 3, 6, 9, 11\}$. Similarly, we obtain $R_{10} = \{3, 5, 7, 11, 12\}$.

To solve the first matrix equation, it is necessary and sufficient that $x_1 + x_6 \in \{0, 9, 10\}$ and $\{2x_1 - x_2, x_1 + x_2 - x_3, x_1 + x_3 - x_4, x_1 + x_4 - x_5, x_1 + x_5 - x_6\} = R_{x_1+x_6}$, with all calculations done modulo 13. To solve the second equation, we need $x_2 + x_5 \in \{0, 9, 10\}$ and $\{x_1 + x_2 - x_3, 2x_2 - x_4, x_2 + x_3 - x_5, x_2 + x_4 - x_6, x_2 + x_6 - x_1\} = R_{x_2+x_5}$. The third equation

requires $x_3+x_4 \in \{0,9,10\}$ and $\{x_3+x_1-x_4, x_3+x_2-x_5, 2x_3-x_6, x_3+x_5-x_1, x_3+x_6-x_2\} = R_{x_3+x_4}$. There are three solution sets satisfying all three conditions, one of which is $x_1 = x_6 = 0$, $x_2 = x_5 = 5$, and $x_3 = x_4 = 11$. (The other solutions are similar).

The case $q = 4$ is also doable. We have $N = 13$, and we may take say $\mathbf{P} = \mathbf{C}(\mathbf{f})$ where $F = \{i : \mathbf{f}_i = 1\} = \{1,4,5,10,12\}$. Proceeding as above, we obtain six sets of equations modulo 21, one solution of which is $x_1 = x_{12} = 0$, $x_2 = x_{11} = 2$, and $x_3 = x_{10} = 9$, $x_4 = x_9 = 6$, $x_5 = x_8 = 14$, and $x_6 = x_7 = 20$.

For q larger, the computations become quite slow.

Example 3: Constructing a plane of order 4 as a union of triangles. We may regard the triangle, with its adjacency matrix $\mathbf{P} = \mathbf{C}(011)$ satisfying $\mathbf{P}\mathbf{P}^t = \mathbf{I}+\mathbf{J}$, as a (degenerate) projective plane of order $q = 1$. If $n \equiv 1 \mod 3$, then 3 divides n^2+n+1. In that case, we can try to construct a projective plane of order n by gluing triangles over an ordinary graph of type $\langle N, n-1, a, b, \times\rangle$ where $N = \frac{n^2+n+1}{3}$, $a = 3$ or $\times$, and $b = 1$ or $\times$.

For $n = 4$, we can use $\mathbf{M} = \mathbf{C}(0110100)$ as an ordinary graph of type $\langle 7,3,\times,1,\times\rangle$. Of course, $\mathbf{S} = \mathbf{C}(010)$. The matrix $\mathbf{H}$ then has the block form $\mathbf{H} = \mathbf{C}(\mathbf{P}, \mathbf{S}^{x_1}, \mathbf{S}^{x_2}, \mathbf{O}, \mathbf{S}^{x_4}, \mathbf{O}, \mathbf{O})$. The equations we obtain from $\mathbf{H}\mathbf{H}^t = \mathbf{J}$ are

$$\mathbf{P} + \mathbf{S}^{2x_1-x_2} = \mathbf{J}$$
$$\mathbf{P} + \mathbf{S}^{2x_2-x_4} = \mathbf{J}$$
$$\mathbf{P} + \mathbf{S}^{2x_4-x_1} = \mathbf{J}$$

Thus we need $2x_1 - x_2 \equiv 2x_2 - x_4 \equiv 2x_4 - x_1 \equiv 0 \mod 3$, with the obvious solution $x_1 = x_2 = x_4 = 0$.

Example 4: Constructing planes of order 7, 13, 16, 19 and 31 as unions of triangles. To construct a plane of order 7 as a union of triangles, we need an ordinary graph of type $\langle 19,6,3,1,\times\rangle$. From Table 2 we find that $\mathbf{M} = \mathbf{C}(0100101101010000000)$ works. The matrix equations are

$$\mathbf{P} + \mathbf{S}^{x_1+x_6-x_7} = \mathbf{J}$$
$$\mathbf{P} + \mathbf{S}^{x_4+x_7-x_{11}} = \mathbf{J}$$
$$\mathbf{P} + \mathbf{S}^{x_9+x_{11}-x_1} = \mathbf{J}$$
$$\mathbf{S}^{x_4-x_6} + \mathbf{S}^{x_7-x_9} + \mathbf{S}^{x_9-x_{11}} = \mathbf{J}$$
$$\mathbf{S}^{x_1-x_4} + \mathbf{S}^{x_4-x_7} + \mathbf{S}^{x_6-x_9} = \mathbf{J}$$
$$\mathbf{S}^{x_1-x_6} + \mathbf{S}^{x_4-x_9} + \mathbf{S}^{x_6-x_{11}} = \mathbf{J}.$$

Thus we need $x_1+x_6-x_7 \equiv x_4+x_7-x_{11} \equiv x_9+x_{11}-x_1 \equiv 0 \mod 3$, and we need that each of the sets $\{x_4-x_6, x_7-x_9, x_9-x_{11}\}$, $\{x_1-x_4, x_4-x_7, x_6-x_9\}$, $\{x_1-x_6, x_4-x_9, x_6-x_{11}\}$ is equivalent to $\{0,1,2\}$ modulo 3. There are two solutions, one of which is $x_4 = x_6 = x_9 = 0$ and $x_1 = x_7 = x_{11} = 1$.

Interestingly, for $n = 10$, we cannot find an ordinary graph of type $\langle 37, 9, 3, 1, \times\rangle$, which could still exist even though a plane of order 10 does not. (Neither have we proved that such a graph does not exist.) There is, however, a circulant ordinary graph of type $\langle 37, 9, 2, 2, \times\rangle$.

To construct a plane of order 13 as a union of triangles, we need an ordinary graph of type $\langle 61, 12, 3, 1, \times\rangle$. One such graph is provided by $\mathbf{M} = \mathbf{C}(\mathbf{e})$ where $\{i : \mathbf{e}_i = 1\} = \{1, 4, 5, 7, 13, 24, 30, 32, 40, 47, 50, 52\}$. We obtain a system of equations modulo 3 having two solutions, including $x_0 = x_{13} = x_{47} = 0$, $x_4 = x_5 = x_7 = x_{24} = x_{30} = x_{52} = 1$, and $x_{32} = x_{40} = x_{50} = 2$.

For a plane of order 16, we need an ordinary graph of type $\langle 91, 15, 3, 1, \times\rangle$. The matrix $\mathbf{M} = \mathbf{C}(\mathbf{e})$ where $\{i : \mathbf{e}_i = 1\} = \{1, 2, 4, 8, 13, 16, 23, 26, 32, 37, 46, 52, 57, 64, 74\}$ works. One solution to the corresponding system of equations is $x_0 = x_4 = x_{16} = x_{23} = x_{64} = x_{74} = 0$, $x_2 = x_8 = x_{32} = x_{37} = x_{46} = x_{57} = 1$, and $x_{13} = x_{26} = x_{52} = 2$.

Likewise, we were able to construct planes of orders 19 and 31 as unions of triangles; the results of the calculations are omitted. For $n \geq 19$, we searched for the ordinary graphs using a method which is not exhaustive and depends on $N = \frac{n^2+n+1}{3}$ being prime. This does not apply when $n = 25$, and produced no results for $n = 22$ and $n = 28$.

On the basis of this evidence, it seems reasonable to conjecture that if q is a prime power and $n \equiv 1 \mod 3$, then the desarguean plane of order q has a decomposition into triangles. So far, we have been unable to prove this, either.

Example 5: Attempts to construct a plane of order $2q^2 + q + 1$. If q and $N = 4q^2 + 3$ are prime powers, then we can try to construct a plane of order $n = 2q^2 + q + 1$ using the quadratic residue tournament of order q^2. The systems of equations we obtain in this case are again tractable. We will show that there is no solution when $q = 2^k$ $(k \geq 1)$ and N is prime.

Assume that $N = 4q^2 + 3$ is prime. The quadratic residue tournament has a circulant matrix $\mathbf{M} = \mathbf{C}(\mathbf{f})$ with $f_i = 1$ if i is a nonzero quadratic residue in $\mathbb{Z}_N$. Let QR_N denote the set of nonzero quadratic residues modulo N. The matrix $\mathbf{H}$ to be constructed is block-circulant of the form $\mathbf{C}(\mathbf{P}, \mathbf{F}_1, \ldots, \mathbf{F}_{N-1})$ with $\mathbf{F}_i = \mathbf{S}^{x_i}$ when $i \in \mathrm{QR}_N$, and $\mathbf{O}$ when $i \notin \mathrm{QR}_N$. Since $N \equiv 3 \mod 4$, we know that -1 is not a quadratic residue modulo N. Hence for each i, exactly one of $\mathbf{F}_i$ and $\mathbf{F}_{N-i}$ is nonzero.

If we consider the $\frac{N-1}{2} = 2q^2 + 1$ equations

$$\mathbf{P}\mathbf{F}_i^t + \mathbf{F}_1\mathbf{F}_{i+1}^t + \cdots + \mathbf{F}_{N-1}\mathbf{F}_{N-1+i}^t = \mathbf{J}$$

where i is a nonzero quadratic residue, then we will have one of each pair of equivalent matrix equations, with no term $\mathbf{P}^t$ involved. These equations simplify to

$$\mathbf{P} + \sum_{\substack{i+k\equiv j \mod N \\ j,k\in \mathrm{QR}_N}} \mathbf{S}^{x_i - x_j + x_k} = \mathbf{J}$$

for each fixed $i \in \mathrm{QR}_N$.

Suppose the matrix $\mathbf{P} = \mathbf{C}(\mathbf{p})$ has 1's in positions $b_1, \ldots, b_{q+1}$. Let $R = \mathbb{Z}_{q^2+q+1} - \{b_1, \ldots, b_{q+1}\}$, and note that $|R| = q^2$. It suffices to find values in $\mathbb{Z}_{q^2+q+1}$ for the $2q^2 + 1$ variables x_j so that, for each nonzero quadratic residue i, the q^2 expressions $x_i - x_j + x_k$ with $i + k \equiv j \mod N$ and $j, k \in \mathrm{QR}_N$ take on each of the values of R exactly once.

Let us formalize this. Let X_N denote the set of variables x_i where i is a nonzero quadratic residue modulo N, and for each such i let T_i be the corresponding set of expressions $x_i - x_j + x_k$ with $i + k \equiv j \mod N$. We seek a map $\varphi : X_N \to \mathbb{Z}_{q^2+q+1}$ such that $\varphi(T_i) = R$ for every $i \in \mathrm{QR}_N$.

If $i \neq k$, then the expression $x_i - x_j + x_k$ occurs twice, once in T_i and once in T_k. However, when 2 is a quadratic residue modulo N, then the expression $2x_i - x_j$ occurs only once (in T_i), while if 2 is a not quadratic residue the expression $2x_i - x_j$ does not occur at all. Recall that, for p prime, 2 is a quadratic residue modulo p if and only if $p \equiv 1$ or $7 \mod 8$. Now $N = 4q^2 + 3 \equiv 3 \mod 8$ when q is even, i.e., $q = 2^k$, and $N \equiv 7 \mod 8$ when q is odd. Let us show that no solution exists in the former case.

Lemma 24. *If $q = 2^k$ and N is prime, then there is no map $\varphi : X_N \to \mathbb{Z}_{q^2+q+1}$ such that $\varphi(T_i) = R$ for every $i \in \mathrm{QR}_N$.*

Proof. There are $2q^2 + 1$ nonzero quadratic residues modulo N. Since each T_i contains q^2 expressions and each expression occurs twice, $\bigcup_i T_i$ contains $\frac{1}{2}(2q^2 + 1)q^2$ expressions. Now $\bigcup_i \varphi(T_i)$ takes on the q^2 values of R. By the pigeonhole principle, there exists an $r \in R$ such that $|\varphi^{-1}(r)| > q^2$, and hence $|\varphi^{-1}(r)| \geq q^2 + 1$. The expressions in $\varphi^{-1}(r)$ occur in two T_i's each, for a total of at least $2q^2 + 2$ times. Hence there is an i_0 such that T_{i_0} contains at least two expressions in $\varphi^{-1}(r)$, so that $\varphi(T_{i_0}) \neq R$. □

We conclude that the special construction we have described does not work in these cases.

Theorem 25. *Let $q = 2^k$ with $k \geq 1$, and let $n = 2q^2 + q + 1$. Assume that $N = 4q^2 + 3$ is prime. Then there is no choice of integers $x_i \in \mathbb{Z}_{q^2+q+1}$ for $i \in \mathrm{QR}_N$ such that the block-circulant matrix $\mathbf{C}(\mathbf{P}, \mathbf{F}_1, \ldots, \mathbf{F}_{N-1})$ is the incidence matrix of a projective plane, where $\mathbf{P}$ is a circulant incidence matrix for a plane of order q, and $\mathbf{F}_i = \mathbf{S}^{x_i}$ when $i \in \mathrm{QR}_N$, and $\mathbf{F}_i = \mathbf{O}$ when $i \notin \mathrm{QR}_N$.*

In particular, this applies when $q = 2$, $n = 11$, $N = 19$ and when $q = 4$, $n = 37$, $N = 67$. It also applies for $q = 2^k$ with $k = 5$, 7, 8, 13 and 14. The proof can be modified to include the case where N is a prime power, but we do not know whether that ever actually occurs.

For q odd, we have no general theorem. However, the considerations in the next section show that the above construction fails when N is a prime power, $n < 2,000,000$ and n is not a prime power.

5. Modifications required to construct nondesarguean planes

The examples we have constructed so far are all desarguean. In fact, Assumptions A, B, C and D together virtually ensure that this will be the case. The authors would like to thank Dean Crnković and Mario-Osvin Pavčević for pointing out to us the nature of these difficulties, which are due to the following elementary result.

Theorem 26. *Let* $\mathbf{H} = (\mathbf{H}_{ij})$ *for* $0 \leq i, j < N$ *be a block-circulant matrix which is the incidence matrix of a symmetric block design. If each* $\mathbf{H}_{ij}$ *is a circulant* $t \times t$ *matrix, then* $\mathbf{H}$ *can be represented as a difference set on the the abelian group* $\mathbb{Z}_N \times \mathbb{Z}_t$.

Section 8 of D. Jungnickel's survey [7] gives a good summary of the difficulties of constructing nondesarguean planar abelian difference sets. Following Jungnickel, we note that there are three main conjectures about how these things work.

Conjecture 27. *Any finite projective plane admitting a Singer group is desarguean. (A Singer group is a group of automorphisms which is regular on the points and lines of a plane.)*

Conjecture 28. *If there is an abelian planar difference set of order* n*, then* n *is a prime power.*

Conjecture 29. *Any abelian planar difference set is cyclic.*

Note that the latter two conjectures would be a consequence of the first. There is substantial evidence supporting these conjectures. The following results are especially relevant.

Theorem 30. *Let* $\mathbf{P}$ *be a finite projective plane with a Singer group* G*. Then either* $\mathbf{P}$ *is desarguean or* G *is a normal subgroup of* Aut $\mathbf{P}$.

Theorem 31. *All cyclic planes of order* m *or* m^2 *with* $m \leq 9$ *are desarguean.*

Theorem 32. *Every abelian difference set of order* $n < 2,000,000$ *has prime power order.*

Theorem 30 is due to U. Ott [8], Theorem 31 is due to R. Bruck [1], and Theorem 32 is a result of D. Gordon [2]) extending earlier work of Keiser, Evans and Mann.

Clearly, our future investigations should eliminate some of the Assumptions from Section 4. Also, much work remains to be done on the existence of ordinary matrices of various types.

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