TRANSITIVE CLOSURES OF BINARY RELATIONS II

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ABSTRACT. Transitive closures of the covering relation in semilattices are investigated.

Vyšetřují se tranzitivní uzávěry pokrývací relace v polosvazech.

This very short note is an immediate continuation of [1]. We therefore refer to [1] as for terminology, notation, various remarks, further references, etc.

1. The covering relation in semilattices

Throughout the note, let $S = S(+)$ be a semilattice (i. e., a commutative idempotent semigroup). Define a relation α on S by $(a, b) \in \alpha$ if and only if $a + b = b$.

1.1. Proposition.

(i) *The relation α is a stable (reflexive) ordering of the semilattice.*

(ii) $(a, a + b) \in \alpha$ *and* $(b, a + b) \in \alpha$ *for all* $a, b \in S$ *(in fact,* $a + b = \sup_\alpha(a, b)$*).*

(iii) *An element $a \in S$ is maximal in $S(\alpha)$ (i. e., a is right α-isolated) if and only if $a = o_S$ is an absorbing element of S; then a is the (unique) greatest element of $S(\alpha)$.*

(iv) *An element $a \in S$ is minimal in $S(\alpha)$ (i. e., a is left α-isolated) if and only if $a \notin (S \setminus \{a\}) + S$ (then the set $(S \setminus \{a\}) + S$ is a proper ideal of S).*

(v) *An element $a \in S$ is the smallest element of $S(\alpha)$ if and only if $a = 0_S$ is a neutral element of S.*

Proof. It is obvious. □

1.2. Lemma.

(i) *Every weakly pseudoirreducible finite α-sequence is pseudoirreducible.*

(ii) *Every weakly pseudoirreducible right (left, resp.) directed infinite α-sequence is pseudoirreducible.*

(iii) *If there exists no pseudoirreducible right directed infinite α-sequence then $o_S \in S$.*

Proof. It is obvious (combine (ii), 1.1(iii) and I.5.4(iii)). □

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1.3. **Lemma.** *Let* $(a, b) \in \alpha$ *and* $I = \mathrm{Int}_\alpha(a, b) = \{c \in S \mid (a, c) \in \alpha, (c, b) \in \alpha\}$. *Then:*

(i) I *is a subsemilattice of* S *and* $\{a, b\} \subseteq I$.
(ii) $a = 0_I$ *and* $b = o_I$.
(iii) $\alpha_I = \alpha_S|I$.

Proof. It is obious. □

In the sequel, put $\beta = \sqrt{\alpha}$ and $\gamma = \mathbf{r}\,\mathbf{t}(\beta)$. Notice that $\mathbf{i}(\gamma) = \mathbf{t}(\beta)$.

1.4. **Proposition.**

(i) β *is totally antitransitive.*
(ii) $\beta \subseteq \gamma \subseteq \alpha$.
(iii) $\beta = \emptyset$ *if and only if either* $|S| = 1$ *or* S *is infinite and for all* $a, b \in S$ *such that* $a + b = b \neq a$ *there exists at least one* $c \in S$ *with* $a + c = c \neq a$ *and* $b + c = b \neq c$.
(iv) γ *is an ordering of* S.
(v) *If* $(a, b) \in \alpha$ *and* $\mathrm{Int}_\alpha(a, b)$ *is finite then* $(a, b) \in \gamma$.

Proof. It is obvious. □

1.5. **Lemma.** *The following conditions are equivalent for* $a, b \in S$*:*

(i) $(a, b) \in \beta$;
(ii) $a + b = b \neq a$ *and* $c \in \{a, b\}$ *whenever* $c \in S$ *is such that* $a + c = c$ *and* $b + c = b$.

Proof. It is obvious. □

We shall say that semilattice $S(+)$ is *resuscitable* if so is the ordering α (i. e., $\alpha = \gamma$).

1.6. **Lemma.** *Let* $(a, b) \in \mathbf{i}(\alpha)$ *be such that there exists no right (left, resp.) directed infinite* $\mathbf{i}(\alpha)$*-sequence* $(a_0, a_1, a_2, \dots)$ $((\dots, b_2, b_1, b_0)$*, resp.) with* $a_0 = a$ $(b_0 = b$*, resp.) and* $(a_i, b) \in \alpha$ $((a, b_i) \in \alpha$*, resp.) for every* $i \geq 1$. *Then there exists at least one* $c \in S$ *such that* $(a, c) \in \alpha$ $((c, b) \in \alpha$*,resp.) and* $(c, b) \in \beta$ $((a, c) \in \beta$*, resp.).*

Proof. If $(a, b) \in \beta$ then we put $c = a$. If $(a, b) \notin \beta$ then there is $a_1 \in S$ with $(a, a_1) \in \mathbf{i}(\alpha)$ and $(a_1, b) \in \mathbf{i}(\alpha)$. If $(a_1, b) \in \beta$ then we put $c = a_1$. If $(a_1, b) \notin \beta$ then there is $a_2 \in S$ with $(a_1, a_2) \in \mathbf{i}(\alpha)$ and $(a_2, b) \in \mathbf{i}(\alpha)$. Proceeding similarly further, we get our result. □

1.7. **Lemma.** *Let* $(a, b) \in \mathbf{i}(\alpha)$ *be such that there exists no right (left, resp.) directed infinite* $\mathbf{i}(\alpha)$*-sequence* $(a_0, a_1, a_2, \dots)$ $((\dots, b_2, b_1, b_0)$*, resp.) with* $a_0 = a$ $(b_0 = b$*, resp.) and* $(a_i, b) \in \alpha$ $((a, b_i) \in \alpha$*, resp.) for every* $i \geq 1$ *and no left (right, resp.) directed infinite* β*-sequence* $(\dots, c_2, c_1, c_0)$ $((d_0, d_1, d_2, \dots)$*, resp.) with* $c_0 = b$ $(d_0 = a$*, resp.) and* $(a, c_j) \in \alpha$ $((d_j, b) \in \alpha$*, resp.) for every* $j \geq 1$. *Then* $(a, b) \in \gamma$.

Proof. According to 1.6, there is $c_1 \in S$ such that $(a, c_1) \in \alpha$ and $(c_1, c_0) \in \beta$, where $c_0 = b$. If $(a, c_1) \in \gamma$ then $(a, b) \in \gamma$. If $(a, c_1) \notin \gamma$ then $(a, c_1) \in \mathbf{i}(\alpha)$, $(a, c_1) \notin \beta$ and, by 1.6 again, there is $c_2 \in S$ with $(a, c_2) \in \alpha$ and $(c_2, c_1) \in \beta$. Proceeding similarly further, we get our result. □

1.8**. Corollary.** *The semilattice S is resuscitable, provided that the following two conditions are satisfied:*

(1) *no right (left, resp.) directed infinite $\mathbf{i}(\alpha)$-sequence is right (left, resp.) bounded in $S(\alpha)$;*
(2) *no left (right, resp.) directed infinite β-sequence is left (right, resp.) bounded in $S(\alpha)$;*

1.9**. Corollary.** *The semilattice S is resuscitable, provided that there exist no right (left, resp.) directed infinite $\mathbf{i}(\alpha)$-sequences and no left (right, resp.) directed infinite β-sequences.*

1.10**. Corollary.** *The semilattice S is resuscitable, provided that it is finite.*

1.11**. Lemma.** *If $(a, b) \in \gamma$ then $\{a, b\} \subseteq \mathrm{Int}_\gamma(a, b) = \{c \,|\, (a, c) \in \gamma, (c, b) \in \gamma\} \subseteq \mathrm{Int}_\alpha(a, b)$.*

Proof. It is obvious. □

1.12**. Example.** Let A be a non-empty set and $\mathcal{S}$ the set of subsets of A. Then $\mathcal{S}(\cup)$ is a semilattice, $\emptyset = 0_{\mathcal{S}}$, $A = o_{\mathcal{S}}$, $(B, C) \in \alpha$ if and only if $B \subseteq C$, $(D, E) \in \beta$ if and only if $D \subseteq E$ and $|E \setminus D| = 1$. This semilattice is resuscitable if and only if A is finite.

2. On when the covering relation is right confluent (or weakly semimodular lattices)

The semilattice S will be called *weakly semimodular* if $d \in \{b, b + c\}$ whenever $a, b, c, d \in S$ are such that $b \neq c$, $(a, b) \in \beta$, $(a, c) \in \beta$, $b + d = d$ and $b + c = d + c$.

2.1**. Lemma.** *The following conditions are equivalent:*

(i) *S is weakly semimodular;*
(ii) *$(b, b + c) \in \beta$ (and $(c, b + c) \in \beta$) whenever $a, b, c \in S$ are such that $(a, b) \in \beta$, $(a, c) \in \beta$ and $b \neq c$;*
(iii) *β is right confluent.*

Proof. It is obvious. □

2.2**. Lemma.** *Assume that S is weakly semimodular. If $a, b, c \in S$ are such that $(a, b) \in \gamma$ and $(a, c) \in \beta$ then $(c, b + c) \in \gamma$ and either $(b, b + c) \in \beta$ or $b = b + c$ (and then $(c, b) \in \gamma$).*

Proof. There is nothing to show for $a = b$. Hence, assume that $a \neq b$ and let $(a_0, a_1, \ldots, a_m)$, $m \geq 1$, be a β-sequence with $a_0 = a$ and $a_m = b$; we will proceed by induction on m.

If $a_1 = c$ then $(c, b+c) = (a_1, b) \in \gamma$ and $b = b+c$. If $a_1 \neq c$ then $(a_1, a_1+c) \in \beta$, $(c, a_1+c) \in \beta$ and $(a_1+c, b+c) = (a_1+c, a_1+c+b) \in \gamma$ by induction, so that $(c, b+c) \in \gamma$. Moreover, either $(b, b+c) = (b, b+a_1+c) \in \beta$ or $b = b+a_1+c = b+c$. □

2.3. **Lemma.** *Assume that S is weakly semimodular. If $a, b, c \in S$ are such that $(a, b) \in \gamma$ and $(a, c) \in \gamma$ then $(b, b+c) \in \gamma$ and $(c, b+c) \in \gamma$.*

Proof. If $a = b$ or $a = c$ then there is nothing to show. Hence, assume that $b \neq a \neq c$ and let $(a_0, a_1, \dots, a_m)$, $m \geq 1$, be a β-sequence with $a_0 = a$ and $a_m = b$. By 2.2, $(a_1, a_1+c) \in \gamma$, and therefore $(b, b+c) = (b+a_1, (b+a_1)+c) \in \gamma$ by induction on m. Quite similarly, $(c, b+c) \in \gamma$. □

2.4. **Corollary.** *If the semilattice S is weakly semimodular then the ordering γ is right strictly confluent.*

2.5. **Lemma.** *Assume that S is weakly semimodular. If $(a, b) \in \gamma$ then there exists no right directed infinite β-sequence $(a_0, a_1, a_2, \dots)$ such that $a_0 = a$ and $(a_i, b) \in \alpha$ for every $i \geq 1$.*

Proof. Let, on the contrary, such a β-sequence exist. If $a = b$ then $(b, a_1) = (a, a_1) \in \beta$, a contradiction with $(a_1, b) \in \alpha$. Thus $a \neq b$ and there is a finite β-sequence $(b_0, b_1, b_2, \dots, b_m)$, $m \geq 1$, with $b_0 = a$ and $b_m = b$. If $m = 1$ then $(a, b) \in \beta$ and, since $(a, a_1) \in \beta$ and $(a_1, b) \in \alpha$, we get $a_1 = b$, and hence $a_2 = a_1$, a contradiction with $(a_1, a_2) \in \beta$. Thus $m \geq 2$ and we shall proceed by induction on m.

If $a_1 = b_1$ then the contradiction follows by induction. On the other hand, if $a_1 \neq b_1$ then $(a_1, a_1+b_1) \in \beta$ and $(b_1, a_1+b_1) \in \beta$; of course, $(a_1+b_1, b) \in \alpha$. If $a_2 = a_1+b_1$ then we use induction once more. Thus $a_2 \neq a_1+b_1$, $(a_2, a_2+b_1) \in \beta$, $(a_1+b_1, a_2+b_1) \in \beta$ and $(a_2+b_1, b) \in \alpha$. Proceeding in this way, we get the β-sequence $(b_1, a_1+b_1, a_2+b_1, a_3+b_1, \dots)$ and we come by induction to our final contradiction. □

2.6. **Lemma.** *Assume that S is weakly semimodular. If $(a, b) \in \gamma$ then there exists no right directed infinite $\mathbf{i}(\gamma)$-sequence $(a_0, a_1, a_2, \dots)$ such that $a_0 = a$ and $(a_i, b) \in \alpha$ for every $i \geq 1$.*

Proof. Use 2.5 and the fact that $\mathbf{i}(\gamma) = \mathbf{t}(\beta)$. □

2.7. **Lemma.** *Assume that S is weakly semimodular. If $(a, b) \in \gamma$ then:*

(i) *$T = \mathrm{Int}_\gamma(a, b)$ is a subsemilattice of S, $a = 0_T$ and $b = o_T$.*
(ii) *T is resuscitable.*
(iii) *$\alpha_T = \gamma_T = \alpha_S|T = \gamma_S|T$ and $\beta_T = \beta_S|T$.*
(iv) *If $(a, c) \in \gamma$ and $(c, b) \in \alpha$ then $c \in T$ (i. e., $(c, b) \in \gamma$).*

Proof.

(i) If $c, d \in T$ then $(a, c) \in \gamma$ and $(a, d) \in \gamma$, and so $(c, c+d) \in \gamma$ by 2.3. Since γ is transitive, we get $(a, c+d) \in \gamma$, Quite similarly, $(c, b) \in \gamma$ and $(c, c+d) \in \gamma$ implies $(c+d, b) = (c+d, b+c+d) \in \gamma$ and we conclude that $c+d \in T$.

(ii) This is easy to see (use 2.3).
(iii) This is also easy to see (use 2.3).
(iv) Use 2.3.

□

2.8. **Example.** Consider the following infinite semilattice S_1:

	0	a	b_1	b_2	b_3	$\dots$	b_m	b_{m+1}	b_{m+2}	$\dots$	o
0	0	a	b_1	b_2	b_3	$\dots$	b_m	b_{m+1}	b_{m+2}	$\dots$	o
a	a	a	o	o	o	$\dots$	o	o	o	$\dots$	o
b_1	b_1	o	b_1	b_1	b_1	$\dots$	b_1	b_1	b_1	$\dots$	o
b_2	b_2	o	b_1	b_2	b_2	$\dots$	b_2	b_2	b_2	$\dots$	o
b_3	b_3	o	b_1	b_2	b_3	$\dots$	b_3	b_3	b_3	$\dots$	o
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
b_m	b_m	o	b_1	b_2	b_3	$\dots$	b_m	b_m	b_m	$\dots$	o
b_{m+1}	b_{m+1}	o	b_1	b_2	b_3	$\dots$	b_m	b_{m+1}	b_{m+1}	$\dots$	o
b_{m+2}	b_{m+2}	o	b_1	b_2	b_3	$\dots$	b_m	b_{m+1}	b_{m+2}	$\dots$	o
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o	o	o	o	o	o	$\dots$	o	o	o	$\dots$	o

Clearly, $S_1(+)$ is weakly semimodular and $\beta = \{(0,a),(a,o),(b_1,o),(b_{i+1},b_i) \mid i \geq 1\}$. Moreover, $(0,o) \in \gamma$, $\mathrm{Int}_\gamma(o,0) = \{0,a,o\}$, $(0,b_1) \notin \gamma$ and $(\dots,b_2,b_1,o)$ is a left bounded left directed infinite β-sequence. Finally, S_1 is not resuscitable.

2.9. **Example.** Consider the following infinite semilattice S_2:

	0	a	$\dots$	b_m	b_{m+1}	b_{m+2}	$\dots$	o
0	0	a	$\dots$	b_m	b_{m+1}	b_{m+2}	$\dots$	o
a	a	a	$\dots$	o	o	o	$\dots$	o
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
b_m	b_m	o	$\dots$	b_m	b_m	b_m	$\dots$	o
b_{m+1}	b_{m+1}	o	$\dots$	b_m	b_{m+1}	b_{m+1}	$\dots$	o
b_{m+2}	b_{m+2}	o	$\dots$	b_m	b_{m+1}	b_{m+2}	$\dots$	o
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o	o	o	$\dots$	o	o	o	$\dots$	o

Clearly, S_2 is weakly semimodular and $\beta = \{(0,a),(a,o),(b_{i+1},b_i) \mid i \in \mathbb{Z}\}$. Moreover, $(0,o) \in \gamma$, $\mathrm{Int}_\gamma(o,0) = \{0,a,o\} \neq S_2 = \mathrm{Int}_\alpha(0,o)$, hence S_2 is not resuscitable. Finally, S_2 contains both left and right (bounded) directed infinite β-sequences.

2.10. **Example.** Consider the following five-element semilattice **P**:

	0	a	b	c	o
0	0	a	b	c	o
a	a	a	o	o	o
b	b	o	b	c	o
c	c	o	c	c	o
o	o	o	o	o	o

Clearly, $\beta = \{(0,a),(0,b),(b,c),(a,o),(c,o)\}$, β is neither right nor left confluent and $\mathbf{P}$ is not weakly semimodular.

3. Semimodular semilattices

The semilattice S will be called *semimodular* if $(a+c, b+c) \in \mathbf{r}(\beta)$ whenever $(a,b) \in \beta$ and $c \in S$.

3.1. Lemma. *The following conditions are equivalent:*

(i) *S is semimodular;*
(ii) *$d \in \{a+c, b+c\}$ whenever $a,b,c,d \in S$ are such that $(a,b) \in \beta$, $a+c \neq b+c$, $a+c+d = d$ and $b+c+d = b+c$;*
(iii) *$\mathbf{r}(\beta)$ is stable.*

Proof. It is obvious. □

3.2. Lemma. *If S is semimodular then it is weakly semimodular and γ is a stable ordering of S.*

Proof. It is obvious. □

3.3. Proposition. *If the semilattice S is resuscitable (e. g., S is finite) then it is semimodular if and only if it is weakly semimodular.*

Proof. Only the converse implication needs a proof. Assume that S is weakly semimodular and take $a,b,c \in S$ such that $(a,b) \in \beta$ and $a+c \neq b+c$. By 3.2 and 2.1, the relation β is right confluent, and so $(a+c, b+c) \in \beta$ follows from I.9.5. □

3.4. Lemma. *Assume that S is semimodular. If $(a,b) \in \gamma$, $(a,c) \in \alpha$ and $(c,b) \in \alpha$ (i. e., $c \in \mathrm{Int}_\alpha(a,b)$) then $(c,b) \in \gamma$.*

Proof. We have $(c,b) = (a+c, a+b) \in \gamma$ by 3.2. □

3.5. Lemma. *Let $(a,c) \in \beta$, $(c,b) \in \beta$, $(a,d) \in \alpha$ and $(d,b) \in \alpha$.*

(i) *If S is weakly semimodular then $(a,d) \in \beta$ implies $(d,b) \in \beta$.*
(ii) *If S is semimodular then $(a,d) \in \beta$ if and only if $(d,b) \in \beta$.*

Proof.

(i) We can assume $c \neq d$. Then $(c, c+d) \in \beta$, $(d, c+d) \in \beta$ and, of course, $(c+d, b) \in \alpha$. Since $(c,b) \in \beta$ we have $c+d = b$ and $(d,b) \in \beta$.

(ii) Assume $c \neq d$ and $(d,b) \in \beta$ (see (i)). Clearly, $a \neq d$. If $e \in S$ is such that $(a,e) \in \alpha$ and $(e,d) \in \alpha$ then either $e = a+e = c+e$ or $(e,c+e) = (a+e,c+e) \in \beta$.

If $e = c+e$ then $(c,e) \in \alpha$, hence $(c,d) \in \alpha$ and $c = d$, since $(c,b) \in \beta$ and $(d,b) \in \beta$, a contradiction. Thus $(e,c+e) \in \beta$. If $c+e = c$ then $(e,c) \in \beta$ and $e = a$, since $(a,e) \in \alpha$ and $(a,c) \in \beta$. On the other hand, if $c+e \neq c$ then $c+e = b$, since $(c,c+e) \in \alpha$ and $(c+e,b) \in \alpha$. Finally, if $c+e = b$ then $(e,b) \in \beta$ and $e = d$, since $(e,d) \in \alpha$ and $(d,b) \in \beta$. We have proved that $e \in \{a,d\}$ and it follows that $(a,d) \in \beta$.

□

3.6. **Example.** The semilattice S_1 (see 2.8) is weakly semimodular but not semimodular.

4. Strongly modular semilattices

The semilattice S will be called *strongly modular* if no subsemilattice of S is a copy of $\mathbf{P}$ (see 2.10).

4.1. **Proposition.** *If S is strongly modular then it is semimodular.*

Proof. Using 3.1, let $(a,b) \in \beta$, $a+c \neq b+c$, $a+c+d = d$ and $b+c+d = b+c$. We have to show that $d \in \{a+c, b+c\}$.

Clearly, $(a,b) \in \alpha$, $(a+c,b+c) \in \alpha$, $(a+c,d) \in \alpha$, $(d,b+c) \in \alpha$ and it follows easily that $T = \{a,b,d,a+c,b+c\}$, is a subsemilattice of S. Moreover, $T \cong \mathbf{P}$, provided that $|T| = 5$. Consequently, since S is strongly modular, we get $|T| \leq 4$.

First, $a+c \neq b+c$ and $(a,b) \in \beta$ implies $a \neq b$. If $a = a+c$ then $b+c = a+b+c = a+b = b$, $d = a+c+d = a+d$, $b = b+c = b+c+d = b+d$, $(a,d) \in \alpha$, $(d,b) \in \alpha$, and hence $d \in \{a,b\} = \{a+c,b+c\}$, since $(a,b) \in \beta$. Furthermore, if $a = b+c$ ($a = d$, resp.) then $a = b+c = b+c+c = a+c$ ($a = d = a+c+d = a+c+a = a+c$, resp.).

Now, we can assume that $a \notin \{b, a+c, d, b+c\}$. If $b = a+c$ then $b = b+b = b+a+c = b+c$ and $a+c = b+c$, a contradiction. If $b = b+c$ then $(a,a+c) \in \alpha$ and $(a+c,b) = (a+c,b+c) \in \alpha$ implies $a+c = b$ (we have $(a,b) \in \beta$ and $a \neq a+c$) and $a+c = b+c$, a contradiction once more. Furthermore, if $b = d$ then $b = d = a+c+d = a+c+b = b+c$, which was already proved to be contradictory.

Finally, we can assume that $a \notin \{b,d,a+c,b+c\}$, $b \notin \{a,d,a+c,b+c\}$, $d \notin \{a,b\}$ and $a+c \neq b+c$. Since $|T| \leq 4$, we obtain $d \in \{a+c,b+c\}$ as desired. □

4.2. **Example.** Let A be a non-empty set and $\mathcal{F}$ the set of non-empty finite subsets of A. Then $\mathcal{F}(\cup)$ is a free semilattice over A, $(B,C) \in \alpha$ if and only if $B \subseteq C$, $(D,E) \in \beta$ if and only if $D \subseteq E$ and $|E \setminus D| = 1$. Moreover, $\mathcal{F}(\cup)$ is semimodular and resuscitable. It is strongly modular if and only if $|A| \leq 3$ (if $|A| \geq 4$ then consider the set $\{\{a_1\},\{a_1,a_2\},\{a_1,a_2,a_3\},\{a_1,a_3,a_4\},\{a_1,a_2,a_3,a_4\}\}$).

4.3. **Example.** Define an operation $\oplus$ on the set $\mathbb{N}_0$ of non-negative integers by $m \oplus n = \mathrm{lcm}(m,n)$. Then $\mathbb{N}_0(\oplus)$ becomes a semilattice, $(m,n) \in \alpha$ if and only if m divides n and $(k,l) \in \beta$ if and only if l/k is a prime number. Clearly, $\mathbb{N}_0(\oplus)$ is semimodular and resuscitable. On the other hand, the set $\{1,4,9,18,36\}$ is a subsemilattice isomorphic to $\mathbf{P}(+)$, and so $\mathbb{N}_0(\oplus)$ is not strongly modular.

5. On when the covering relation is regular

5.1. **Proposition.** *If the semilattice $S(+)$ is weakly semimodular then the covering relation β is regular.*

Proof. Let $(a,b) \in \gamma$ and $T = \mathrm{Int}_\gamma(a,b)$. By 2.7 and 3.3, T is a semimodular and resuscitable semilattice. Moreover, $a = 0_T$ and $b = o_T$. In particular, b is right α_T-isolated. We have $\beta_T = \beta_S|T$, $\alpha_T = \gamma_T = \gamma_S|T = \mathbf{r}\,\mathbf{t}(\beta_T)$ and $(c,b) \in \alpha_T$ for every $c \in T$. The relation β_T is right confluent (on T) and β_T is regular by I.8.3. Now, our result easily follows. □

5.2. **Example.** Put $S = \mathbf{P}$ (see 2.10). Then β is not regular.

5.3. **Example.** Consider the following six-element semilattice $S_3(+)$:

	0	a	b	c	d	o
0	0	a	b	c	d	o
a	a	a	b	o	o	o
b	b	b	b	o	o	o
c	c	o	o	c	d	o
d	d	o	o	d	d	o
o	o	o	o	o	o	o

Clearly, $\beta = \{(0,a),(0,c),(a,b),(c,d),(b,o),(d,o)\}$ and β is regular. On the other hand, S_3 is not weakly semimodular.

5.4. *Remark.* Assume that β is regular. If $(a,b) \in \mathbf{i}(\gamma)$ $(= \mathbf{t}(\beta))$ then all the β-sequences from a to b have the same length, say $m \geq 1$, and we put $\mathrm{dist}_\gamma(a,b) = m$. We put also $\mathrm{dist}_\gamma(c,c) = 0$ for every $c \in S$.

5.5. **Lemma.** *Assume that β is regular. If $(a,b) \in \gamma$ and $(b,c) \in \gamma$ then* $\mathrm{dist}_\gamma(a,c) = \mathrm{dist}_\gamma(a,b) + \mathrm{dist}_\gamma(b,c)$.

Proof. It is obvious. □

6. Further results

6.1. **Lemma.** *Assume that S is semimodular. If $(a,b) \in \gamma$, $(a,c) \in \alpha$ and $(c,b) \in \alpha$ then $(a,c) \in \gamma$ and $(c,b) \in \gamma$.*

Proof. We have $(c,b) \in \gamma$ by 3.4 and the covering relation β is regular by 5.1. Put $m = \mathrm{dist}_\gamma(a,b)$. If $m = 0$ then $a = c = b$ and there is nothing to prove. If $m = 1$ then $(a,b) \in \beta$ and either $c = a$ or $c = b$ and there is nothing to prove again. Consequently, assume that $m \geq 2$ and proceed by induction on m.

There is a β-sequence $(a_0, a_1, \dots, a_m)$ such that $a_0 = a$ and $a_m = b$. Now, $(a_1, a_1{+}c) \in \alpha$, $(a_1{+}c, b) \in \alpha$, $\mathrm{dist}_\gamma(a_1, b) = m-1$ and we get $(a_1, a_1{+}c) \in \gamma$ by induction. According to 5.5, $m-1 = \mathrm{dist}_\gamma(a_1, b) = \mathrm{dist}_\gamma(a_1, a_1+c) + \mathrm{dist}_\gamma(a_1+c, b)$. If $\mathrm{dist}_\gamma(a_1+c, b) \geq 1$ then $\mathrm{dist}_\gamma(a_1, a_1+c) \leq m-2$, $\mathrm{dist}_\gamma(a, a_1+c) = 1 + \mathrm{dist}_\gamma(a_1, a_1+c) \leq m-1$ and $(a, c) \in \gamma$ by induction (we have $(c, a_1+c) \in \alpha$).

Now, consider the case $\mathrm{dist}_\gamma(a_1+c, b) = 0$. Then $a_1 + c = b$ and we get $(c, b) = (a_0 + c, a_1 + c) \in \mathbf{r}(\beta)$. If $c = b$ then $(a, c) \in \gamma$ trivially, and hence, let $(c, b) \in \beta$ and $(a, c) \notin \gamma$. Then there is $d \in S$ with $(a, d) \in \mathbf{i}(\alpha)$ and $(d, c) \in \mathbf{i}(\alpha)$. If $(a, d) \notin \gamma$ then, according to the preceding part of the proof, we get $(d, b) \in \beta$, and so $d = c$, a contradiction. Thus $(a, d) \in \gamma$ and we have $m = \mathrm{dist}_\gamma(a, b) = \mathrm{dist}_\gamma(a, d) + \mathrm{dist}_\gamma(d, b)$. Since $a \neq d$, it follows that $\mathrm{dist}_\gamma(d, b) \leq m-1$, and therefore $(d, c) \in \gamma$ by induction. Consequently, $(a, c) \in \gamma$, a contradiction. □

6.2. **Lemma.** *Assume that S is semimodular. If $(a, b) \in \gamma$, $(a, c) \in \alpha$, $(c, d) \in \alpha$ and $(d, b) \in \alpha$ then $(a, c) \in \gamma$, $(c, d) \in \gamma$ and $(d, b) \in \gamma$.*

Proof. We have $(a, d) \in \gamma$ and $(d, b) \in \gamma$ by 6.1. Then $(a, c) \in \gamma$ and $(c, d) \in \gamma$ by 6.1 again. □

6.3. **Proposition.** *Assume that $S(+)$ is weakly semimodular. Let $(a, b) \in \gamma$ and $T = \mathrm{Int}_\gamma(a, b)$. Then:*

(i) *T is a subsemilattice of S, $a = 0_T$ and $b = o_T$.*
(ii) *T is semimodular and resuscitable.*
(iii) *$\beta_T = \beta_S|T$ and $\alpha_T = \gamma_T = \alpha_S|T = \gamma_S|T$.*
(iv) *Every non-empty subset A of T contains at least one element that is maximal in $A(\alpha)$ and at least one element that is minimal in $A(\alpha)$.*
(v) *Every subchain of $T(\alpha)$ is finite and of length at most $\mathrm{dist}_\gamma(a, b)$.*
(vi) *$T \subseteq \mathrm{Int}_\alpha(a, b)$ and $c \in T$, provided that $(a, c) \in \gamma$ and $(c, b) \in \alpha$.*
(vii) *$T = \mathrm{Int}_\alpha(a, b)$, provided that S is semimodular.*

Proof.

(i) This is 2.6 (i).
(ii) T is resuscitable by 2.6 (ii), and hence it is semimodular by 3.3.
(iii) This is 2.6 (iii).
(iv) Use 5.1 and 5.5.
(v) Use 5.1 and 5.5.
(vi) This is 2.6 (iv).
(vii) See 6.1.

□

6.4. **Proposition.** *The following conditions are equivalent:*

(i) *S is weakly semimodular, no right (left, resp.) directed infinite $\mathbf{i}(\alpha)$-sequence is right (left, resp.) bounded in $S(\alpha)$ and no left (right, resp.) directed infinite β-sequence is left (right, resp.) bounded in $S(\alpha)$.*

(ii) *S is semimodular and resuscitable.*
(iii) *S is weakly semimodular and every right and left bounded subchain of $S(\alpha)$ is finite.*

Proof. (i) implies (ii). The semilattice S is resuscitable by 1.8, and so it is semimodular by 3.3.

(ii) implies (iii). Let C be a non-empty subchain of $S(\alpha)$ such that there exist $a, b \in S$ with $(a, c) \in \alpha$ and $(c, b) \in \alpha$ for every $c \in C$. Then $C \subseteq \mathrm{Int}_\alpha(a, b) = \mathrm{Int}_\gamma(a, b)$ and C is finite by 6.3 (ix).

(iii) implies (i). Every right (left, resp.) directed $\mathbf{i}(\alpha)$-sequence is left (right, resp.) bounded in $S(\alpha)$. The rest is clear. □

References

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