COMPACT ELEMENTS IN THE LATTICE OF VARIETIES

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Abstract. We prove that the lattice of varieties contains almost no compact elements.

The lattice $\mathcal{L}_\sigma$ of varieties of σ-algebras (for a signature σ) has been investigated in many papers. For example, the series of papers [1] is a treatment of definability in this lattice (for an arbitrary σ); from the results of [2] and [3] it follows that the problem (still open) to characterize the lattices isomorphic to a principal ideal of $\mathcal{L}_\sigma$ will be very difficult to solve.

In the present paper we are going to investigate compact elements of this lattice. Recall that an element a of a complete lattice L is said to be compact if the following holds: whenever a is below the join of a subset S of L then a is below the join of a finite subset of S. For other terminology and basics of universal algebra the reader is referred to the monograph [4].

The dual of the lattice of varieties is isomorphic to the lattice of equational theories of signature σ. This second lattice is algebraic, i.e., its every element is the join of a set of compact elements. On the other hand, the lattice of varieties itself is not algebraic, with the only exception when the signature is finite and contains no other symbols than those of arity 0: we will see that if there is at least one symbol of arity at least 1 then it contains no compact elements except the least element.

Let F be an operation symbol of arity $n \geq 1$ in the signature σ. By a term we mean a term (of this signature) over the infinitely countable set X of variables. Recall that equations are ordered pairs of terms. For two terms u, v we write $u \leq v$ if there exists a substitution f (i.e., an endomorphism of the algebra of terms) such that $f(u)$ is a subterm of v. By the length of a term t we mean the total number of occurrences of variables and operation symbols in t.

For every term t and every nonnegative integer n we define a term $U_n(t)$ by induction on n as follows: $U_0(t) = t$; $U_{n+1}(t) = F(U_n(t), t, \dots, t)$. Let x be a variable. For $n > 0$ denote by V_n the variety determined by the equation $(U_n(x), x)$ and by E_n the corresponding equational theory.

Lemma 1. (1) *Let $n, m > 0$ be such that n divides m. Then $V_n \subseteq V_m$.*

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(2) *The join of the varieties* V_n $(n > 0)$ *is the variety of all* σ*-algebras.*

Proof. (1)Clearly, if n divides m then the equation $(U_m(x), x)$ is a consequence of $(U_n(x), x)$.

(2) For $n > 0$ define a σ-algebra A_n in this way: its underlying set is the set of the terms t such that $U_n(x) \nleq t$ (clearly, $X \subseteq A_n$); for a k-ary operation symbol G of σ and a k-tuple $a_1, \dots, a_k$ of elements of A_n let

$$G^{(A_n)}(a_1, \dots, a_k) = \begin{cases} G(a_1, \dots, a_k) \text{ if } G(a_1, \dots, a_k \in A_n, \\ b \text{ if } G(a_1, \dots, a_k) = U_n(b) \text{ for a term } b \in A_n. \end{cases}$$

(Observe that the operations are correctly defined.)

Let $b \in A_n$. For $m < n$ we have $U_n(x) \nleq U_m(b)$, so that $U_m(b) \in A_n$ and the value of $U_m(x)$ in A_n under the interpretation sending x to b is $U_m(b)$. Since this is true for $m = n - 1$, the value of $U_n(x)$ in A_n under that interpretation is b. This means that the equation $(U_n(x), x)$ is satisfied in A_n and hence $A_n \in V_n$. (From this it follows that A_n is the free algebra in V_n over the set X; but we do not need this fact.)

Denote by V the join of the varieties V_n $(n > 0)$. If an equation (u, v) is satisfied in V then it is satisfied in A_n for all numbers $n > 0$; but if n is chosen to be larger than the length of both u and v, then this is possible only if $u = v$. We see that V does not satisfy any nontrivial equation, so that V is the variety of all σ-algebras. □

For every $n > 0$ denote by E'_n the set of the equations (u, v) such that either $u = v$ or both u and v are terms of length at least n. Also, denote by V'_n the variety of models of E'_n. One can easily check that E'_n is an equational theory, so that en equation is satisfied in V'_n if and only if it belongs to E'_n.

Lemma 2. (1) *Let* $n \leq m$. *Then* $V'_n \subseteq V'_m$.

(2) *The join of the varieties* V'_n *is the variety of all* σ*-algebras.*

Proof. This is easy to see. □

Theorem 3. *Let* σ *be a signature containing a symbol* F *of arity* $n \geq 1$. *The trivial variety is the only compact element of the lattice of varieties of* σ*-algebras.*

Proof. Let C be a compact element of the lattice of varieties. It follows from 1 and 2 that there exists a positive integer n such that $C \subseteq V_n$ and $C \subseteq V'_n$. However, the intersection $V_n \cap V'_n$ is the trivial variety. □

Theorem 4. *Let* σ *be a signature consisting of nullary symbols only. If* σ *is finite then the lattice* $\mathcal{L}_\sigma$ *is finite and every element of* $\mathcal{L}_\sigma$ *is compact. If* σ *is infinite then* $\mathcal{L}_\sigma$ *has precisely two compact elements: the trivial variety and the only minimal variety of* σ*-algebras.*

Proof. The finite case is clear, so let σ be an infinite set of nullary symbols. Denote by K the lattice of equivalences on σ. The lattice $\mathcal{L}_\sigma$ is isomorphic to the ordinal sum of the one-element lattice and the dual of K. So, it is sufficient to prove that K has no dually compact elements except the all-equivalence $\sigma \times \sigma$. For this it will be sufficient to construct, for any equivalence $E \neq \sigma \times \sigma$, an infinite chain $E_0 \supset E_1 \supset E_2 \supset \dots$ of equivalences such that $E_0 \cap E_1 \cap \dots \subseteq E$ but $E_i \nsubseteq E$ for all i. We need to distinguish two cases.

Suppose first that E has an infinite block B. Take an infinite sequence $a_0, a_1, \dots$ of pairwise distinct elements of B and an element $c \in \sigma - B$. We can define E_i for any $i \geq 0$ by $E_i = \mathrm{id}_\sigma \cup (H_i \times H_i)$ where $H_i = \{c, a_i, a_{i+1}, a_{i+2}, \dots\}$.

The remaining case is when all the blocks of E are finite, so that there are infinitely many of them. Clearly, there exists an infinite sequence $a_0, b_0, a_1, b_1, \dots$ of elements of σ such that $(a_i, b_i) \notin E$ for all i. We can define E_i for any $i \geq 0$ by $E_i = \{(a_i, b_i), (a_{i+1}, b_{i+1}), \dots\}$. $\square$

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