# Definability in substructure orderings, I: finite semilattices

Jaroslav Ježek and Ralph McKenzie

Abstract. We investigate definability in the set of isomorphism types of finite semilattices ordered by embeddability; we prove, among other things, that every finite semilattice is a definable element in this ordered set. Then we apply these results to investigate definability in the closely related lattice of universal classes of semilattices; we prove that the lattice has no non-identical automorphisms, the set of finitely generated and also the set of finitely axiomatizable universal classes are definable subsets and each element of the two subsets is a definable element in the lattice.

## 1. Introduction

Let $K$ be a fixed class of structures of some given finite signature. Consider the collection $L$ of all subclasses of $K$ that are axiomatizable by a selected type $T$ of axioms (like equations, quasi-equations, or universal sentences). Usually, this collection is a complete lattice with respect to inclusion. Let us say that $L$ has positive definability if one can prove the following statements:

(1) the collection of all finitely $T$-axiomatizable subclasses of $K$ is a definable subset of $L$ and every element of that collection is definable in $L$ up to the automorphisms of $L$;
(2) the collection of all finitely generated $T$-subclasses of $K$ is a definable subset of $L$ and every element of that collection is definable in $L$ up to the automorphisms of $L$;
(3) the collection of the subclasses of $K$ that are axiomatizable by a single $T$-axiom is a definable subset of $L$;
(4) $L$ has no other automorphisms than the obvious, syntactically defined ones.

It has been proved in the series of papers [1] that the lattice of subvarieties (i.e., equationally definable subclasses) of the variety of all universal algebras of a given signature has positive definability. So far, no other variety has been found to have positive definability for its lattice of subvarieties (except for trivial cases). The paper [3] contains some partial results for the lattice of varieties of semigroups. In [2] the investigation was started but not finished for the lattice of varieties of

1991 *Mathematics Subject Classification*: 03C40,06A12,08C10.
*Key words and phrases*: definable, semilattice, lattice.
While working on this paper, the first author (Ježek) and the second author (McKenzie) were supported by US NSF grant DMS-0604065. The first author was also supported by the institutional grant MSM0021620839 financed by MSMT and partially supported by the grant GAČR 201/05/0002.

commutative groupoids. In [4] it has been proved that the lattice of varieties of commutative semigroups has uncountably many automorphisms, so that it has negative definability.

As far as the authors know, no attempt has been done to investigate definability in lattices of quasivarieties or universal classes. In the present paper we are going to establish positive definability for the lattice of universal subclasses of the variety of semilattices. We will start with the investigation of definability for the closely related quasi-ordered set of finite semilattices, where the ordering is embeddability; we will prove that every element of this quasi-ordered set is definable.

Let $\langle Q, \leq \rangle$ be a quasi-ordered set (i.e., the binary relation $\leq$ is reflexive on $Q$ and transitive). An $n$-ary relation $R$ on $Q$ is said to be definable (in $\langle Q, \leq \rangle$) if there exists a first-order formula $\phi(x_1, \dots, x_n)$ with free variables $x_1, \dots, x_n$ in the language of $\leq$ such that for any $a_1, \dots, a_n \in Q$, $\phi(a_1, \dots, a_n)$ holds in $\langle Q, \leq \rangle$ if and only if $\langle a_1, \dots, a_m \rangle \in R$. A subset of $Q$ is said to be definable if it is definable as a unary relation. An element $a$ of $Q$ is said to be definable if the set $\{x \in Q : x \leq a \text{ and } a \leq x\}$ is definable.

By a semilattice we mean a meet semilattice. We denote by $\mathcal{S}$ the quasi-ordered set of finite semilattices, where the quasi-ordering is given by $A \leq B$ if and only if $A$ is isomorphic to a subsemilattice of $B$ (i.e., there exists an embedding of $A$ into $B$). Strictly speaking, $\mathcal{S}$ is not a set but a proper class. This can be corrected by considering only those finite semilattices the underlying set of which belongs to a fixed countably infinite set. The best candidate for such a set is the hereditarily finite universum (the smallest set containing the empty set and closed under the binary operations $\{x, y\}$ and $x \cup y$). Thus $\mathcal{S}$ becomes a countable set. Or we could consider, instead of $\mathcal{S}$, the ordered set of isomorphism types of finite semilattices; this equivalent approach would have some technical difficulties.

For two elements $A, B$ of $\mathcal{S}$ write $A < B$ if $A \leq B$ and $B \not\leq A$. We have $A \simeq B$ if and only if $A \leq B$ and $B \leq A$. If all elements of $\mathcal{S}$ with a certain property are isomorphic to a given element $A$ of $\mathcal{S}$, we say that $A$ is essentially the only element with that property.

The least element of a finite semilattice $A$ will be denoted by $0_A$. A finite semilattice has the largest element if and only if it is a lattice. If it exists, the largest element of $A$ will be denoted by $1_A$.

For other concepts of universal algebra and lattice theory the reader is referred to [5].

## 2. The ordered set of finite semilattices

An element $A$ of $\mathcal{S}$ is said to be covered by an element $B$ of $\mathcal{S}$ if $A < B$ and there is no $C \in \mathcal{S}$ with $A < C < B$. We write $A \prec B$ and also say that $B$ is a cover of $A$ or that $A$ is a subcover of $B$.

**2.1. Proposition.** *Let $A, B \in \mathcal{S}$. Then $A \prec B$ if and only if $A \leq B$ and $|B| = |A| + 1$.*

*Proof.* Let $A \prec B$. Clearly, $|A| < |B|$. Suppose that $|A| \leq |B| - 2$. Take a maximal element $c$ in $B - A$. Then $c$ is meet-irreducible in $B$, so that $C = B - \{c\}$ is a subsemilattice of $B$. We get $A < C < B$, a contradiction. Thus $|B| = |A| + 1$. The converse implication is obvious. $\square$

For every $n \geq 0$ we denote by $\mathbf{C}_n$ the (essentially unique) chain with $n+1$ elements. The height $\mathbf{h}(A)$ of a finite semilattice $A$ is the largest nonnegative integer $n$ such that $\mathbf{C}_n \leq A$. By a flat semilattice we mean a finite semilattice of height at most 1. For every $n \geq 0$ we denote by $\mathbf{A}_n$ the (essentially unique) flat semilattice with $n$ atoms. Thus $\mathbf{C}_0 \simeq \mathbf{A}_0$ is the trivial semilattice.

**2.2. Proposition.** $\mathbf{C}_n$ $(n \geq 0)$ *and* $\mathbf{A}_n$ $(n \geq 0)$ *are essentially the only elements $A$ of $\mathcal{S}$ such that $(A]$ (the principal ideal of $\mathcal{S}$ generated by $A$) is a chain. The set of finite chains is a definable subset of $\mathcal{S}$ and each finite chain is a definable element of $\mathcal{S}$. The set of flat semilattices is a definable subset of $\mathcal{S}$ and each flat semilattice is a definable element of $\mathcal{S}$.*

*Proof.* Suppose that $(A]$ is a chain and that $A$ is not a chain, so that $A$ contains two incomparable elements $x, y$. If there exists an element $z < x \wedge y$ then $\{x, y, x \wedge y\}$ and $\{z, x \wedge y, x\}$ are two incomparable subsemilattices of $A$, a contradiction. Thus $x \wedge y = 0_A$ for any pair $x, y$ of incomparable elements. If $x, y$ are incomparable and there exists an element $z > x$ then $\{x, y, 0_A\}$ and $\{0_A, x, z\}$ are two incomparable subsemilattices of $A$, a contradiction again. Thus $A \simeq \mathbf{A}_n$ for some $n$.

It follows that the union of the set of finite chains with the set of flat semilattices is a definable subset. The chain with four elements has seven covers in $\mathcal{S}$, while the flat semilattice with four elements has only five covers in $\mathcal{S}$. Thus each of the two sets is definable; since each of the two sets is a subchain of $\mathcal{S}$, their elements are definable elements of $\mathcal{S}$. $\square$

**2.3. Proposition.** *The following binary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B \rangle \in R$ if and only if $B$ is the chain of the same height as $A$.*

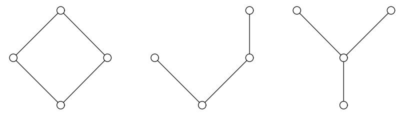
*Proof.* $B$ is essentially the only maximal element of $\mathcal{S}$ such that $B$ is a chain and $B \leq A$. $\square$

For every $n \geq 1$ denote by $\mathbf{B}_n$ the Boolean lattice of height $n$ (so that it has $n$ atoms and $2^n$ elements). For every $n \geq 1$ denote by $\mathbf{Y}_n$ the semilattice with elements $a_0, \dots, a_{n+1}$ and covers $a_0 < a_1 < \dots < a_{n-1} < a_n$ and $a_{n-1} < a_{n+1}$. For every $n \geq 1$ denote by $\mathbf{V}_n$ the semilattice with elements $a_0, \dots, a_{n+1}$ and covers $a_0 < a_1 < \dots < a_n$ and $a_0 < a_{n+1}$. We have $|\mathbf{Y}_n| = |\mathbf{V}_n| = n+2$ and $\mathbf{h}(\mathbf{Y}_n) = \mathbf{h}(\mathbf{V}_n) = n$.

By a tree we mean a finite semilattice $A$ such that $\mathbf{B}_2 \not\leq A$.

**2.4. Proposition.** *Every semilattice with at most four elements is definable.*

*Proof.* Except for chains and flat semilattices, there are essentially only three at most four-element semilattices: the semilattices $\mathbf{B}_2$, $\mathbf{V}_2$ and $\mathbf{Y}_2$. The set consisting of these three semilattices is definable, since they are (essentially) the only covers of



$\mathbf{B}_2$ $\mathbf{V}_2$ $\mathbf{Y}_2$

Fig. 1

$\mathbf{C}_2$ (or also of $\mathbf{A}_2$) that are not either chains or flat semilattices. Now $\mathbf{Y}_2$ is the only one of them that is below only one of the five covers of $\mathbf{A}_3$, so that $\mathbf{Y}_2$ is definable. $\mathbf{B}_2$ is the only one of the remaining two semilattices that has only six covers. Thus $\mathbf{B}_2$ is definable. The remaining semilattice $\mathbf{V}_2$ is then also definable. □

**2.5. Proposition.** *The set of trees is definable.*

*Proof.* Since (by 2.4) $\mathbf{B}_2$ is definable, the set of trees is definable. □

**2.6. Lemma.** *A finite semilattice $A$ is isomorphic to $\mathbf{Y}_n$ for some $n \geq 1$ if and only if $\mathbf{A}_2 \leq A$, $\mathbf{B}_2 \nleq A$, $\mathbf{V}_2 \nleq A$ and $\mathbf{A}_3 \nleq A$. Consequently, the set $\{\mathbf{Y}_n : n \geq 1\}$ is definable and its every element is definable. Moreover, the following binary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B\rangle \in R$ iff $A \simeq \mathbf{Y}_n$ and $B \simeq \mathbf{C}_n$ for some $n \geq 1$.*

*Proof.* Clearly, each $\mathbf{Y}_n$ has these properties. Let $A$ have these properties. Since $\mathbf{A}_2 \leq A$, there are two incomparable elements $a, b$ in $A$. Since $\mathbf{B}_2 \nleq A$ and $\mathbf{V}_2 \nleq A$, the elements $a, b$ are both maximal. Put $c = a \wedge b$. Since $\mathbf{V}_2 \nleq A$, both $a$ and $b$ are covers of $c$. If there is an element $d > c$ different from both $a$ and $b$, it must be incomparable with both $a$ and $b$ and the set $\{a, b, c, d\}$ is a subsemilattice isomorphic with $\mathbf{A}_3$, a contradiction. Thus $a, b$ are the only elements strictly above $c$. Suppose that there is an element $e$ incomparable with $c$. We can suppose that $e$ covers $c \wedge e$, since otherwise we could replace $e$ by a cover of $c \wedge e$ that is below $e$. Clearly, $e$ cannot be below both $a$ and $b$. Without loss of generality, $e \nleq a$. Then $a \wedge e = c \wedge e$ and $\{a, c, c \wedge e, e\}$ is a subsemilattice isomorphic with $\mathbf{V}_2$, a contradiction.

Thus $c$ is comparable with all elements of $A$. If there are two incomparable elements below $c$ then $A$ has a subsemilattice isomorphic with $\mathbf{B}_2$, a contradiction. Thus the principal ideal generated by $c$ is a chain and $A$ is isomorphic to $\mathbf{Y}_n$ for some $n$.

For $A \simeq \mathbf{Y}_n$, $\mathbf{C}_n$ is essentially the only chain $B$ with $\mathbf{h}(B) = \mathbf{h}(A)$. □

Denote by $\mathbf{U}$ the semilattice with five elements $a, b, c, d, e$ and coverings $a < b < c$ and $a < d < e$.

**2.7. Lemma.** $\mathbf{U}$ *is a definable element of $\mathcal{S}$.*

*Proof.* $\mathbf{U}$ is essentially the only finite semilattice $A$ such that for any $B$, $B < A$ if and only if $B \leq \mathbf{V}_2$. □

**2.8. Lemma.** *A finite semilattice $A$ is isomorphic to $\mathbf{V}_n$ for some $n \geq 1$ if and only if $\mathbf{A}_2 \leq A$, $\mathbf{B}_2 \nleq A$, $\mathbf{Y}_2 \nleq A$, $\mathbf{A}_3 \nleq A$ and $\mathbf{U} \nleq A$. Consequently, the set*

*$\{\mathbf{V}_n : n \geq 1\}$ is definable and its every element is definable. Moreover, the following binary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B\rangle \in R$ iff $A \simeq \mathbf{V}_n$ and $B \simeq \mathbf{C}_n$ for some $n \geq 1$.*

*Proof.* Clearly, each $\mathbf{V}_n$ has these properties. Let $A$ have these properties. Since $\mathbf{A}_2 \leq A$, there are two incomparable elements $a, b$ in $A$. Since $\mathbf{Y}_2 \nleq A$, $a \wedge b = 0_A$. We can assume that $a, b$ are atoms. Since $\mathbf{A}_3 \nleq A$, $a$ and $b$ are the only two atoms of $A$. Since $\mathbf{B}_2 \nleq A$ and $\mathbf{U} \nleq A$, at least one of the elements $a, b$ is maximal in $A$. Without loss of generality, $a$ is maximal. All the other elements of $A$ are above $b$. If two elements above $b$ are incomparable then $\mathbf{Y}_2 \leq A$, a contradiction. Thus the elements above $b$ form a chain and $A$ is isomorphic to $\mathbf{V}_n$ for some $n$.

For $A \simeq \mathbf{V}_n$, $\mathbf{C}_n$ is essentially the only chain $B$ with $\mathbf{h}(B) = \mathbf{h}(A)$. □

For $n \geq 1$ and $0 \leq m < n$ denote by $\mathbf{V}_{n,m}$ the semilattice with elements $a_0, \dots, a_{n+1}$ and covers $a_0 < a_1 < \dots < a_n$ and $a_m < a_{n+1}$. Thus $|\mathbf{V}_{n,m}| = n + 2$ and $\mathbf{h}(\mathbf{V}_{n,m}) = n$. We have $\mathbf{V}_n = \mathbf{V}_{n,0}$ and $\mathbf{Y}_n = \mathbf{V}_{n,n-1}$.

**2.9. Lemma.** *A finite semilattice $A$ is isomorphic to $\mathbf{V}_{n,m}$ for some $n, m$ if and only if $A$ is a tree, $A$ is not a chain and there exists a chain $C$ that is covered by $A$. Consequently, the set $\{\mathbf{V}_{n,m} : n \geq 1, \quad 0 \leq m < n\}$ is definable and its every element is definable. Moreover, the following ternary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B, C\rangle \in R$ iff $A \simeq \mathbf{V}_{n,m}$, $B \simeq \mathbf{C}_n$ and $C \simeq \mathbf{C}_m$ for some $n, m$.*

*Proof.* Clearly, each $\mathbf{V}_{n,m}$ has these properties. Let $A$ have these properties. There exists a largest element $a$ in $A$ such that $a$ is comparable with every element of $A$. Since $A$ is a tree, $[0, a]$ is a chain. Since $A$ is not a chain, the element $a$ has at least two covers $b$ and $c$. Since $A$ is a tree that becomes a chain after removing just one element, the elements $b$ and $c$ are the only covers of $a$, one of them is maximal and all the remaining elements of $A$ form a chain above the second one of them. Thus $A \simeq \mathbf{V}_{n,m}$ for some $n, m$.

For $A \simeq \mathbf{V}_{n,m}$, $\mathbf{C}_n$ is essentially the only chain $B$ with $\mathbf{h}(B) = \mathbf{h}(A)$ and $\mathbf{C}_m$ is the maximal chain such that $\mathbf{Y}_{m+1} \leq A$. □

**2.10. Theorem.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B, C\rangle \in R$ if and only if $A, B, C$ are chains and $\mathbf{h}(C) = \mathbf{h}(A) + \mathbf{h}(B)$.*

*Proof.* We have $\langle A, B, C\rangle \in R$ if and only if there exist $D, E, F, G \in \mathcal{S}$ such that the following conditions are satisfied:
$D \simeq \mathbf{V}_{n,m}$ for some $n, m$;
$E$ is the largest element of $\mathcal{S}$ such that $E \leq D$ and $E \simeq \mathbf{V}_k$ for some $k$;
$F$ is the largest element of $\mathcal{S}$ such that $F \leq D$ and $F \simeq \mathbf{Y}_r$ for some $r$;
$A$ is a chain of the same height as $E$;
$B$ is a chain of the same height as $F$;
$G$ is a chain of the same height as $D$;
$C$ is a chain and $C$ covers $G$ in $\mathcal{S}$. □

For $n \geq 2$ and $2 \leq m \leq n$ denote by $\mathbf{D}_{n,m}$ the semilattice with elements $a_0, \dots, a_{n+1}$ and covers $a_0 < a_1 < \dots < a_n$ and $a_0 < a_{n+1} < a_m$. Thus $\mathbf{D}_{2,2} = \mathbf{B}_2$.

**2.11. Lemma.** *A finite semilattice $A$ is isomorphic to $\mathbf{D}_{n,m}$ for some $n, m$ if and only if $\mathbf{B}_2 \leq A$, $\mathbf{Y}_2 \nleq A$ and there exists a chain $C$ that is covered by $A$. Consequently, the set $\{\mathbf{D}_{n,m} : n \geq 2, \quad 2 \leq m \leq n\}$ is definable. Moreover, the following ternary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B, C\rangle \in R$ iff $A \simeq \mathbf{D}_{n,m}$, $B \simeq \mathbf{C}_n$ and $C \simeq \mathbf{C}_m$ for some $n, m$.*

*Proof.* Clearly, each $\mathbf{D}_{n,m}$ has these properties. Let $A$ have these properties. Since $\mathbf{Y}_2 \nleq A$ and $A$ is not a chain, there are at least two atoms $a, b$ in $A$. Since $\mathbf{Y}_2 \nleq A$, the principal filter generated by any non-zero element is a chain. If $A$ has more than two atoms then it has more than two maximal elements; but this is not possible, since $A$ becomes a chain after removing just one element. Thus $a, b$ are the only two atoms. Since $\mathbf{B}_2 \leq A$, there is an element above both $a$ and $b$; consequently, the two atoms have a join $d$ in $A$. Since $A$ becomes a chain after removing just one element (that one element must be either $a$ or $b$), $d$ is a cover of either $a$ or $b$ (without loss of generality, $d$ is a cover of $b$) and $A$ becomes a chain after removing $b$. Thus $A \simeq \mathbf{D}_{n,m}$ for some $n, m$.

For $A \simeq \mathbf{D}_{n,m}$, $\mathbf{C}_n$ is essentially the only chain $B$ with $\mathbf{h}(B) = \mathbf{h}(A)$ and $\mathbf{C}_m$ is the maximal chain such that $\mathbf{V}_{m-1} \leq A$. □

**2.12. Lemma.** *The set of all $A \in \mathcal{S}$ such that $A$ has at least two atoms is definable. The set of all $A \in \mathcal{S}$ such that $A$ has precisely one atom is definable.*

*Proof.* It is sufficient to prove the definability of the first set. A finite semilattice $A$ has at least two atoms if and only if it satisfies the following condition: there exists a $B \in \mathcal{S}$ such that $B \leq A$, $\mathbf{h}(B) = \mathbf{h}(A)$ and either $B \simeq \mathbf{V}_n$ for some $n$ or $B \simeq \mathbf{D}_{n,m}$ for some $n, m$. □

For two finite semilattices $A, B$ denote by $A \oplus B$ the finite semilattice with underlying set the disjoint union of $A$ and $B$, such that $A$ and $B$ are subsemilattices and $x < y$ whenever $x \in A$ and $y \in B$. We call $A \oplus B$ the ordinal sum of $A$ and $B$.

For two finite semilattices $A, B$ we denote by $A +_c B$ the semilattice $C$ with the underlying set the almost disjoint union of $A$ and $B$, with just the two least elements glued into a single one, such that both $A$ and $B$ are subsemilattices and $a \wedge b = 0_C$ for all $a \in A$ and $b \in B$. The semilattice $A +_c B$ is called the (amalgamated) cardinal sum of $A$ and $B$. We have $|A +_c B| = |A| + |B| - 1$. Put $A_1 +_c \dots +_c A_k = ((A_1 +_c A_2) +_c \dots) +_c A_k$. For example, $\mathbf{A}_n$ is isomorphic to the cardinal sum of $n$ copies of $\mathbf{C}_1$.

**2.13. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable: $\langle A, B, C\rangle \in R$ if and only if $A$ has at least two atoms, $B$ is a chain and $C \simeq B \oplus A$.*

*Proof.* $B \oplus A$ is essentially the only element $C$ of $\mathcal{S}$ with these two properties:
$\mathbf{h}(C) = \mathbf{h}(A) + \mathbf{h}(B) + 1$;
$A$ is the largest element of $\mathcal{S}$ such that $A \leq C$ and $A$ has at least two atoms. □

**2.14. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable: $\langle A, B, C\rangle \in R$ if and only if $B$ is a chain and $C \simeq B \oplus A$.*

*Proof.* $B \oplus A$ is essentially the only element $C$ of $\mathcal{S}$ such that either $A$ is a chain and $C$ is a chain of height $\mathbf{h}(A) + \mathbf{h}(B) + 1$, or else $A$ has at least two atoms and $C \simeq B \oplus A$, or else there exist $D, E \in \mathcal{S}$ with the following properties:

$D$ has at least two atoms;

$E$ is a chain;

$A \simeq E \oplus D$;

$C \simeq (E \oplus B) \oplus D$. □

**2.15. Lemma.** *The following binary relation $R$ on $\mathcal{S}$ is definable: $\langle A, B\rangle \in R$ if and only if $A$ has at least two atoms and $B$ is the (essentially unique) chain of height $|A|$.*

*Proof.* By a component of an element $C \in \mathcal{S}$ we will mean (just in this proof) a maximal element $D$ of $\mathcal{S}$ with the following properties: $D \le C$, $D$ is not a chain and $D$ has precisely one atom. Each component $D$ of $C$ can be written uniquely as $D = P \oplus Q$ where $P$ is a chain and $Q$ has at least two atoms. $P$ will be called the lower part and $Q$ will be called the upper part of $C$. If $D_1$ and $D_2$ are two components of $C$, then we say that $D_1$ dominates $D_2$ if the upper part of $D_1$ is a cover of the upper part of $D_2$ and the lower part of $D_2$ is a cover of the lower part of $D_1$. (Thus $|D_1| = |D_2|$.)

We have $\langle A, B\rangle \in R$ if and only if $A$ has at least two atoms, $B$ is a chain and there exist $C, E \in \mathcal{S}$ with the following properties:

(1) $E$ is a chain;
(2) $E \oplus A$ is a component of $C$;
(3) for every component $D_1$ of $C$ such that the upper part of $D_1$ is not isomorphic to $\mathbf{V}_1$ there exists another component $D_2$ of $C$ such that $D_1$ dominates $D_2$;
(4) there is precisely one component $D$ of $C$ the upper part of which is isomorphic to $\mathbf{V}_1$; the lower part of this component is a chain of height $\mathbf{h}(E) + \mathbf{h}(B) - 3$.

If this is true then (using the previous results) it is clear that $R$ is definable. Of course, this needs to be proved. Put $n = |A|$.

Let us start with the direct implication. Let $\langle A, B\rangle \in R$. Take a chain $E$ of height larger than $\mathbf{h}(A)$. There exists a sequence $A_0, \dots, A_{n-3}$ of elements of $\mathcal{S}$ such that $A_0 = A$ and every $A_{i+1}$ is a subsemilattice of $A_i$ obtained from $A_i$ by deleting a maximal element; if $A_i$ has a maximal element that is not an atom, one such should be chosen to be deleted. Since $A_0$ has $n$ elements, $A_{n-3}$ has three elements and is isomorphic with $\mathbf{V}_1$; each $A_i$ has at least two atoms. For each $i$ denote by $E_i$ the chain of height $\mathbf{h}(E) + i$. Denote by $C$ the cardinal sum of the semilattices $E_i \oplus A_i$ ($i = 0, \dots, n-3$). Since $E$ is a long chain, each $E_i \oplus A_i$ is a component of $C$. Clearly, there are no other components of $C$. Now it is easy to check that the conditions (1) through (4) are satisfied.

Conversely, let $C$ and $E$ have all the four properties. It follows from (2) and (3) that there exists a sequence $D_0, \dots, D_m$ of components of $C$ for some $m \ge 0$ such that $D_0 = E \oplus A$, each $D_i$ dominates $D_{i+1}$ and $D_m$ has upper component isomorphic to $\mathbf{V}_1$. Since the cardinalities of the upper parts are decreasing by (1), we have $m = n - 3$. Denote by $F$ the lower part of $D_m$. Since the cardinalities of the lower parts are increasing by (1), we have $\mathbf{h}(F) = \mathbf{h}(E) + |A| - 3$. By (4) we have $\mathbf{h}(F) = \mathbf{h}(E) + \mathbf{h}(B) - 3$. Thus $\mathbf{h}(B) = |A|$. □

**2.16. Theorem.** *The following binary relations $R, R', R''$ on $\mathcal{S}$ are definable:*
$\langle A, B\rangle \in R$ *iff $B$ is the chain of height $|A|$;*
$\langle A, B\rangle \in R'$ *iff $|A| = |B|$;*
$\langle A, B\rangle \in R''$ *iff $|A| \le |B|$.*

*Proof.* It follows easily from 2.15. □

Definability of various similar relations, like $|A| > |B| + 9$, also follows immediately.

Let $n \ge 2$ and let $A$ be the flat semilattice with $n$ atoms (so that $A \simeq \mathbf{A}_n$). We denote by $A'$ the semilattice with underlying set $A \cup \{o, a\}$ such that $A$ is a principal filter in $A'$ and the only new covers are $o < a$ and $o < 0_A$. We denote by $A''$ the semilattice with underlying set $A \cup \{o, a\}$ such that $A$ is a principal filter in $A''$ and the only new covers are $o < a < b$ and $o < 0_A$, where $b$ is one of the atoms of $A$.

$\mathbf{A}'_3$ $\mathbf{A}''_3$

Fig. 2

**2.17. Lemma.** *The following two binary relations on $\mathcal{S}$ are definable:*
$\langle A, B\rangle \in R_1$ *iff $A \simeq \mathbf{A}_n$ and $B \simeq \mathbf{A}'_n$ for some $n \ge 2$;*
$\langle A, B\rangle \in R_2$ *iff $A \simeq \mathbf{A}_n$ and $B \simeq \mathbf{A}''_n$ for some $n \ge 2$.*

*Proof.* $\mathbf{A}'_n$ and $\mathbf{A}''_n$ are essentially the only two covers $B$ of $\mathbf{C}_0 \oplus \mathbf{A}_n$ such that $\mathbf{h}(B) = 2$ and $\mathbf{A}_{n+1} \not\le B$. Moreover, $\mathbf{A}''_n$ is the only one of them which is above $\mathbf{B}_2$. □

**2.18. Lemma.** *A finite semilattice $A$ has the largest element and at least two coatoms if and only if there exists a finite semilattice $B$ with the following properties:*

(1) *$A < B$;*
(2) *where $n = |A|$, $n \ge 4$, $\mathbf{A}_n \le B$ and $\mathbf{A}_{n+1} \not\le B$;*
(3) *where $m = \mathbf{h}(A)$, $\mathbf{C}_{m-1} \oplus \mathbf{A}_n \le B$;*
(4) *$\mathbf{A}'_n \not\le B$ and $\mathbf{A}''_n \not\le B$;*
(5) *the elements of $\mathcal{S}$ between $A$ and $B$ form (essentially) a chain;*
(6) *$|B| = 2n$ and $\mathbf{h}(B) = m + 1$.*

*Consequently, the set of finite semilattices with the largest element and at least two coatoms is a definable subset of $\mathcal{S}$.*

*Proof.* Let $A$ have the largest element $1_A$ and at least two coatoms. Put $n = |A|$ and $m = \mathbf{h}(A)$, so that $n \geq 4$ and $m \geq 2$. Define $B$ as the disjoint union of its subsemilattice $A$ with an $n$-element set $\{b_1, \dots, b_n\}$, with the only covers not in $A$ being $1_A < b_i$ $(1 \leq i \leq n)$. It will be clear that the conditions are satisfied if we prove that whenever $U$ is a subsemilattice of $B$ isomorphic with $A$ then either $U = A$ or $U = (A - \{1_A\}) \cup \{b_i\}$ for some $i$. Since $U$ has the largest element, it can contain at most one of the elements $b_i$. If it contains no element $b_i$ then clearly $U = A$. Let $b_i$ be the only element of $U$ not belonging to $A$. Then there is precisely one element $a \in A$ not in $U$. If $a \neq 1_A$ then $U$ contains only one coatom (namely $1_A$), so that it is not isomorphic to $A$. Thus $a = 1_A$ and $U = (A - \{1_A\}) \cup \{b_i\}$.

Conversely, let there exist a $B$ with the five properties. Let $U$ be any subsemilattice of $B$ isomorphic with $A$. By (3), $B$ contains a subsemilattice with elements $a_0, \dots, a_m, b_1, \dots, b_n$ and covers $a_0 < a_1 < \dots < a_m$ and $a_m < b_i$ $(i = 1, \dots, n)$; since $B$ has height $m + 1$, all these covers are also covers in $B$ and every $b_i$ is a maximal element of $B$; so, if some $b_i$ belongs to $U$ then it is also maximal in $U$.

Suppose that there exists an element $c \in B$ such that $c < b_i$ for some $i$ and $c$ is incomparable with $a_m$. Then $c \wedge b_i = c \wedge a_m < a_m$. For every $j \neq i$ we have $c \wedge b_j = c \wedge b_i \wedge b_j = c \wedge a_m$. Thus $\{b_1, \dots, b_n, a_m, c, c \wedge a_m\}$ is a subsemilattice isomorphic to $\mathbf{A}''_n$, a contradiction.

Thus $c < b_i$ implies $c \leq a_m$ for any $c \in B$ and any $i$.

Suppose that an element $c$ of $B$ is incomparable with $a_m$. Then for every $i$, $c$ is incomparable with $b_i$ and we have $c \wedge b_i < a_m$, so that $c \wedge b_i = c \wedge a_m$. Thus $\{b_1, \dots, b_n, a_m, c, c \wedge a_m\}$ is a subsemilattice isomorphic to $\mathbf{A}'_n$, a contradiction.

Thus every element of $B$ is comparable with $a_m$. Since every element larger than $a_m$ is maximal in $B$ and $\mathbf{A}_{n+1} \not\leq B$, every element of $B$ except the elements $b_i$ is $\leq a_m$.

Suppose that for some $i$, both $b_i$ and $a_m$ belong to $U$. There exists an index $j$ such that $b_j \notin U$. Also, there exists an element $a \in U$ such that $a < a_m$. Denote by $k$ the number of maximal elements in the subsemilattice $U \cup \{a\}$ of $B$. The number of maximal elements in the subsemilattice $U \cup \{b_j\}$ is $k+1$, so that the two subsemilattices are non-isomorphic; since they are of the same cardinality, they are incomparable elements of $\mathcal{S}$; this is a contradiction with (5).

Thus if $U$ contains an element of $\{b_1, \dots, b_n\}$ then it contains only one such element and $a_m \notin U$. From this it follows that either $U = (a_m]_B$ or else $U = ((a_m]_B - \{a_m\}) \cup \{b_i\}$ for some $i$. In particular, $U$ (and thus $A$) contains a largest element. If $A$ contained only one coatom $c$, then we would have two incomparable elements of $\mathcal{S}$ between $A$ and $B$: the first with the coatom removed and two covers of $1_A$ added, the other just with one cover of $1_A$ added. $\square$

**2.19. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B, C\rangle \in R$ iff $A$ has the largest element and at least two coatoms, $B$ is a chain and $C \simeq A \oplus B$.*

*Proof.* Let $A$ have the largest element and at least two coatoms, and let $B$ be a chain. It is sufficient to prove that $A \oplus B$ is essentially the only element $C$ of $\mathcal{S}$ with the following properties:

(1) $|C| = |A| + |B|$;
(2) $\mathbf{h}(C) = \mathbf{h}(A) + \mathbf{h}(B) + 1$;
(3) $A$ is the largest element of $\mathcal{S}$ such that $A \leq C$, $A$ is a lattice and $A$ has at least two coatoms.

Clearly, $A \oplus B$ has all these properties. Let $C$ satisfy (1), (2) and (3). By (3), $A \leq C$ and we can assume that $A$ is a subsemilattice of $C$. Put $n = |A|$, $m = \mathbf{h}(A)$ and $k = \mathbf{h}(B)$. By (2), there is a maximal subchain $a_0 < \dots < a_{m+k+1}$ of $C$. At most $m+1$ members of this chain belong to $A$, so that at least $k+1$ members belong to $C - A$. But (by (1)) $|C - A| = k+1$. Thus precisely $k+1$ members of the chain belong to $C - A$, precisely $m+1$ members belong to $A$ and $C - A$ is a chain. Since $\mathbf{h}(A) = m$, those $m+1$ members are a maximal subchain of $A$; in particular, the largest element $1_A$ of $A$ is among them. From this it follows that $a_{m+k+1}$ is the largest element of $C$. It follows easily from (3) that all the $k+2$ elements of the chain $(C - A) \cup \{1_A\}$ are above all elements of $A - \{1_A\}$, so that $C$ is isomorphic to $A \oplus B$. □

**2.20. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:* $\langle A, B, C \rangle \in R$ *iff* $B$ *is a chain and* $C \simeq A \oplus B$.

*Proof.* It follows easily from 2.19. □

**2.21. Theorem.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:* $\langle A, B, C \rangle \in R$ *iff* $C \simeq A \oplus B$.

*Proof.* Let $A_1$ be the chain of the same height as $A$ and let $B_1$ be the chain of the same height as $B$. Then $A \oplus B$ is the smallest element $C$ of $\mathcal{S}$ such that $\mathbf{h}(C) = \mathbf{h}(A) + \mathbf{h}(B) + 1$, $|C| = |A| + |B|$, $A \oplus B_1 \leq C$ and $A_1 \oplus B \leq C$. □

**2.22. Theorem.** *The set of semilattices with the largest element* (*i.e., lattices*) *is definable.*

*Proof.* It follows from 2.21. □

For every $n, m \geq 1$ we denote by $\mathbf{W}_{n,m}$ the cardinal sum of $m$ copies of $\mathbf{C}_n$. Thus $|\mathbf{W}_{n,m}| = nm + 1$ and $\mathbf{h}(W_{n,m}) = n$.

**2.23. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:* $\langle A, B, C \rangle \in R$ *iff* $A \simeq \mathbf{C}_n$, $B \simeq \mathbf{C}_m$ *and* $C \simeq \mathbf{W}_{n,m}$ *for some* $n, m \in \omega$.

*Proof.* $\mathbf{W}_{n,m}$ is essentially the only element $C$ of $\mathcal{S}$ with these properties: $\mathbf{A}_m \leq C$; $\mathbf{A}_{m+1} \not\leq C$; $\mathbf{Y}_2 \not\leq C$; $\mathbf{B}_2 \not\leq C$; $\mathbf{h}(C) = n$; every subsemilattice of $C$ with at least $|C| - m + 1$ elements is of height $n$. □

**2.24. Theorem.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:* $\langle A, B, C \rangle \in R$ *iff* $A \simeq \mathbf{C}_n$, $B \simeq \mathbf{C}_m$ *and* $C \simeq \mathbf{C}_{nm}$ *for some* $n, m \in \omega$.

*Proof.* It follows from 2.23. □

By a regular semilattice we mean a finite semilattice $A$ such that where $a_1, \dots, a_k$ are all the (pairwise distinct) maximal elements of $A$, the subsemilattices $(a_i]$ are pairwise incomparable and $A = (a_1] +_c \dots +_c (a_k]$.

By a max-sublattice of a finite semilattice $A$ we mean any element $B$ of $\mathcal{S}$ that is maximal with respect to the property that $B \le A$ and $B$ is a lattice. Clearly, every max-sublattice of $A$ is isomorphic to $(a]_A$ for a maximal element of $A$; the converse is not true.

**2.25. Lemma.** *The set of regular semilattices is a definable subset of $\mathcal{S}$.*

*Proof.* The subset can be defined as follows. An element $A$ of $\mathcal{S}$ is regular if and only if the following two conditions are satisfied:

(1) for every $C \le A$, if every max-sublattice of $A$ is $\le C$ then $C \simeq A$;
(2) whenever $B, C$ are two non-isomorphic max-sublattices of $A$ and $D$ is an element of $\mathcal{S}$ such that $B \le D \le A$ and $C \le D$, then $|D| \ge |B| + |C| - 1$.

The direct implication is easy. For the converse, let $A$ satisfy (1) and (2). Denote by $A_1, \dots, A_k$ all pairwise non-isomorphic maximal sublattices of $A$. There are pairwise distinct maximal elements $a_1, \dots, a_k$ of $A$ such that $(a_i] \simeq A_i$. Put $C = (a_1] \cup \dots \cup (a_k]$. Then $C$ is a subsemilattice of $A$ and every max-subsemilattice of $A$ is $\le C$, so that $C = A$ by (1). Thus $a_1, \dots, a_k$ are the only maximal elements of $A$. Let $i \ne j$. The subsemilattice $(a_i] \cup (a_j]$ of $A$ is above both $A_i$ and $A_j$, so that by (2) we have $|(a_i] \cup (a_j]| \ge |(a_i]| + |(a_j]| - 1$ and consequently $a_i \wedge a_j = 0_A$. Thus $A = (a_1] +_c \dots +_c (a_k]$. □

Let $A$ be a finite semilattice. Put $A^+ = \mathbf{C}_0 \oplus \mathbf{A}_2 \oplus \mathbf{A}_2 \oplus A \oplus \mathbf{C}_0$, so that $A^+$ is a lattice and $|A^+| = |A| + 8$. Denote by $r$ the least positive integer such that $\mathbf{A}_r \not\le A^+$. For every $k \ge 1$ we denote by $\beta_k(A)$ the regular semilattice with max-sublattices $A^+$ and $D_i = \mathbf{C}_{k-i} \oplus \mathbf{A}_{r+i-1} \oplus \mathbf{C}_0$ $(i = 1, \dots, k)$. (Clearly, these $k+1$ lattices are pairwise incomparable.) We have $|D_i| = k + r + 2$ for all $i$.

**2.26. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B, C\rangle \in R$ iff $B \simeq \mathbf{C}_k$ for some $k \ge 1$ and $C \simeq \beta_k(A)$.*

*Proof.* The proof, using 2.25, is obvious. □

Let $A$ be a finite semilattice and $a_1, \dots, a_k$ be a nonempty finite sequence of elements of $A$. We denote by $\gamma(A; a_1, \dots, a_k)$ the finite semilattice obtained from $\beta_k(A)$ by adding new elements $b_1, \dots, b_k$ and relations $1_{D_i} < b_i$ and $a_i < b_i$ $(i = 1, \dots, k)$. The cardinality of this semilattice is $|\beta_k(A)| + k$. The semilattice has $k+1$ maximal elements $1_{A^+}, b_1, \dots, b_k$ and $(b_i]$ are its $k+1$ max-sublattices.

For example, let $A = \mathbf{V}_2$ (elements $a_0, a_1, a_2, a_3$ and covers $a_0 < a_1 < a_2$ and $a_0 < a_3$. The semilattice $\gamma(A; a_3, a_2, a_2, a_0)$ is pictured in Fig. 3.

**2.27. Lemma.** *The following ternary relation $R$ on $\mathcal{S}$ is definable:*
*$\langle A, B, C\rangle \in R$ iff there exist a $k \ge 1$ and a sequence $a_1, \dots, a_k$ of elements of $A$ such that $B \simeq \mathbf{C}_k$ and $C \simeq \gamma(A; a_1, \dots, a_k)$.*

*Proof.* Let $A$ be a finite semilattice and $k$ be a positive integer; put $E = \beta_k(A)$ and let $r$ and $D_i$ have the same meaning as above; put $A' = \mathbf{C}_0 \oplus \mathbf{A}_2 \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$. We are going to prove that $\langle A, \mathbf{C}_k, C\rangle \in R$ if and only if the following conditions are satisfied:

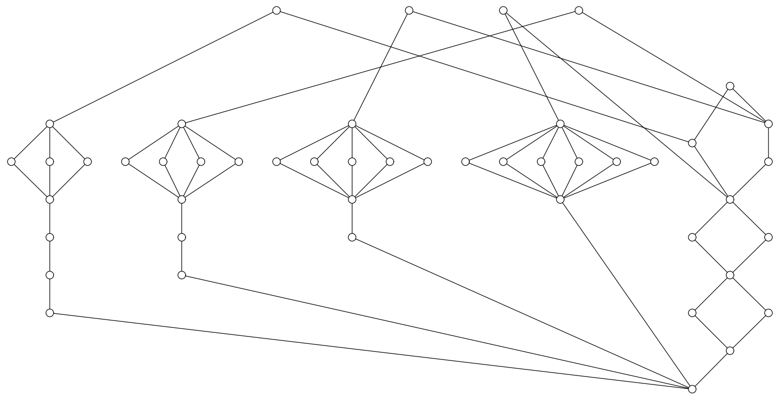


Fig. 3

(1) $E < C$;
(2) $|C| = |E| + k$;
(3) $A^+$ is a max-sublattice of $C$;
(4) for every $i = 1, \dots, k$ there exists (up to isomorphism) precisely one max-sublattice of $C$ larger than $D_i$, and it is not isomorphic to $A^+$;
(5) whenever $D$ is a max-sublattice of $C$ and $D \not\simeq A^+$ then $D \simeq (D_i +_c H) \oplus \mathbf{C}_0$ for precisely one $i$ and for a lattice $H$ with $A' \leq H$ and $\mathbf{A}_r \not\leq H$;
(6) $\mathbf{C}_0 \oplus \mathbf{A}_2 \oplus \mathbf{A}_2 \oplus \mathbf{A}_r \not\leq C$.

It will follow that $R$ is definable.

The direct implication is easy.

Let (1) through (6) be satisfied. By (4), for every $i$ there is a maximal element $e_i$ of $C$ such that $D_i < (e_i]$ and $(e_i]$ is a max-sublattice of $C$. Since $D_i$ is incomparable with $A^+$, we have $(e_i] \not\simeq A^+$ and thus, by (5), $(e_i] \simeq (D_j +_c H) \oplus \mathbf{C}_0$ for precisely one $j$ and some lattice $H$ with $A' \leq H$ and $\mathbf{A}_r \not\leq H$. Suppose that $i \neq j$. There exists an embedding $f$ of $D_i$ into $(D_j +_c H) \oplus \mathbf{C}_0$. Denote by $g_i$ the meet of the antichain in $D_i$. It follows from (6) that $f(g_i)$ does not belong to $H$; it cannot be equal to $e_i$, since $e_i$ is maximal; so, $f(g_i) \in D_j$. But this is possible only if $i = j$. Thus $1_{D_i} \leq e_i$. But $1_{D_i}$ is not above the largest element of $A^+$ and thus $1_{D_i} < e_i$. It follows that $e_i \in C - E$ and then, by (2), that $C - E = \{e_1, \dots, e_k\}$.

Denote by $d_i$ the largest element of $D_i$. For each $i$ we have $(e_i] = (d_i] +_c H_i$ where $H_i$ is a lattice. Denote by $a_i$ the largest element of $H_i$. Obviously, $a_i$ belongs to $A^+$ (because it cannot belong anywhere else). Since $A' \leq H_i$, the largest element of $A'$, i.e., the least element of $A$, is below $a_i$. Since the largest element of $A^+$ is a maximal element in $C$, we get $a_i \in A$. Thus $C \simeq \gamma(A, a_1, \dots, a_k)$. □

Let $A$ be a finite semilattice and $a_1, \dots, a_k$ be a sequence of its pairwise distinct elements, so that $k = |A|$. Take variables $X, Y, Z, Y_1, \dots, Y_k, Z_1, \dots, Z_k$ and denote by $\Phi$ the formula in these variables expressing that if $X$ is $A$ then $Y \simeq \gamma(A; b_1, \dots, b_k)$ for a sequence $b_1, \dots, b_k$ of elements of $A$, $Z \simeq A^+$, $Y_i \simeq D_i$

and $Z_i \simeq H_i$ for all $1 \leq i \leq k$. For every pair $i, j$ of different elements of $\{1, \dots, k\}$ denote by $\Psi_{i,j}$ the formula in the same variables expressing that if $U$ is the least element of $\mathcal{S}$ below $Y$ and above the two max-sublattices of $Y$ that are above $D_i$ and $D_j$, then $|U| = |Z_j| + |Y_i| + |Y_j|$. Let $\Gamma$ be the conjunction of the formulas $\Psi_{i,j}$ for all $i \neq j$ such that $a_i \leq a_j$ and of the formulas $\neg\Psi_{i,j}$ for all the remaining pairs $i, j \in \{1, \dots, k\}$ with $i \neq j$. Denote by $\Lambda(X)$ the following formula with one free variable $X$: there exist $Y, Z, Y_1, \dots, Y_k, Z_1, \dots, Z_k$ such that $\Phi$ and $\Gamma$. Now it is easy to see that $\Lambda(X)$ is satisfied in $\mathcal{S}$ if and only if $X \simeq A$. Thus we have proved:

**2.28. Theorem.** *Every element of $\mathcal{S}$ is definable. Consequently, the ordered set of isomorphism types of finite semilattices, ordered by embeddability, has only the identical automorphism.*

## 3. Universal classes of semilattices

A class $K$ of semilattices is called universal if it satisfies any of the following equivalent conditions:

(1) $K$ is axiomatizable and closed under subsemilattices;
(2) $K$ is closed under isomorphic images, subsemilattices and ultraproducts;
(3) $K$ is closed under subsemilattices and contains every semilattice $A$ such that all finite subsemilattices of $A$ belong to $K$.

We denote by $\mathcal{U}$ the lattice of all universal classes of semilattices (with respect to inclusion). Its least element is the empty class, the largest element is the class of all semilattices. Since the union of finitely many universal classes is a universal class, the lattice $\mathcal{U}$ is distributive; of course, it is a complete algebraic lattice.

As it is easy to see, the mapping $K \mapsto K \cap \mathcal{S}$ is an isomorphism of the lattice $\mathcal{U}$ onto the lattice of order-ideals of $\mathcal{S}$ (including the empty order-ideal). Every universal class of semilattices is generated by the set of its finite members.

We are well aware that it is illegal to speak about lattices of proper classes. One way how to make $\mathcal{U}$ legal would be to replace it by the lattice of order-ideals of $\mathcal{S}$.

A universal class $K \in \mathcal{U}$ is said to be finitely generated if it is generated (as a universal class) by finitely many finite semilattices. Clearly, $K \in \mathcal{U}$ is finitely generated if and only if it contains up to isomorphism only finitely many semilattices; also, if and only if it does not contain any infinite semilattice; also, if and only if it has only finitely many universal subclasses.

**3.1. Theorem.** *The set of finitely generated universal classes of semilattices is a definable subset of $\mathcal{U}$.*

*Proof.* An element $K$ of $\mathcal{U}$ belongs to the set if and only if for every $X \leq K$ with $X > 0_{\mathcal{U}}$ there exists an $Y$ with $Y \prec X$. □

For every finite semilattice $A$ denote by $\mathbf{U}(A)$ the universal class generated by $A$, i.e., the class of all semilattices isomorphic to a subsemilattice of $A$. Universal classes obtained in this way from individual finite semilattices will be called finitely one-generated.

**3.2. Theorem.** *A finitely generated universal class of semilattices is finitely one-generated if and only if it is a completely join-irreducible element of $\mathcal{U}$. Consequently, the set of finitely one-generated universal classes is a definable subset of $\mathcal{U}$.*

*Proof.* It is obvious. □

**3.3. Theorem.** *Every finitely generated universal class of semilattices is a definable element of $\mathcal{U}$. The lattice $\mathcal{U}$ has no non-identical automorphisms.*

*Proof.* The definable subset of finitely one-generated universal classes is an ordered set isomorphic to the quasiordered set $\mathcal{S}$ factored by the equivalence $\equiv$, where $X \equiv Y$ means $X \leq Y$ and $Y \leq X$. Thus it follows from2.28 that every finitely one-generated universal class is a definable element of $\mathcal{U}$ and every automorphism of $\mathcal{U}$ is identity on $F = \{\mathbf{U}(A) : A \in \mathcal{S}\}$. Since every element $K$ of $\mathcal{U}$ is the join of $\{\mathbf{U}(A) : A \in K \cap \mathcal{S}\}$, and this set is finite if $K$ is finitely generated, it follows that every automorphism of $\mathcal{U}$ is the identity and every finitely generated universal class is a definable element. □

Denote by $\mathbf{C}$ the class of all chains and by $\mathbf{A}$ the class of all flat semilattices.

**3.4. Theorem.** *$\mathbf{C}$ and $\mathbf{A}$ are definable elements of $\mathcal{U}$. The set of not finitely generated universal classes of semilattices is the union $(\mathbf{C}]_{\mathcal{U}} \cup (\mathbf{A}]_{\mathcal{U}}$ of two principal filters of $\mathcal{U}$.*

*Proof.* Clearly, $\mathbf{C}$ and $\mathbf{A}$ are not finitely generated universal classes. Let $K$ be a not finitely generated universal class of semilattices not containing all chains. Then $K$ does not contain all finite chains and thus all finite semilattices in $K$ are of a bounded height. But $K$ contains infinitely many non-isomorphic finite semilattices, so that (as it is easy to see) $K$ contains all the semilattices $\mathbf{A}_n$ and thus it contains all flat semilattices.

From this it follows easily that $\mathbf{C}$ and $\mathbf{A}$ are definable elements. □

**3.5. Lemma.** *The following conditions are equivalent for a universal class $K$ of semilattices:*

1. *$K$ is finitely axiomatizable;*
2. *$K$ is axiomatizable by one universal sentence;*
3. *the set of the minimal finite semilattices not belonging to $K$ is finite up to isomorphism.*

*Proof.* The equivalence of the first two conditions is obvious.

Let $K$ be the class of all semilattices satisfying the universal closure of a quantifier-free formula $\phi(x_1, \ldots, x_n)$ and let $A$ be a minimal semilattice not belonging to $K$. There exists an $n$-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $A$ such that $\phi(a_1, \ldots, a_n)$ is not true in $A$. But then $\phi(a_1, \ldots, a_n)$ is not true in the subsemilattice $B$ of $A$ generated by $a_1, \ldots, a_n$. Since $|B| < 2^n$ and $A$ is minimal, we get $|A| < 2^n$.

Conversely, let there exist a finite sequence $A_1, \ldots, A_n$ of finite semilattices such that $K$ is the class of all semilattices avoiding $A_1, \ldots, A_n$. For every $i = 1, \ldots, n$ it is easy to find a quantifier-free formula in $|A_i|$ variables that is satisfied under all interpretations in a semilattice $B$ if and only if $A_i$ is not embeddable into $B$. □

**3.6. Theorem.** *The set of finitely axiomatizable universal classes of semilattices is a definable subset of $\mathcal{U}$. Every finitely axiomatizable universal class of semilattices is a definable element of $\mathcal{U}$.*

*Proof.* It follows from 3.5 that a universal class $K$ is finitely axiomatizable if and only if there exists a finitely one-generated element $\mathbf{U}(A)$ of $\mathcal{U}$ such that $\mathbf{U}(B) \leq \mathbf{U}(A)$ for every finitely one-generated element $\mathbf{U}(B)$ of $\mathcal{U}$ such that $\mathbf{U}(B)$ is minimal among the elements of $\mathcal{U}$ that are not below $K$. The second statement also follows easily from 3.5 and 2.28. □

## 4. Open problems

**4.1. Problem.** For which positive integers $n$ is it true that every semilattice with $n$ elements is uniquely determined up to isomorphism by the isomorphism types of its proper subsemilattices?

We know that the answer to this question is no when $n = 4$, and yes for both $n = 5$ and $n = 6$. If it is true for all numbers $n \geq 5$, we would have a more simple proof of Theorem 2.28. (We would get the definability of a finite semilattice $A$ just by induction on the number of elements of $A$.)

We have seen in Theorem 3.4 that the set of not finitely generated universal classes of semilattices is the union of two principal filters of $\mathcal{U}$. It is easy to see that the set of not finitely generated universal classes of distributive lattices is a principal filter in the lattice of universal classes of distributive lattices (the principal filter generated by the class of chains). Thus it is natural to ask:

**4.2. Problem.** Which locally finite universal classes $K$ (or just varieties) have the property that the set of not finitely generated universal subclasses of $K$ is the union of finitely many principal filters in the lattice of all universal subclasses of $K$?

## References

[1] J. Ježek, *The lattice of equational theories I, II, III, IV.* Czechoslovak Math. J. **31**, 1981, 127–152; **31**, 1981, 573–603; **32**, 1982, 129–164; **36**, 1986, 331–341.
[2] J. Ježek, *The ordering of commutative terms.* Czechoslovak Math. J. **56**, 2006, 133–154.]]
[3] J. Ježek and R. McKenzie, *Definability in the lattice of equational theories of semigroups.* Semigroup Forum **46**, 1993, 199–245.
[4] A. Kisielewicz, *Definability in the lattice of equational theories of commutative semigroups.* Trans. Amer. Math. Soc. **356**, 2004, 3483–3504.
[5] R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties, Volume I.* Wadsworth & Brooks/Cole, Monterey, CA, 1987.

Department of Mathematics, Charles University, Prague, Czech Republic
*E-mail address*: `jezek@karlin.mff.cuni.cz`

Department of Mathematics, Vanderbilt University, Nashville, TN 37235, USA
*E-mail address*: `ralph.mckenzie@vanderbilt.edu`