THE FACTOR OF A SUBDIRECTLY IRREDUCIBLE ALGEBRA THROUGH ITS MONOLITH

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Abstract. A nontrivial algebra with at least one at least binary operation is isomorphic to the factor of a subdirectly irreducible algebra through its monolith if and only if the intersection of all its ideals is nonempty.

0. Introduction

Subdirectly irreducible algebras play a very important role in many aspects of universal algebra. It is well known that every algebra of a signature containing at least one at least binary operation symbol can be embedded into a subdirectly irreducible one. It stands as a contrast to this fact that (for an arbitrary signature) there exist algebras that are not homomorphic images of any subdirectly irreducible algebra. The investigation of homomorphic images of subdirectly irreducible algebras was started in the papers [1], [2] and [3]. The purpose of the present paper is to characterize such homomorphic images and, in particular, to find a necessary and sufficient condition for an algebra to be isomorphic to the factor of a subdirectly irreducible algebra through its monolith. However, we will succeed only in the case of a signature containing at least one at least binary operation symbol. In Section 2 we prove that an algebra A of such a rich signature is a homomorphic image of a subdirectly irreducible algebra if and only if the intersection of all ideals of A is nonempty. In Section 3 we modify the construction to obtain the subdirectly irreducible algebra finite, given that A is finite. Finally, in Section 4, we formulate some remarks related to the remaining case of a signature containing only unary symbols.

For the necessary background of universal algebra, the reader is referred to [4].

1. Preliminaries

By a subdirectly irreducible algebra we mean an algebra A such that there exists the least congruence among its nonidentical congruences; this

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congruence is then called the monolith of A. Of course, A is then necessarily nontrivial, i.e., of cardinality at least 2.

By an ideal of an algebra A we will mean a nonempty subset I such that $F(a_1,\dots,a_k)\in I$ whenever F is a k-ary fundamental operation of A and $a_1,\dots,a_k\in A$ are elements with $a_i\in I$ for at least one index i. Clearly, if I is an ideal of A, then the relation $\mathrm{id}_A\cup(I\times I)$ is a congruence of A.

1.1. **Proposition.** *Let A be an algebra of a signature σ. The intersection of any nonempty family of ideals of A is either empty or an ideal of A. If σ is rich (i.e., contains at least one at least binary operation symbol) then the intersection of any finite nonempty family of ideals of A is nonempty. In particular, if σ is rich and A is finite, then the intersection of all ideals of A is nonempty.*

Proof. It is easy. □

1.2. **Proposition.** *Let A be a subdirectly irreducible algebra of a rich signature. Then the intersection of all ideals of A is nonempty.*

Proof. It is easy. □

1.3. **Proposition.** *Let f be a homomorphism of an algebra A onto an algebra B. Denote by I the intersection of all ideals of A and by J the intersection of all ideals of B. Then $f(I)\subseteq J$. Moreover, if I is nonempty, then $f(I)=J$.*

Proof. It is easy. □

1.4. **Proposition.** *Let A be an algebra of a rich signature. If A is a homomorphic image of a subdirectly irreducible algebra, then the intersection of all ideals of A is nonempty.*

Proof. It follows from 1.2 and 1.3. □

1.5. **Example.** Denote by N the additive semigroup of nonnegative integers. Since the intersection of all ideals of N is empty, it follows that N is not a homomorphic image of any subdirectly irreducible algebra.

2. The general case

The aim of this section is to prove the following result.

2.1. **Theorem.** *The following three conditions are equivalent for a nontrivial algebra A of a signature containing at least one operation symbol of arity ≥ 2:*

(1) *A is a homomorphic image of a subdirectly irreducible algebra.*
(2) *A is isomorphic to the factor of a subdirectly irreducible algebra through its monolith.*
(3) *The intersection of all ideals of A is nonempty.*

Clearly, (2) implies (1). According to 1.4, (1) implies (3). In order to prove that (3) implies (2), let A be an algebra such that the intersection I

of all ideals of A is nonempty. (This means that I is the least ideal of A.) Let us fix one operation symbol G of arity $n_G \geq 2$ in the signature of A, and consider $x \circ y$ an abbreviation for $G(x, y, y, \dots, y)$.

For every $n = 0, 1, 2, \dots$ we are going to define, by induction on n, an algebra $A^{(n)}$ extending A and a homomorphism $\varphi^{(n)}$ of $A^{(n)}$ onto A extending the identity on A. Put $A^{(0)} = A$ and $\varphi^{(0)} = \mathrm{id}_A$. Now assume that $A^{(n-1)}$ and $\varphi^{(n-1)}$ have been defined. Then $A^{(n)}$ is the union of $A^{(n-1)}$ with the set of the following new elements:

(1) $d^{(n)}_{x,y,i,\varepsilon}$ for any $x, y \in A^{(n-1)}$, $i \in I$ and $\varepsilon \in \{1, 2, 3, 4, 5\}$ with $x \neq y$;
(2) $e^{(n)}_{x,y,u,v,i,j}$ for any $x, y, u, v \in A^{(n-1)}$ and $i, j \in I$ such that $x \neq y$, $\varphi^{(n-1)}(x) = \varphi^{(n-1)}(y) = i$ and $\varphi^{(n-1)}(u) = \varphi^{(n-1)}(v) = i \circ j$;
(3) $f^{(n)}_{F,x_1,\dots,x_k,y,\varepsilon}$ for any operation symbol F of arity k in the given signature and elements $x_1, \dots, x_k \in A$, $y \in A^{(n-1)}$, $\varepsilon \in \{1, \dots, k\}$ such that $x_\varepsilon \in I$ and $\varphi^{(n-1)}(y) = F(x_1, \dots, x_k)$.

We define $\varphi^{(n)}$ on the new elements by

(1) $\varphi^{(n)}(d^{(n)}_{x,y,i,\varepsilon}) = \begin{cases} i & \text{for } \varepsilon \in \{1,2\}, \\ \varphi^{(n-1)}(x) \circ i & \text{for } \varepsilon \in \{3,4\}, \\ \varphi^{(n-1)}(y) \circ i & \text{for } \varepsilon = 5; \end{cases}$
(2) $\varphi^{(n)}(e^{(n)}_{x,y,u,v,i,j}) = j$;
(3) $\varphi^{(n)}(f^{(n)}_{F,x_1,\dots,x_k,y,\varepsilon}) = x_\varepsilon$.

Finally, the operations on $A^{(n)}$ are defined as follows:

(1) $x \circ d^{(n)}_{x,y,i,1} = d^{(n)}_{x,y,i,3}$, $x \circ d^{(n)}_{x,y,i,2} = d^{(n)}_{x,y,i,4}$, $y \circ d^{(n)}_{x,y,i,1} = y \circ d^{(n)}_{x,y,i,2} = d^{(n)}_{x,y,i,5}$;
(2) $x \circ e^{(n)}_{x,y,u,v,i,j} = u$, $y \circ e^{(n)}_{x,y,u,v,i,j} = v$;
(3) $F(x_1, \dots, x_{\varepsilon-1}, f^{(n)}_{F,x_1,\dots,x_k,y,\varepsilon}, x_{\varepsilon+1}, \dots, x_k) = y$;
(4) $F(x_1, \dots, x_k) = F(\varphi^{(n)}(x_1), \dots, \varphi^{(n)}(x_k))$ whenever the left side is not yet defined.

It should be clear from these definitions that the union B of the chain of algebras $A = A^{(0)} \subseteq A^{(1)} \subseteq \dots$ is an algebra and the union φ of the chain of homomorphisms $\varphi^{(0)} \subseteq \varphi^{(1)} \subseteq \dots$ is a homomorphism of B onto A extending the identity on A. So, A is isomorphic to the factor $B/\ker(\varphi)$. We are going to show that B is subdirectly irreducible, with monolith $\ker(\varphi)$. We have $\ker(\varphi) \neq \mathrm{id}_B$: for any $i \in I$, there are infinitely many elements $x \in B$ with $\varphi(x) = i$, e.g., the elements d for various indexes. (On the other hand, for $a \in A - I$ we have $\varphi(x) = a$ if and only if $x = a$.) Let α be a nonidentical congruence of B. In order to prove that $\ker(\varphi)$ is the monolith of A, it remains to show that $\ker(\varphi) \subseteq \alpha$.

2.2. **Lemma.** *There is a pair $(x, y) \in \alpha$ with $x \neq y$ and $\varphi(x) = \varphi(y) \in I$.*

Proof. There are two elements $p, q \in B$ with $(p, q) \in \alpha$ and $p \neq q$. Take $i \in I$ arbitrarily and let n be such that $p, q \in A^{(n-1)}$. Then $(p \circ d^{(n)}_{p,q,i,1}, q \circ d^{(n)}_{p,q,i,1}) \in \alpha$ and $(p \circ d^{(n)}_{p,q,i,2}, q \circ d^{(n)}_{p,q,i,2}) \in \alpha$, i.e., $(d^{(n)}_{p,q,i,3}, d^{(n)}_{p,q,i,5}) \in \alpha$ and $(d^{(n)}_{p,q,i,4}, d^{(n)}_{p,q,i,5}) \in \alpha$, so that $(x, y) \in \alpha$ for $x = d^{(n)}_{p,q,i,3}$ and $y = d^{(n)}_{p,q,i,4}$. We have $x \neq y$ and $\varphi(x) = \varphi(y) = \varphi(p) \circ i \in I$. □

2.3. **Lemma.** *Let x and y be as in 2.2. Put $i = \varphi(x) = \varphi(y)$ and let $j \in I$. Then $(u, v) \in \alpha$ for all $u, v \in B$ with $\varphi(u) = \varphi(v) = i \circ j$.*

Proof. We have $(u, v) = (x \circ e^{(n)}_{x,y,u,v,i,j}, y \circ e^{(n)}_{x,y,u,v,i,j})$ for a sufficiently large n. □

2.4. **Lemma.** *Let $i \in I$ be such that $(x, y) \in \alpha$ for all $x, y \in B$ with $\varphi(x) = \varphi(y) = i$. Let $j \in I$ be such that there exist an operation symbol F (in the given signature) of some arity k and an index $\varepsilon \in \{1, \dots, k\}$ with $j = F(x_1, \dots, x_{\varepsilon-1}, i, x_{\varepsilon+1}, \dots, x_k)$ for some $x_1, \dots, x_k \in A$. Then $(u, v) \in \alpha$ for all $u, v \in B$ with $\varphi(u) = \varphi(v) = j$.*

Proof. Put $x_\varepsilon = i$ and take n sufficiently large. We have

$$(f^{(n)}_{F,x_1,\dots,x_k,u,\varepsilon}, f^{(n)}_{F,x_1,\dots,x_k,v,\varepsilon}) \in \varphi^{-1}(\{i\}) \subseteq \alpha$$

and hence

$$\begin{aligned}(&F(x_1, \dots, x_{\varepsilon-1}, f^{(n)}_{F,x_1,\dots,x_k,u,\varepsilon}, x_{\varepsilon+1}, \dots, x_k,\\ &F(x_1, \dots, x_{\varepsilon-1}, f^{(n)}_{F,x_1,\dots,x_k,v,\varepsilon}, x_{\varepsilon+1}, \dots, x_k) \in \alpha,\end{aligned}$$

i.e., $(u, v) \in \alpha$. □

Now denote by J the set of the elements $i \in I$ such that $(u, v) \in \alpha$ for all $u, v \in B$ with $\varphi(u) = \varphi(v) = i$. By 2.2 and 2.3, J is nonempty. By 2.4, J is an ideal of B. Hence $I \subseteq J$ and, consequently, $\ker(\varphi) \subseteq \alpha$. The proof of Theorem 2.1 is thus finished.

3. The finite case

The aim of this section is to prove the following result.

3.1. **Theorem.** *Let A be a finite, nontrivial algebra of a signature containing at least one operation symbol of arity ≥ 2. Then A is isomorphic to the factor of a finite, subdirectly irreducible algebra through its monolith.*

Let $A = \{a_0, \dots, a_{N-1}\}$, so that $N \geq 2$. Denote by I the intersection of all ideals of A. Since A is finite, I is nonempty. As before, let us fix one operation symbol G of arity $n_G \geq 2$ in the signature of A, and consider $x \circ y$ an abbreviation for $G(x, y, y, \dots, y)$. Put $S = \{0, 1, \dots, N-1\}$; for $s \in S$ put $s' = s + 1 \mod N$. Put $B = A \cup (I \times S)$ and define a mapping φ of B onto A by $\varphi(a) = a$ for $a \in A$ and $\varphi(i, s) = i$ for $(i, s) \in I \times S$. Define the operations on B as follows:

(1) A is a subalgebra of B;

(2) $a_r \circ (i,s) = \begin{cases} i \circ i & \text{for } a_r = i \text{ and } s = 0; \\ (a_r \circ i, r+s) & \text{otherwise} \quad (r+s \text{ taken } \bmod N); \end{cases}$

(3) $(i,s) \circ (j,t) = \begin{cases} (i \circ i, s) & \text{for } (j,t) = (i,s'); \\ (i \circ j, s') & \text{otherwise;} \end{cases}$

(4) if $F(x_1, \dots, x_k)$ is not yet defined and $x_\varepsilon \in I \times S$ for precisely one ε, put $(i,s) = x_\varepsilon$, $j = F(x_1, \dots, x_{\varepsilon-1}, i, x_{\varepsilon+1}, \dots, x_k)$ and define $F(x_1, \dots, x_k) = (j,s)$;

(5) put $F(x_1, \dots, x_k) = F(\varphi(x_1), \dots, \varphi(x_k))$ if the left side is not yet defined.

Clearly, φ is a homomorphism of B onto A.

Let $\sim$ be a nonidentical congruence of B.

3.2. **Lemma.** *There exist two elements $x, y \in B$ with $x \sim y$, $x \neq y$ and $\varphi(x), \varphi(y) \in I$.*

Proof. There exist two distinct elements u, v with $u \sim v$. If both $\varphi(u) \in I$ and $\varphi(v) \in I$, we can put $x = u$ and $y = v$.

Consider first the case when $u \in A$ and $v \in A$. Take $i \in I$ arbitrarily, and take $s \in S - \{0\}$. We have $u \circ (i,s) \sim v \circ (i,s)$, i.e., $(u \circ i, p+s) \sim (v \circ i, q+s)$ where $u = a_p$ and $v = a_q$; put $x = (u \circ i, p+s)$ and $y = (v \circ i, q+s)$.

It remains to consider the case $u \in A$ and $v = (i,s) \in I \times S$. We have $u \circ i \sim (i,s) \circ i$, i.e., $u \circ i \sim (i \circ i, s)$; put $x = u \circ i$ and $y = (i \circ i, s)$. □

3.3. **Lemma.** *There exist two elements $z, w \in B$ with $z \sim w$, $z \neq w$ and $\varphi(z) = \varphi(w) \in I$.*

Proof. Let x and y be as in 3.2. We can assume that $\varphi(x) \neq \varphi(y)$; otherwise, we could take $z = x$ and $w = y$.

Consider first the case $x = i \in I$ and $y = j \in I$. We have $i \circ i \sim j \circ i$ and $i \circ (i,0) = i \circ i \sim j \circ (i,0) = (j \circ i, r)$ where $j = a_r$, so that $j \circ i \sim (j \circ i, r)$; put $z = j \circ i$ and $w = (j \circ i, r)$.

Now consider the case $x = i \in I$ and $y = (j,s) \in I \times S$ (where $i \neq j$). We have $i \circ i \sim (j,s) \circ i = (j \circ i, s)$ and $i \circ (i,0) = i \circ i \sim (j,s) \circ (i,0) = (j \circ i, s')$, so that $(j \circ i, s) \sim (j \circ i, s')$; put $z = (j \circ i, s)$ and $w = (j \circ i, s')$.

It remains to consider the case $x = (i,s)$ and $y = (j,t)$ (where $i \neq j$). We have $(i,s) \circ (i,s) \sim (j,t) \circ (i,s)$ and $(i,s) \circ (i,s') \sim (j,t) \circ (i,s')$, i.e., $(i \circ i, s') \sim (j \circ i, t')$ and $(i \circ i, s) \sim (j \circ i, t')$, so that $(i \circ i, s) \sim (i \circ i, s')$; put $z = (i \circ i, s)$ and $w = (i \circ i, s')$. □

3.4. **Lemma.** *Let $(i,s) \sim (i,t)$ where $s \neq t$. Then $(i \circ i, s) \sim (i \circ i, s') \sim (i \circ i, t) \sim (i \circ i, t')$.*

Proof. We have $(i,s) \circ i \sim (i,t) \circ i$, i.e., $(i \circ i, s) \sim (i \circ i, t)$. If $N = 2$, this means that also $(i \circ i, s') \sim (i \circ i, t')$. If $N > 2$, then for $u \in I - \{s', t'\}$ we have $(i,s) \circ (i,u) \sim (i,t) \circ (i,u)$, i.e., $(i \circ i, s') \sim (i \circ i, t')$. Hence $(i \circ i, s') \sim (i \circ i, t')$ in all cases. We also have $(i,s) \circ (i,s') \sim (i,t) \circ (i,s')$, i.e., $(i \circ i, s) \sim (i \circ i, t')$. □

3.5. **Lemma.** *Let $i \sim (i,s)$. Then $(i \circ i, s) \sim (i \circ i, s')$.*

Proof. We have $(i,s)\circ i \sim (i,s)\circ(i,s)$, i.e., $(i\circ i,s)\sim(i\circ i,s')$. $\square$

3.6. **Lemma.** *For every $i\in I$ there exist $s,t\in S$ with $s\neq t$ and $(i,s)\sim(i,t)$.*

Proof. Denote by J the set of all $i\in I$ for which this is true. It is sufficient to prove that J is an ideal of A. By 3.3 and 3.5, J is nonempty. Let F be a symbol of arity k and $x_1,\dots,x_k\in A$ be elements such that $x_\varepsilon = i$ for some ε and some $i\in J$. Put $j=F(x_1,\dots,x_k)$, so that $j\in I$. We need to prove $j\in J$.

There are two elements $s,t\in S$ with $s\neq t$ and $(i,s)\sim(i,t)$, and it is sufficient to prove

$$F(x_1,\dots,x_{\varepsilon-1},(i,s),x_{\varepsilon+1},\dots,x_k)\neq F(x_1,\dots,x_{\varepsilon-1},(i,t),x_{\varepsilon+1},\dots,x_k).$$

This is certainly true if both these elements are defined by (4). If not, then only (2) can apply, $F=G$, G is of arity 2 and what we need to prove is that $a_r\circ(i,s)\neq a_r\circ(i,t)$. If $a_r=i$ and either $s=0$ or $t=0$, then one of the two elements belongs to I while the other one belongs to $I\times S$. In all other cases we have $a_r\circ(i,s)=(a_r\circ i,r+s)\neq(a_r\circ i,r+t)=a_r\circ(i,t)$. $\square$

Define a mapping P of I into I by $P(i)=i\circ i$. An element $i\in I$ is said to be cyclic if $P^m(i)=i$ for at least one (and hence many) positive integer m. Clearly, I contains at least one cyclic element.

3.7. **Lemma.** *Let i be a cyclic element of I. Then $(i,s)\sim(i,t)$ for all $s,t\in S$.*

Proof. By 3.6 there exist two different elements $s_0,t_0\in S$ with $(i,s_0)\sim(i,t_0)$. By 3.4 we have $(P^1(i),s_0)\sim(P^1(i),s_0')$, $(P^2(i),s_0)\sim(P^2(i),s_0')\sim(P^2(i),s_0'')$, etc., so that for all sufficiently large positive integers m we get $(P^m(i),s)\sim(P^m(i),t)$ for all $s,t\in S$. Since i is cyclic, there exist sufficiently large numbers m with $P^m(i)=i$. $\square$

3.8. **Lemma.** *Let i be a cyclic element of I. Then $i\sim(i,s)$ for all $s\in S$.*

Proof. By 3.7, $(i,s)\sim(i,t)$ for all $s,t\in S$. In particular, $(i,0)\sim(i,t)$ for $t\neq 0$. Hence $i\circ(i,0)\sim i\circ(i,t)$, i.e., $i\circ i\sim(i\circ i,u)$ for some $u\in S$. By 3.7 we have $(i\circ i,u)\sim(i\circ i,s)$ for all $s\in S$, and thus $i\circ i\sim(i\circ i,s)$ for all s. But $i\circ i$ is an arbitrary cyclic element of I. $\square$

3.9. **Lemma.** *Let $i\in I$. Then $i\sim(i,s)$ for all $s\in S$.*

Proof. Denote by J the set of all $i\in I$ for which this is true. It is sufficient to prove that J is an ideal of A. By 3.8, J is nonempty. Let $i\in J$; let F be a k-ary symbol and $x_1,\dots,x_k\in A$ be elements such that $x_\varepsilon=i$ for some ε. Put $j=F(x_1,\dots,x_k)$, so that $j\in I$. We need to prove $j\in J$. We have $i\sim(i,s)$ for all $s\in S$ and hence

$$j\sim F(x_1,\dots,x_{\varepsilon-1},(i,s),x_{\varepsilon+1},\dots,x_k)$$

for all $s\in S$. If this last expression is defined by (4), we get $j\sim(j,s)$ for all $s\in S$, and we are through. Otherwise, $F=G$ is of arity 2 and we get

$j \sim a_r \circ (i,s)$ for all $s \in S$, where a_r is an element such that $j = a_r \circ i$. If $a_r \neq i$, this means $j \sim (j, r+s)$ for all $s \in S$ and hence $j \sim (j,t)$ for all $t \in S$. The case $a_r = i$ remains; then $j = i \circ i$. We get $j \sim (j, r+s)$ for all $s \neq 0$. Since $(i,r) \sim (i,r')$, by 3.4 we have $(i \circ i, r) \sim (i \circ i, r')$. Hence also $j \sim (j, r+s)$ for $s = 0$, and we get $j \sim (j, r+s)$ for all $s \in S$. Then $j \sim (j,t)$ for all $t \in S$ and $j \in J$. □

By 3.9 we get $\ker(\varphi) \subseteq \sim$ for any nonidentical congruence $\sim$ of B, so that $\ker(\varphi)$ is the monolith of B and B is subdirectly irreducible. Since A is isomorphic to $B/\ker(\varphi)$, the proof of Theorem 3.1 is thus finished.

4. Algebras with unary operations only

Let us now consider a signature containing no operation symbols of arity ≥ 2. Since nullary operations play no role in the investigation of congruences, we can assume without loss of generality that the signature consists of unary operation symbols only. If the signature is empty, then there exists, up to isomorphism, just one subdirectly irreducible algebra, the two-element one. If the signature contains precisely one unary symbol, then we can also describe all the subdirectly irreducible algebras:

4.1. **Proposition.** *An algebra of the signature consisting of a single unary operation symbol F is subdirectly irreducible if and only if it is isomorphic to one of the algebras in the following list:*

(1) $U_\infty = \{0,1,2,\dots\}$ *with* $F(i) = \max(0, i-1)$;
(2) $U_n = \{0,1,\dots,n\}$ *for* $n \geq 1$, *with* $F(i) = \max(0, i-1)$;
(3) $V_{p,k} = \{0,1,\dots,p^k-1\}$ *for p a prime number and* $k \geq 1$, *with* $F(i) = i+1 \mod p^k$;
(4) $W_0 = \{0,1\}$ *with* $F(0) = 0$, $F(1) = 1$;
(5) $W_{p,k} = \{0,1,\dots,p^k\}$ *for p a prime number and* $k \geq 1$, *with* $F(i) = i+1$ *for* $i < p^k$ *and* $F(p^k) = p^k$.

An algebra of the given signature is a homomorphic image of a subdirectly irreducible algebra if and only if it is isomorphic to the factor of a subdirectly irreducible algebra through its monolith if and only if it is either trivial or subdirectly irreducible itself.

Proof. It is easy. □

4.2. **Example.** Define an algebra $A = \{0,1,\dots,5\}$ with a single unary operation F by $F(i) = i+1 \mod 6$. The intersection of the ideals of A is nonempty (it equals A). On the other hand, A is not a homomorphic image of any subdirectly irreducible algebra.

In the case of a signature containing at least two unary (and no other) operation symbols, the situation is more complicated.

4.3. **Example.** For every $n \geq 1$, define an algebra $Z_n = \{a, b, c_1, \dots, c_n\}$ with unary operations $F, G, \dots$ in this way: for every $P = F, G, \dots$ put

$P(a) = P(b) = c_1$, $P(c_i) = c_{i+1}$ if $i < n$, and $P(c_n) = c_n$. The intersection of all ideals of Z_n is nonempty: it is the one-element set $\{c_n\}$. While Z_1 is isomorphic to the factor of a four-element subdirectly irreducible algebra through its monolith, we are going to show that for $n \geq 2$, the algebra Z_n cannot be embedded into the factor of any subdirectly irreducible algebra through its monolith (even if we allow the subdirectly irreducible algebra to be infinite).

Suppose that there exists a subdirectly irreducible algebra S with monolith μ such that Z_n can be embedded into S/μ. Denote by $A, B, C_1, \dots, C_n$ the blocks of μ corresponding to the elements $a, b, c_1, \dots, c_n$, respectively. Then $\alpha = ((C_{n-1} \cup C_n) \times (C_{n-1} \cup C_n)) \cup \mathrm{id}_S$ is a nonidentical congruence of S, from which we get $|A| = |B| = |C_1| = \dots = |C_{n-2}| = 1$. Further, if $|C_n| \geq 2$, then $(C_n \times C_n) \cup \mathrm{id}_S$ is a nonidentical congruence of S and we get $|C_1| = 1$. But then $\beta = ((A \cup B) \times (A \cup B)) \cup \mathrm{id}_S$ is a nonidentical congruence of S and $\alpha \cap \beta = \mathrm{id}_S$, a contradiction. Thus $|C_n| = 1$, $|C_1| \geq 2$ and $n = 2$. If $C_2 = \{w\}$ and u, v are two different elements of C_1, then $\gamma = (\{u, w\} \times \{u, w\}) \cup \mathrm{id}_S$ and $\delta = (\{v, w\} \times \{v, w\}) \cup \mathrm{id}_S$ are two nonidentical congruences of S and $\gamma \cap \delta = \mathrm{id}_S$, a contradiction.

4.4. **Problem.** For a signature containing at least two unary operation symbols (and no other ones), characterize those algebras that are isomorphic to the factor of a subdirectly irreducible algebra through its monolith.

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