Definability in substructure orderings, III: finite distributive lattices

Jaroslav Ježek and Ralph McKenzie

Abstract. Let $\mathcal{D}$ be the ordered set of isomorphism types of finite distributive lattices, where the ordering is by embeddability. We study first-order definability in this ordered set. We prove among other things that for every finite distributive lattice $\mathbf{D}$, the set $\{d, d^{\text{opp}}\}$ is definable, where d and d^{opp} are the isomorphism types of $\mathbf{D}$ and its opposite ($\mathbf{D}$ turned upside down). We prove that the only non-identity automorphism of $\mathcal{D}$ is the opposite map. Then we apply these results to investigate definability in the closely related lattice of universal classes of distributive lattices. We prove that this lattice has only one non-identity automorphism, the opposite map; that the set of finitely generated and also the set of finitely axiomatizable universal classes are definable subsets of the lattice; and that for each element K of the two subsets, $\{K, K^{\text{opp}}\}$ is a definable subset of the lattice.

1. Introduction

The set $\mathcal{D}$ of isomorphism types of finite distributive lattices is denumerable. This set becomes a poset under the order induced by the substructure relation— we put $l_0 \leq l_1$, where l_i is the type of the finite distributive lattice $\mathbf{L}_i$, iff $\mathbf{L}_0$ is isomorphic to a sub-lattice of $\mathbf{L}_1$. In this way we obtain a poset $\langle \mathcal{D}, \leq \rangle$. In this paper, we explore the scope of first-order definitions in the structure $\langle \mathcal{D}, \leq \rangle$. It is an interesting topic because that scope is surprisingly wide.

The preceding remarks illustrate, by way of example, what we mean by the phrase "definability in substructure orderings". This paper is the third in a series of four exploring definability in substructure orderings. The paper [4] dealt with finite semilattices; [5] deals with finite posets; and [6] treats finite lattices. The idea for these explorations arose during our study of some combinatorial properties of these sub-structure orderings (see [1], [2]). We realized also that certain kinds of results on definability in substructure orderings would yield definitive results on definability in the lattice of universal classes of the structures.

By a *universal class* of distributive lattices we mean a class K defined by a set of first-order universal sentences, equivalently, a class K closed under forming substructures and ultraproducts. Let $\mathcal{D}$ be the ordered set of isomorphism types of finite distributive lattices, where the ordering is by embeddability. Since distributive lattices constitute a locally finite variety, the lattice of universal classes of

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distributive lattices is naturally isomorphic with the lattice of order-ideals of $\langle \mathcal{D}, \leq \rangle$, and within this lattice, the principal order-ideals are the same as the strictly join-irreducible elements of the lattice, and they constitute a definable subset of the lattice that is order-isomorphic with $\mathcal{D}$. Thus every subset or relation over the elements of $\mathcal{D}$ that can be shown to be definable in $\langle \mathcal{D}, \leq \rangle$ gives rise to a definable subset or relation in the lattice of universal classes.

A simple but important property of distributive lattices is that for every finite collection F of finite distributive lattices, there is a finite distributive lattice $\mathbf{U}$ such that all members of F are embeddable into $\mathbf{U}$. From this fact, it is clear that a universal class of distributive lattices is finitely generated iff it is contained in a strictly join-irreducible member of the lattice of universal classes. Thus the set of finitely generated universal classes is a definable subset of the lattice. It is easy to show that a universal class K of distributive lattices is finitely axiomatizable (in the first-order language of lattices) iff up to isomorphism, there are only a finite number of minimal (in the sense of embedding) finite distributive lattices lying outside of K. Thus it is easy to write a first-order definition in the language of lattice theory for the class of finitely axiomatizable universal classes: A universal class K is finitely axiomatizable iff there is a strictly join-irreducible universal class O such that for every universal class M, $M \not\leq K \Rightarrow M \cap O \not\leq K$.

We have just proved two of the principal results about universal classes of distributive lattices announced in the abstract. The remaining results, that $\{K, K^{\text{opp}}\}$ is definable in the lattice of universal classes whenever K is a finitely generated, or a finitely axiomatizable, universal class, and that the lattice of universal classes has exactly two automorphisms, are not so easy. Our approach is to exhibit two five-element isomorphism types, e_1 and e_1^{opp}, and show that $\{e_1, e_1^{\text{opp}}\}$ is definable in $\langle \mathcal{D}, \leq \rangle$, and that when e_1 is taken as a parameter, every member of $\mathcal{D}$ becomes definable. After this has been accomplished, the paper is quickly concluded with the derivation of the two results about the lattice of universal classes.

In [4], [5] and [6], the authors have obtained analogous results in the domains of semilattices and ordered sets, and also for finite lattices (although the definability results for finite lattices under embeddability do not lift to the lattice of universal classes of lattices). The chief result of [5] is much stronger than what we prove here for finite distributive lattices. There it is proved that every isomorphism-invariant relation among finite posets that is definable by a second-order formula over the domain of finite posets is, after reduction to isomorphism types, definable by a first-order formula in the substructure ordering of isomorphism types. We believe that the analogous result is true for finite distributive lattices (and also true in the domain of finite lattices, and in the domain of finite semilattices). We hope that someone will be intriqued by this possibility and either prove or disprove our supposition.

2. Notation and basic results

Our principal object of investigation will actually be $\langle \text{QDLATT}, \leq \rangle$, the quasi-ordered set whose members are all the distributive lattices $\mathbf{A} = \langle A, \wedge, \vee \rangle$ in which

A is a finite subset of the non-negative integers, and in which $\mathbf{A} \leq \mathbf{B}$ means that there is a one-to-one lattice homomorphism $f : \mathbf{A} \to \mathbf{B}$.

For $\mathbf{A}, \mathbf{B} \in$ Qdlatt we have that $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A}$ if and only if $\mathbf{A} \cong \mathbf{B}$ ($\mathbf{A}$ and $\mathbf{B}$ are isomorphic). We write $\mathbf{A} < \mathbf{B}$ to denote that $\mathbf{A} \leq \mathbf{B}$ and the two lattices are not isomorphic. The top and bottom elements of a finite lattice will frequently be denoted as $\top$ and $\bot$.

We denote by $\mathbf{E}_1$ a five-element member $\langle \{0, 1, 2, 3, 4\}, \wedge, \vee \rangle$ of Qdlatt in which the covers are $0 \prec 1 \prec 2 \prec 4$ and $1 \prec 3 \prec 4$. We put $\mathbf{E}_0 = \mathbf{E}_1^{\mathrm{opp}}$, the opposite of $\mathbf{E}_1$; i.e., $\mathbf{E}_0 = \langle \{0, 1, 2, 3, 4\}, \vee, \wedge \rangle$. We shall show below that $\{\mathbf{A} \in \text{Qdlatt} : \mathbf{A} \cong \mathbf{E}_0 \text{ or } \mathbf{A} \cong \mathbf{E}_1\}$ is first-order definable in $\langle \text{Qdlatt}, \leq \rangle$. Now we set Qdlatt$'$ equal to the pointed quasi-ordered set $\langle \text{Qdlatt}, \leq, \mathbf{E}_1 \rangle$.

When we say that a subset of Qdlatt or a relation over Qdlatt is definable in Qdlatt$'$, we shall mean definable by a formula in the first-order language with two non-logical symbols, $\leq$ and $\mathbf{E}_1$, and without the equality symbol. Our task, for the next several sections of this paper, will be to demonstrate the definability in Qdlatt$'$ of a sequence of subsets and relations over Qdlatt. As noted above, $\{(\mathbf{A}, \mathbf{B}) : \mathbf{A} \cong \mathbf{B}\}$ is definable in Qdlatt$'$, and it is easily proved (say by induction on the complexity of formulas) that for every formula $\varphi(x_0, \ldots, x_{n-1})$ in this language and for $\mathbf{A}_0 \cong \mathbf{B}_0, \ldots, \mathbf{A}_{n-1} \cong \mathbf{B}_{n-1}$ in Qdlatt we have Qdlatt$' \models \varphi(\mathbf{A}_0, \ldots, \mathbf{A}_{n-1})$ if and only if Qdlatt$' \models \varphi(\mathbf{B}_0, \ldots, \mathbf{B}_{n-1})$. (Thus with our convention about the language—omitting equality—first-order definability in Qdlatt$'$ is only "up to isomorphism".) In particular, $\{\mathbf{E}_1\}$ is not definable, although $\{\mathbf{A} : \mathbf{A} \cong \mathbf{E}_1\}$ is definable. However, we write that "$\mathbf{E}_1$ is a definable member of Qdlatt$'$", meaning that it is definable up to isomorphism; and we shall generally use the language in this way with respect to all definable elements, definable subsets and definable relations over Qdlatt$'$.

The relation of isomorphism, definable in Qdlatt$'$, is an equivalence relation over Qdlatt that gives rise to the pointed ordered set of isomorphism types, $\mathcal{D}' = \langle \mathcal{D}, \leq, e_1 \rangle$ (in which e_1 is the isomorphism type of $\mathbf{E}_1$). Via the map sending $\mathbf{A} \in$ Qdlatt to $\mathbf{A}/\cong \in \mathcal{D}$, definable relations over Qdlatt$'$ become definable relations over $\mathcal{D}'$, and conversely. Thus working over Qdlatt$'$ is simply a convenient means to give a more concrete feel to the study of definability over $\mathcal{D}'$. By working in Qdlatt$'$ (or $\mathcal{D}'$) rather than in $\langle \text{Qdlatt}, \leq \rangle$ (or $\mathcal{D}$), we avoid some simple complications that arise from the existence of the automorphism $\mathbf{A} \mapsto \mathbf{A}^{\mathrm{opp}}$. Among other results, we shall find that all elements of Qdlatt are definable in Qdlatt$'$.

For every $n \geq 0$ we write $\mathbf{C}_n$ for the member of Qdlatt whose underlying ordered set is

$$\langle \{0, 1, \ldots, n\}, \leq \rangle$$

in which $\leq$ is the usual order. This is a chain of height n. The set of chains is a definable subset of Qdlatt$'$, which we denote by $\mathcal{C}$. $\mathbf{L} \in$ Qdlatt is a chain iff there is $\mathbf{L}' \in$ Qdlatt, $\mathbf{L} \leq \mathbf{L}'$ such that $\mathbf{L}'$ has at least five non-isomorphic sublattices and for every pair $\mathbf{A}, \mathbf{B} \leq \mathbf{L}'$ either $\mathbf{A} \leq \mathbf{B}$ or $\mathbf{B} \leq \mathbf{A}$.

For $\mathbf{L} \in$ QDLATT, we write $\mathrm{ht}(\mathbf{L})$ (called the *height of* $\mathbf{L}$) for the largest n such that $\mathbf{C}_n \leq \mathbf{L}$. Obviously, the relation

$$\{(\mathbf{L}, \mathbf{C}) : \mathbf{C} \cong \mathbf{C}_n,\ n = \mathrm{ht}(\mathbf{L})\}$$

is definable. Also, for each $n \geq 0$, the lattice $\mathbf{C}_n$ is definable in QDLATT$'$. Further, for each $\mathbf{A} \in \mathcal{D}$, the height of $\mathbf{A}$ is definable, relative to $\mathbf{A}$, as the largest n such that $\mathbf{A}$ has $n+1$ non-isomorphic sublattices that are chains.

We should also observe that $\mathrm{ht}(\mathbf{A}) = |P|$ where $P = \mathbf{A}^\partial$ is the dual of $\mathbf{A}$—the ordered set of join-irreducible elements in $\mathbf{A}$.

We write $\mathbf{A} \times \mathbf{B}$ for the direct (or Cartesian) product of lattices $\mathbf{A}$ and $\mathbf{B}$, and we write $\mathbf{A} \oplus \mathbf{B}$ for the glued ordinal sum of $\mathbf{A}$ and $\mathbf{B}$ in which the top element of $\mathbf{A}$ is identified with the bottom element of $\mathbf{B}$. Note that $\mathbf{E}_1$, defined above, is isomorphic to $\mathbf{C}_1 \oplus (\mathbf{C}_1 \times \mathbf{C}_1)$. Also, $\mathbf{E}_0 = \mathbf{E}_1^{\mathrm{opp}}$ is isomorphic to $(\mathbf{C}_1 \times \mathbf{C}_1) \oplus \mathbf{C}_1$.

We write $\wp(n)$ for the power set lattice of an n-element set, so that $\wp(0) \cong \mathbf{C}_0$, $\wp(1) \cong \mathbf{C}_1$, $\wp(2) \cong \mathbf{C}_1 \times \mathbf{C}_1$. Note that for each n, $\wp(n)$ is definable, as the largest (in the sense of embeddability) member of QDLATT having height n. The set $\{\wp(n) : n \in \omega\}$ is obviously definable, as well.

We have noted that $\mathbf{C}_1 \times \mathbf{C}_1 = \wp(2)$ is definable. Also, the set $\{\mathbf{E}_0, \mathbf{E}_1\}$ is definable, as its members are precisely the covers in QDLATT of the element $\mathbf{C}_3$ that are not isomorphic to $\mathbf{C}_4$.

3. Tight embeddings

For $\mathbf{A}, \mathbf{B} \in$ QDLATT, we write $\mathbf{A} \leq_t \mathbf{B}$ to denote that $\mathbf{A} \leq \mathbf{B}$ and $\mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{B})$. Obviously, this relation, called *tight embeddability*, is definable in QDLATT$'$. When $\mathbf{A} \leq \mathbf{B}$, we say that $\mathbf{A}$ is *tightly embedded in* $\mathbf{B}$ if $\mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{B})$.

Proposition 3.1. *Assume that* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{B})$. *Then*

(1) *Up to isomorphism,* $\mathbf{A}^\partial = Q$ *is obtained from* $\mathbf{B}^\partial = P$ *by keeping the same set* P *of points and augmenting the order of* P.
(2) *We have* $\mathbf{A} \prec \mathbf{B}$ *(sub-cover) in* QDLATT *iff where* $\mathbf{B}^\partial = \langle P, \leq \rangle$, *we have* $\mathbf{A}^\partial \cong \langle P, \leq_0 \rangle$ *where* $\leq \subseteq \leq_0$ *and* $|\leq_0| = |\leq| + 1$.
(3) *Every element of* $\mathbf{A}$ *has the same height (and the same depth) in* $\mathbf{A}$, *as it has in* $\mathbf{B}$.

Proof. The proofs are pretty obvious, relying on the most basic properties of distributive lattices. •

4. Tight covers of chains

We put $\mathbf{E}_{i,j} = \mathbf{C}_i \oplus (\mathbf{C}_1 \times \mathbf{C}_1) \oplus \mathbf{C}_j$ where $0 \leq i, j$. Thus $\mathbf{E}_0$ (defined above) is $\mathbf{E}_{0,1}$ and $\mathbf{E}_1$ is $\mathbf{E}_{1,0}$. Obviously, $\mathrm{ht}(\mathbf{E}_{i,j}) = i + j + 2$ and $\mathbf{E}_{i,j}$ is a cover of $\mathbf{C}_{i+j+2}$ in QDLATT. Recall that the set $\{\mathbf{E}_0, \mathbf{E}_1\}$ is definable in QDLATT, hence $\mathbf{E}_0$ and $\mathbf{E}_1$ are both definable members of QDLATT$'$.

Theorem 4.1. *Given* $i, j \geq 0$, $\mathbf{E}_{i,0}$ *is the unique cover of* $\mathbf{C}_{i+2}$ *of height* $i+2$ *that does not embed* $\mathbf{E}_0$*;* $\mathbf{E}_{0,j}$ *is the unique cover of* $\mathbf{C}_{j+2}$ *of height* $j+2$ *that does not embed* $\mathbf{E}_1$*; and* $\mathbf{E}_{i,j}$ *is the unique cover of* $\mathbf{C}_{i+j+2}$ *of height* $i+j+2$ *that embeds* $\mathbf{E}_{i,0}$ *but does not embed* $\mathbf{E}_{i+1,0}$*. Thus* $\mathbf{E}_{i,j}$ *is definable in* QDLATT'*, and likewise, the relation*

$$\{(\mathbf{C}_i, \mathbf{C}_j, \mathbf{E}_{i,j}) : 0 \leq i, j\}$$

is definable.

Proof. Straightforward. •

Corollary 4.2. *The covers of* $\mathbf{C}_n$ *in* QDLATT*, where* $n \geq 2$*, are* $\mathbf{C}_{n+1}$ *and the* $\mathbf{E}_{i,j}$ *where* $n = i + j + 2$*.*

5. Definability of addition

The relation $\{(\mathbf{C}_i, \mathbf{C}_j, \mathbf{C}_n) : i + j = n\}$ is definable (in QDLATT'). Indeed, $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ belongs to this relation iff $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are chains and there is a lattice $\mathbf{E}_{i,j}$ (in the definable family introduced above) such that $\text{ht}(\mathbf{E}_{i,j}) = \text{ht}(\mathbf{C})+2$, $\text{ht}(\mathbf{E}_{i,0}) = \text{ht}(\mathbf{A}) + 2$ and $\text{ht}(\mathbf{E}_{0,j}) = \text{ht}(\mathbf{B}) + 2$.

6. Definability of $\mathbf{A} \oplus \mathbf{B}$

For $\mathbf{A} \in$ QDLATT and $c \in A$, we say that c is a *cut-point* of $\mathbf{A}$ iff c is comparable to all elements of $\mathbf{A}$—equivalently, where k is the height of c in $\mathbf{A}$, we have that c is the unique element of height k in $\mathbf{A}$. Clearly, $\mathbf{A}$ has a cut-point of height k, where $0 < k < \text{ht}(\mathbf{A})$, iff $\mathbf{A} \cong \mathbf{C} \oplus \mathbf{D}$ for some $\mathbf{C}, \mathbf{D}$ with $\text{ht}(\mathbf{C}) = k$. Recall that $\mathcal{C} \subseteq$ QDLATT denotes the set of chains.

Lemma 6.1. *The relation*

$$\{(\mathbf{C}, \mathbf{A}) : \mathbf{C} \in \mathcal{C} \text{ and } \mathbf{A} \text{ has a cut-point } c \text{ of height equal to } \text{ht}(\mathbf{C})\}$$

is definable over QDLATT'*. Also, the relations*

$$R_0 = \{(\mathbf{C}, \mathbf{U}, \mathbf{A}) : \mathbf{C} \in \mathcal{C},\ \mathbf{A} \cong \mathbf{C} \oplus \mathbf{U},\ \text{and } \mathbf{U} \text{ has at least two atoms}\},$$

$$R_1 = \{(\mathbf{U}, \mathbf{C}, \mathbf{A}) : \mathbf{C} \in \mathcal{C},\ \mathbf{A} \cong \mathbf{U} \oplus \mathbf{C},\ \text{and } \mathbf{U} \text{ has at least two co-atoms}\}$$

are definable.

Proof. $\mathbf{A}$ has a cut-point of height k iff $k = 0$ or $k = \text{ht}(\mathbf{A})$ or $0 < k < \text{ht}(\mathbf{A})$ and $\mathbf{E}_{k-1,\text{ht}(\mathbf{A})-k-1} \not\leq \mathbf{A}$. To see that this is so, suppose first that $0 < k < \text{ht}(\mathbf{A})$ and $\mathbf{A}$ has no cut-point of height k. Let c be an element of $\mathbf{A}$ of height k. Then there is $u \in A$ incomparable to c. We can assume that $u \prec u + c$ in $\mathbf{A}$. Then $c \succ cu$ and there is d such that $cu \prec d \leq u$. Here $cd = cu$ and $\{c, d, cd, c + d\}$ is a sublattice isomorphic to $\wp(2)$ with the interval $I[cd, c+d]$ having height 2. There is a maximal chain of $\mathbf{A}$ containing the elements $cd, c, c + d$, giving a sublattice of $\mathbf{A}$ isomorphic to $\mathbf{E}_{k-1,\text{ht}(\mathbf{A})-k-1}$.

Conversely, if $\mathbf{L}$ is a sublattice of $\mathbf{A}$ isomorphic to $\mathbf{E}_{k-1,\mathrm{ht}(\mathbf{A})-k-1}$ then this is a tight sublattice of $\mathbf{A}$ and it has two incomparable elements of height k, and thus so does $\mathbf{A}$.

Now to define R_0, we ask the reader to verify that $(\mathbf{C},\mathbf{U},\mathbf{A})$ belongs to R_0 iff $\mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{C}) + \mathrm{ht}(\mathbf{U})$, $\mathbf{C} \in \mathcal{C}$, and where $k = \mathrm{ht}(\mathbf{U})$, $\ell = \mathrm{ht}(\mathbf{C})$: $k \geq 2$ and $\mathbf{E}_{0,k-2} \leq \mathbf{U}$; $\mathbf{U} \leq \mathbf{A}$ and whenever $\mathbf{U}' \leq \mathbf{A}$ and $\mathrm{ht}(\mathbf{U}') \geq 2$ and $\mathbf{E}_{0,\mathrm{ht}(\mathbf{A}')-2} \leq \mathbf{U}'$ then $\mathbf{U}' \leq \mathbf{U}$; if $\ell > 0$ then $\mathbf{E}_{\ell-1,k-1} \not\leq \mathbf{A}$ (so that $\mathbf{A}$ has a cut-point of height ℓ); and in fact, if $\ell > 0$ then for all $0 \leq i \leq \ell - 1$, $\mathbf{E}_{i,k+\ell-i-2} \not\leq \mathbf{A}$.

A first-order definition of R_1 is obtained via dual considerations. •

Lemma 6.2. *The relations*

$$S_0 = \{(\mathbf{C},\mathbf{U},\mathbf{A}) : \mathbf{C} \in \mathcal{C},\ \mathbf{A} \cong \mathbf{C} \oplus \mathbf{U}\}, \text{ and}$$
$$S_1 = \{(\mathbf{U},\mathbf{C},\mathbf{A}) : \mathbf{C} \in \mathcal{C},\ \mathbf{A} \cong \mathbf{U} \oplus \mathbf{C}\}$$

are definable.

Proof. We have that $(\mathbf{C},\mathbf{U},\mathbf{A}) \in S_0$ iff $\mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{C}) + \mathrm{ht}(\mathbf{U})$, $\mathbf{C} \in \mathcal{C}$, and where $k = \mathrm{ht}(\mathbf{U})$, $\ell = \mathrm{ht}(\mathbf{C})$: either $\mathbf{U} \in \mathcal{C}$ and $\mathbf{A} \in \mathcal{C}$ or there are $\mathbf{D},\mathbf{E},\mathbf{U}' \in$ QDLATT such that $(\mathbf{D},\mathbf{U}',\mathbf{U}) \in R_0$ and $(\mathbf{E},\mathbf{U}',\mathbf{A}) \in R_0$ and $\mathrm{ht}(\mathbf{C}) + \mathrm{ht}(\mathbf{D}) = \mathrm{ht}(\mathbf{E})$. The relation S_1 is shown to be first-order definable through a dual definition. •

Theorem 6.3. *The relation*

$$\{(\mathbf{A},\mathbf{B},\mathbf{S}) : \mathbf{S} \cong \mathbf{A} \oplus \mathbf{B}\}$$

is definable.

Proof. We claim that $\mathbf{S} \cong \mathbf{A} \oplus \mathbf{B}$ iff where $k = \mathrm{ht}(\mathbf{A})$ and $\ell = \mathrm{ht}(\mathbf{B})$, the lattice $\mathbf{S}$ has a height k cut-point, $\mathbf{C}_k \oplus \mathbf{B} \leq_t \mathbf{S}$, $\mathbf{A} \oplus \mathbf{C}_\ell \leq_t \mathbf{S}$, and if $\mathbf{A} \leq \mathbf{A}'$ and $\mathbf{B} \leq \mathbf{B}'$ and the stated properties hold for $\mathbf{A}',\mathbf{B}'$ in place of $\mathbf{A},\mathbf{B}$, then $\mathbf{A} \cong \mathbf{A}'$ and $\mathbf{B} \cong \mathbf{B}'$. The proof that this is so is quite straightforward. Combined with the results of the previous two lemmas, this claim proves the theorem. •

7. Definability of diamond-chains and rectangles

For $n \geq 0$ we denote by $\mathbf{D}_n$ the lattice which is the $\oplus$-sum of n copies of $\wp(2)$. We call the lattices $\mathbf{D}_n$ *diamond-chains*. Lattices isomorphic to products of two chains are called *rectangles*.

Theorem 7.1. *The set* $\{\mathbf{D}_n : n \geq 0\}$ *is definable and the relation*

$$\{(\mathbf{C}_n,\mathbf{D}_n) : n \geq 0\}$$

is definable.

Proof. A finite distributive lattice $\mathbf{A}$ is isomorphic to $\mathbf{D}_n$ iff $\mathrm{ht}(\mathbf{C}_n \oplus \mathbf{C}_n) = \mathrm{ht}(\mathbf{A})$ $(= 2n)$ and either $n = 0$ and $\mathbf{A} \cong \mathbf{C}_0$ or else $n \geq 1$, $\mathbf{E}_{0,2n-2} \leq \mathbf{A}$ and for all i with $0 \leq i < 2n-2$ it is the case that $\mathbf{E}_{i,2n-i-2} \leq \mathbf{A}$ iff $\mathbf{E}_{i+1,2n-i-3} \not\leq \mathbf{A}$. •

Theorem 7.2. *The set* $\{\mathbf{C}_m \times \mathbf{C}_n : m,n \geq 0\}$ *is definable and the relation* $\{(\mathbf{C}_m,\mathbf{C}_n,\mathbf{C}_m \times \mathbf{C}_n) : m,n \geq 0\}$ *is definable.*

Proof. We claim that a finite distributive lattice $\mathbf{A}$ is isomorphic to $\mathbf{C}_m \times \mathbf{C}_n$ for some $m, n > 0$ iff $\wp(2) \leq \mathbf{A}$, $\wp(3) \not\leq \mathbf{A}$ and for all $\mathbf{B}$, if $\mathbf{A} <_t \mathbf{B}$ (i.e., if $\mathbf{A} < \mathbf{B}$ and $\mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{B})$) then $\wp(3) \leq \mathbf{B}$.

To see this, observe that each of the following conditions is equivalent to the property that $\wp(3) \not\leq \mathbf{A}$: each member of $\mathbf{A}$ has at most two covers; the dual poset of $\mathbf{A}$, $\mathbf{A}^\partial$, has no three-element anti-chain; $\mathbf{A}^\partial$ is the union of two chains (by Dilworth's theorem); the order on $\mathbf{A}^\partial$ has a sub-order under which it is a chain or the cardinal sum of two chains; $\mathbf{A} \leq_t \mathbf{C}_m \times \mathbf{C}_n$ for some m, n (for which $m + n = \mathrm{ht}(\mathbf{A})$).

For the correctness of the maximality clause in our definition, suppose that $\mathbf{C}_m \times \mathbf{C}_n = \mathbf{A} <_t \mathbf{B}$. Then $\mathbf{B}^\partial$ is isomorphic to an ordered set obtained from $\mathbf{A}^\partial$ by weakening the order while keeping the same set of points. Since $\mathbf{A}^\partial$ is ordered as the disjoint union of an m-element chain and an n-element chain, then $\mathbf{B}^\partial$ (in this situation) is easily seen to have a three-element anti-chain.

The relation $\{(\mathbf{C}_m, \mathbf{C}_n, \mathbf{C}_m \times \mathbf{C}_n) : m, n \geq 0\}$ is definable in this way: $\mathbf{A} \cong \mathbf{C}_m \times \mathbf{C}_n$ iff $\mathbf{A} \cong \mathbf{C} \times \mathbf{D}$ for some $\mathbf{C}, \mathbf{D} \in \mathcal{C}$, $\mathrm{ht}(\mathbf{A}) = m + n$, and where $k = \min(m, n)$, we have that k is the largest integer ℓ such that $\mathbf{D}_\ell \leq \mathbf{A}$. •

8. Definability of (k, ℓ)-products

Assume that $\mathbf{C}, \mathbf{D} \in \mathcal{C}$. We say that $\mathbf{A}$ is a $(\mathbf{C}, \mathbf{D})$-*product* (or a (k, ℓ)-*product* if $\mathbf{C} = \mathbf{C}_k$ and $\mathbf{D} = \mathbf{C}_\ell$) iff $\mathbf{C} \times \mathbf{D} \leq_t \mathbf{A}$.

Theorem 8.1. *The relation*

$$\{(\mathbf{C}, \mathbf{D}, \mathbf{A}) : \{\mathbf{C}, \mathbf{D}\} \subseteq \mathcal{C} \text{ and } \mathbf{A} \cong \mathbf{R} \times \mathbf{S} \text{ for some}$$

$$\mathbf{R} \text{ and } \mathbf{S} \text{ with } \mathrm{ht}(\mathbf{R}) = \mathrm{ht}(\mathbf{C}) \text{ and } \mathrm{ht}(\mathbf{S}) = \mathrm{ht}(\mathbf{D})\}$$

is definable. In fact $(\mathbf{C}, \mathbf{D}, \mathbf{A})$ *belongs to this relation iff* $\{\mathbf{C}, \mathbf{D}\} \subseteq \mathcal{C}$ *and* $\mathbf{C} \times \mathbf{D} \leq_t \mathbf{A}$.

Proof. First, if $\mathbf{A} = \mathbf{R} \times \mathbf{S}$ and $\mathbf{C}$ is a maximal chain in $\mathbf{R}$ and $\mathbf{D}$ is a maximal chain in $\mathbf{S}$ then $\mathbf{C} \times \mathbf{D} \leq_t \mathbf{A}$ and, of course, $\mathrm{ht}(\mathbf{C}) = \mathrm{ht}(\mathbf{R})$ and $\mathrm{ht}(\mathbf{D}) = \mathrm{ht}(\mathbf{S})$.

Conversely, suppose that $\{\mathbf{C}, \mathbf{D}\} \subseteq \mathcal{C}$ and $\mathbf{C} \times \mathbf{D} \leq_t \mathbf{A}$. Now $\mathbf{C} \times \mathbf{D}$ has a pair of complementary elements e, f of heights $\mathrm{ht}(\mathbf{C})$, $\mathrm{ht}(\mathbf{D})$ respectively. Since $\mathbf{C} \times \mathbf{D}$ is isomorphic to a tight sublattice of $\mathbf{A}$, then $\mathbf{A}$ has such a pair of complements e', f' with the same heights. Since our lattices are distributive, $\mathbf{A}$ is isomorphic to the product of its interval sublattices $I[0, e']$, $I[0, f']$. These lattices have the required heights.

We remark that of course, $\mathbf{A}$ is also isomorphic to the product of the intervals $I[e', 1]$ and $I[f', 1]$. •

9. Definability of products

Lemma 9.1. *The relation* $\{(\mathbf{C}_m, \mathbf{C}_n, \wp(m) \times \mathbf{C}_n) : 1 \leq m, 1 \leq n\}$ *is definable.*

Proof. Recall that $\wp(m)$ is the $\leq$-largest member of QDLATT of height m, equivalently, the $\leq$-largest member for which $\mathbf{C}_m$ is its $\leq$-largest embeddable chain. The dual poset of $\wp(m)\times\mathbf{C}_n$ is the cardinal sum of an m-element discretely ordered set and an n-element chain. The lattice $\wp(m)\times\mathbf{C}_n$ is a tight sublattice of $\wp(m+n)$ and each tight sublattice of $\wp(m+n)$ is determined by its dual poset which, up to isomorphism, is an arbitrary ordered set whose elements constitute a fixed set of $m+n$ elements.

These observations allow one to verify that $\mathbf{U}\cong\wp(m)\times\mathbf{C}_n$ iff the following hold:

$\mathbf{U}\leq_t\wp(m{+}n)$. If $n=1$ then $\mathbf{U}\cong\wp(m{+}1)$. Assume that $n>1$. Then (1) $\mathbf{U}$ has precisely $m{+}1$ atoms and $m{+}1$ co-atoms—equivalently, $\wp(m{+}1)\oplus\mathbf{C}_{n-1}\leq_t\mathbf{U}$ and $\wp(m{+}2)\oplus\mathbf{C}_{n-2}\not\leq_t\mathbf{U}$, and dually, $\mathbf{C}_{n-1}\oplus\wp(m{+}1)\leq_t\mathbf{U}$ and $\mathbf{C}_{n-2}\oplus\wp(m{+}2)\not\leq_t\mathbf{U}$; (2) whenever $\mathbf{L}<_t\mathbf{U}$ then either $\mathbf{L}$ has at most m atoms, or at most m co-atoms; and (3) if $\mathbf{A}\in$ QDLATT, $\mathrm{ht}(\mathbf{A})=n-1$, and $\wp(m+1)\oplus\mathbf{A}\leq_t\mathbf{U}$, then $\mathbf{A}\cong\mathbf{C}_{n-1}$.

We remark that when $n>1$, a lattice $\mathbf{U}\in$ QDLATT satisfies (1) and (2), iff the ordered set $\mathbf{U}^\partial$ is, for some $1\leq k\leq m+1$ with $k+1\leq n$, the cardinal sum of an antichain of size $m-k+1$ and an ordered set P of size $n+k-1$ which consists of a chain C of size $n-k-1$ (so possibly empty) together with a k element antichain T of elements above all elements of C (the maximal elements of P), and a k element antichain B of elements below all the elements of $C\cup T$ (the minimal elements of P).

All the unproved claims above are left for the reader to validate. •

Lemma 9.2. *The relation*

$$\{(\mathbf{C},\mathbf{D},\mathbf{U}):\mathbf{C},\mathbf{D}\in\mathcal{C},\mathbf{C}>\mathbf{D}\ \textit{and}\ \mathbf{U}\cong\mathbf{A}\times\mathbf{C}\ \textit{for some}$$
$$\mathbf{A}\in\text{QDLATT}\ \textit{with}\ \mathrm{ht}(\mathbf{A})=\mathrm{ht}(\mathbf{D})\}$$

is definable.

Proof. $(\mathbf{C},\mathbf{D},\mathbf{U})$ belongs to this relation iff $\{\mathbf{C},\mathbf{D}\}\subseteq\mathcal{C}$, $\mathbf{D}<\mathbf{C}$, $\mathbf{U}$ is a $(\mathrm{ht}(\mathbf{D}),\mathrm{ht}(\mathbf{C}))$-product, either $\mathrm{ht}(\mathbf{D})=0$ and $\mathbf{U}\cong\mathbf{C}$, or else $\mathrm{ht}(\mathbf{D})=m\geq 1$ and, say, $\mathbf{C}\cong\mathbf{C}_n$ and $\mathbf{U}\leq_t\wp(m)\times\mathbf{C}_n$. Indeed, if these conditions hold then we can assume that $\mathbf{U}$ is a tight sublattice of $\wp(m)\times\mathbf{C}_n$. Since $\mathbf{U}$ is a (m,n)-product, it has a pair of complements (e,f) so that e has height m and f has height n (in $\mathbf{U}$, and hence in $\wp(m)\times\mathbf{C}_n$). We have, say, $e=(i,j)$ and $f=(i',j')$. Since they are complements in $\wp(m)\times\mathbf{C}_n$ and $\mathbf{C}_n$ is a chain, then $\{j,j'\}=\{\bot,\top\}$ (the set of extreme elements of $\mathbf{C}_n$). If $j=\top$ then $\mathrm{ht}(e)=\mathrm{ht}(i)+\mathrm{ht}(j)\geq n$, contradicting that $m<n$. Thus $j'=\top$. Then $n=\mathrm{ht}(f)=\mathrm{ht}(i')+\mathrm{ht}(j')=\mathrm{ht}(i')+n$, implying that $f=(\bot,\top)$ and consequently, $e=(\top,\bot)$. Since $I[\bot,f]$ in $\mathbf{U}$ is embeddable into $\mathbf{C}_n$, and has height n, then $I[\bot,f]$ in $\mathbf{U}$ is isomorphic to $\mathbf{C}_n$. It follows that $\mathbf{U}\cong\mathbf{A}\times\mathbf{C}_n$ for some $\mathbf{A}$, as claimed. •

Lemma 9.3. *The relation* $\{(\mathbf{A},\mathbf{C},\mathbf{U}):\mathbf{U}\cong\mathbf{A}\times\mathbf{C}$ *and* $\mathbf{C}\in\mathcal{C}\}$ *is definable.*

Proof. Assume that $\mathbf{C}=\mathbf{C}_n$ and $\mathrm{ht}(\mathbf{A})=m$. We claim that $\mathbf{U}\cong\mathbf{A}\times\mathbf{C}$ iff:

(a) $\mathbf{U}$ is a (m,n)-product and $\mathbf{C}_n$ divides $\mathbf{U}$ in the sense that $\mathbf{U}\cong\mathbf{A}\times\mathbf{C}_n$ for some $\mathbf{A}$;

(b) either $\mathbf{A} \in \mathcal{C}$ and $\mathbf{U} \cong \mathbf{A} \times \mathbf{C}$, or else $\mathbf{A} \notin \mathcal{C}$ and there exists a lattice $\mathbf{Q} \in$ QDLATT with the following four properties:
  (1) $\mathbf{Q}$ is a $(2n+m, n)$-product and if $n = 0$ then $\mathbf{Q} \cong \mathbf{A} \cong \mathbf{U}$.
  (2) $\mathbf{Q}$ has no tight sublattice of the form $\mathbf{C} \oplus \wp(3) \oplus \mathbf{L}$ (for any $\mathbf{L}$) with $\mathbf{C} = \mathbf{C}_i$, $0 \leq i < 2n$.
  (3) $(\mathbf{C}_n \times \mathbf{C}_n) \oplus (\mathbf{C}_n \oplus \mathbf{A}) \leq_t \mathbf{Q}$ and if $\mathbf{W} \in$ QDLATT satisfies $(\mathbf{C}_n \times \mathbf{C}_n) \oplus (\mathbf{C}_n \oplus \mathbf{W}) \leq_t \mathbf{Q}$ then $\mathbf{W} \leq \mathbf{A}$.
  (4) $\mathbf{C}_{2n} \oplus \mathbf{U} \leq_t \mathbf{Q}$, and if $\mathbf{W} \in$ QDLATT and $\mathbf{C}_{2n} \oplus \mathbf{W} \leq_t \mathbf{Q}$ and $\mathbf{W}$ is a (m, n)-product which is furthermore directly divisible by $\mathbf{C}_n$ if $n > m$, then $\mathbf{W} \leq \mathbf{U}$.

To prove the claim, we first tackle the sufficiency of the condition. The sufficency is trivial if $\mathbf{A} \in \mathcal{C}$ or $n = 0$. So we assume that $\mathbf{A} \notin \mathcal{C}$, that $n > 0$, that (a) is true, and that $\mathbf{Q} \in$ QDLATT satisfies (1) – (4). We have to prove that $U \cong \mathbf{A} \times \mathbf{C}_n$.

Our first step is to show that, setting $\widehat{\mathbf{Q}} = (\mathbf{C}_{2n} \oplus \mathbf{A}) \times \mathbf{C}_n$, then $\mathbf{Q} \cong \widehat{\mathbf{Q}}$ follows from statements (1), (2) and (3).

We have that $\mathbf{Q} \cong \mathbf{R} \times \mathbf{S}$ with $\mathrm{ht}(\mathbf{R}) = 2n+m$, $\mathrm{ht}(\mathbf{S}) = n$ (by (1)). Property (2) is easily seen to be equivalent to the statement that every element of height $< 2n$ in $\mathbf{Q}$ has at most two covers. This implies that $\mathbf{S}$ is a chain and $\mathbf{R} \cong \mathbf{C}_{2n} \oplus \mathbf{T}$ for some $\mathbf{T}$. Thus we can, without losing generality, suppose that $\mathbf{Q} = (\mathbf{C}_{2n} \oplus \mathbf{T}) \times \mathbf{C}_n$.

Clearly, $(\mathbf{C}_n \times \mathbf{C}_n) \oplus (\mathbf{C}_n \oplus \mathbf{T}) \leq_t \mathbf{Q}$, and therefore by (3), $\mathbf{T} \leq \mathbf{A}$. To fully use (3), we notice that in the lattice $\mathbf{Q}$, the elements of height $2n$ are the pairs $(2n - j, j)$ in $\mathbf{C}_{2n} \times \mathbf{C}_n$, where $0 \leq j \leq n$. If $\mathbf{C}_n \times \mathbf{C}_n \leq_t (2n - j, j)\!\downarrow$ then $\mathbf{C}_n \times \mathbf{C}_n \leq_t \mathbf{C}_{2n-j} \times \mathbf{C}_j$, implying that $\mathbf{C}_{2n-j} \times \mathbf{C}_j$ is an (n, n)-product. But $\mathbf{C}_p$ is directly indecomposable for all $p > 0$ and we have the unique factorization property for direct products of finite distributive lattices. This means that $2n - j = n = j$. Thus there is a unique element e in $\mathbf{Q}$ of height $2n$ with $\mathbf{C}_n \times \mathbf{C}_n \leq_t e\!\downarrow$—namely, $e = (n, n)$ where the second n is the top element of $\mathbf{C}_n$. This implies that for any $\mathbf{W}$, we have $(\mathbf{C}_n \times \mathbf{C}_n) \oplus \mathbf{W} \leq_t \mathbf{Q}$ iff $\mathbf{W} \leq_t (n, n)\!\uparrow$, i.e, $\mathbf{W} \leq_t \mathbf{C}_n \oplus \mathbf{T}$. Thus by (3), $\mathbf{C}_n \oplus \mathbf{A} \leq \mathbf{C}_n \oplus \mathbf{T}$. This combined with $\mathbf{T} \leq \mathbf{A}$ yields that $\mathbf{C}_n \oplus \mathbf{T} \cong \mathbf{C}_n \oplus \mathbf{A}$, and then $\mathbf{T} \cong \mathbf{A}$. Thus we do have $\mathbf{Q} \cong \widehat{\mathbf{Q}}$.

Henceforth we assume that $\mathbf{Q} = (\mathbf{C}_{2n} \oplus \mathbf{A}) \times \mathbf{C}_n$; i.e., the two lattices are not just isomorphic, but equal. Now Observe that $\mathbf{C}_{2n} \oplus (\mathbf{A} \times \mathbf{C}_n) \leq_t \mathbf{Q}$. Thus by (4), $\mathbf{A} \times \mathbf{C}_n \leq \mathbf{U}$, and this is a tight embedding (the heights are equal). It still remains to show that $\mathbf{U} \cong \mathbf{A} \times \mathbf{C}_n$, or equivalently, that $\mathbf{U} \leq \mathbf{A} \times \mathbf{C}_n$. Property (4) tells us that $\mathbf{C}_{2n} \oplus \mathbf{U} \leq_t \mathbf{Q}$. Hence there is an element $e \in \mathbf{Q}$ of height $2n$ with $\mathbf{U}$ tightly embeddable into the interval $e\!\uparrow$ in $\mathbf{Q}$. We have that $e = (2n - j, j)$ for some $0 \leq j \leq n$, and

$$\mathbf{U} \leq_t e\!\uparrow \cong (\mathbf{C}_j \oplus \mathbf{A}) \times \mathbf{C}_{n-j} \,.$$

If $j = 0$, the displayed formula reduces to $\mathbf{U} \leq_t \mathbf{A} \times \mathbf{C}_n$, as desired. If $j = n$, then the formula reduces to $\mathbf{U} \leq_t \mathbf{C}_n \oplus \mathbf{A}$. Since $\mathbf{U}$ is an (m, n)-product, by (a), then so is $\mathbf{C}_n \oplus \mathbf{A}$, but this is absurd. So we can assume that $n > j > 0$. Since $\mathbf{A} \times \mathbf{C}_n \leq_t \mathbf{U}$, the formula above yields

$$\mathbf{A} \times \mathbf{C}_n \leq_t (\mathbf{C}_j \oplus \mathbf{A}) \times \mathbf{C}_{n-j} \,.$$

This formula leads to a contradiction, as follows. The formula implies the existence of a complementary pair (e, f) in $(\mathbf{C}_j \oplus \mathbf{A}) \times \mathbf{C}_{n-j}$ with $\mathbf{A} \leq_t e{\downarrow}$ and $\mathbf{C}_n \leq_t f{\downarrow}$. Since each of $\mathbf{C}_j \oplus \mathbf{A}$ and $\mathbf{C}_{n-j}$ has a unique element of height 1, then $\{e, f\} = \{(\perp, \top), (\top, \perp)\}$, the only complementary pair of elements in this product, neither of which is the bottom element of the lattice. Since $\mathbf{A}$ is not a chain, then $\mathbf{A} \not\leq_t (\perp, \top){\downarrow}$. Thus $e = (\top, \perp)$, but this gives $\mathbf{A} \leq_t \mathbf{C}_j \oplus \mathbf{A}$. Since $\mathrm{ht}(\mathbf{A}) = m < j + m = \mathrm{ht}(\mathbf{C}_j \oplus \mathbf{A})$, this is impossible. The contradiction finishes our proof of sufficiency.

For necessity, we assume that $U \cong \mathbf{A} \times \mathbf{C}_n$. Clearly, (a) is true. In proving (b), we can certainly assume that $\mathbf{A} \notin \mathcal{C}$, i.e., that $\mathbf{A} \ncong \mathbf{C}_m$; and we can, and do, assume that $n > 0$. We put

$$\mathbf{Q} = (\mathbf{C}_{2n} \oplus \mathbf{A}) \times \mathbf{C}_n\,.$$

We need to verify statements (1) – (4).

The truth of (1) is obvious. Statement (2) is equivalent to: every element of height less than $2n$ in $\mathbf{Q}$ has at most two covers in $\mathbf{Q}$. This is easily seen to be true, and so (2) is true. The only element $e \in \mathbf{Q}$ such that $e{\downarrow}$ tightly embeds $\mathbf{C}_n \times \mathbf{C}_n$ is $e = (n, n)$. For this element, $e{\uparrow} \cong \mathbf{C}_n \oplus \mathbf{A}$. From these considerations, the truth of (3) follows.

To prove (4), note that $c = (2n, 0)$ is an element of height $2n$ in $\mathbf{Q}$ such that $c{\uparrow} \cong \mathbf{U}$; thus $\mathbf{C}_{2n} \oplus \mathbf{U} \leq_t \mathbf{Q}$. Next, suppose that $\mathbf{W} \in$ QDLATT and $\mathbf{C}_{2n} \oplus \mathbf{W} \leq_t \mathbf{Q}$ and $\mathbf{W}$ is a (m, n)-product which is furthermore directly divisible by $\mathbf{C}_n$ if $n > m$. We need to prove that $\mathbf{W} \leq \mathbf{U}$.

We have an element $c = (2n - j, j)$ in $\mathbf{Q}$ with $\mathbf{W} \leq_t c{\uparrow}$. If $j = 0$ then of course $\mathbf{W} \leq_t \mathbf{U}$ as desired. If $j = n$ then $\mathbf{W} \leq_t c{\uparrow} \cong \mathbf{C}_n \oplus \mathbf{A}$. Since $\mathbf{W}$ is an (m, n)-product, this is impossible. Thus suppose that $c = (2n - j, j)$, $j > 0$. Then

$$\mathbf{W} \leq_t c{\uparrow} \cong (\mathbf{C}_j \oplus \mathbf{A}) \times \mathbf{C}_{n-j}\,.$$

We can write $\mathbf{W} \cong \mathbf{J} \times \mathbf{C}$, $\mathrm{ht}(\mathbf{J}) = m$, $\mathrm{ht}(\mathbf{C}) = n$, $\mathbf{C} \cong \mathbf{C}_n$ if $m < n$. This implies the existence of a a complementary pair (e, f) in $(\mathbf{C}_j \oplus \mathbf{A}) \times \mathbf{C}_{n-j}$, with $\mathbf{J} \leq_t e{\downarrow}$ and $\mathbf{C} \leq_t f{\downarrow}$. Now, as above, $\{e, f\} = \{(\perp, \top), (\top, \perp)\}$.

Since $(\perp, \top){\downarrow}$ has height $n - j < n$, then $f = (\top, \perp)$, $e = (\perp, \top)$. This implies that $m = \mathrm{ht}(\mathbf{J}) = n - j < n$. Thus $\mathbf{C} \cong \mathbf{C}_n$ (as follows from our assumptions about $\mathbf{W}$). Also, $\mathbf{J} \leq_t (\perp, \top){\downarrow} \cong \mathbf{C}_{n-j}$, implying that $\mathbf{J}$ is a chain. From the height of $\mathbf{J}$ we see that $\mathbf{J} \cong \mathbf{C}_m$. Hence, finally,

$$\mathbf{W} \cong \mathbf{J} \times \mathbf{C} \cong \mathbf{C}_m \times \mathbf{C}_n \leq \mathbf{A} \times \mathbf{C}_n \cong U$$

and we have our desired conclusion. •

Lemma 9.4. *The relation*

$$\{(\mathbf{A}, \mathbf{B}, \mathbf{U}) : \mathbf{U} \cong \mathbf{A} \times (\mathbf{C}_1 \oplus \mathbf{B}) \text{ and } \mathrm{ht}(\mathbf{A}) \leq \mathrm{ht}(\mathbf{B})\}$$

is definable in QDLATT$'$.

Proof. Let $\mathbf{A}, \mathbf{B}$ in QDLATT be given, with say, $\mathrm{ht}(\mathbf{A}) = m \leq n = \mathrm{ht}(\mathbf{B})$. Define $\widehat{\mathbf{Q}} = \mathbf{A} \times (\mathbf{C}_1 \oplus \mathbf{B})$. For any $\mathbf{Q} \in$ QDLATT we claim that $\mathbf{Q} \cong \widehat{\mathbf{Q}}$ iff the following hold:

(i) $\mathbf{Q}$ is a $(m, n+1)$-product.
(ii) $\mathbf{C}_m \times (\mathbf{C}_1 \oplus \mathbf{B}) \leq_t \mathbf{Q}$ and $\mathbf{A} \times \mathbf{C}_{n+1} \leq_t \mathbf{Q}$.
(iii) Whenever $\mathbf{C}_m \times \mathbf{J} \leq_t \mathbf{Q}$ we have $\mathbf{J} \leq_t \mathbf{C}_1 \oplus \mathbf{B}$; and whenever $\mathbf{K} \times \mathbf{C}_{n+1} \leq_t \mathbf{Q}$ we have $\mathbf{K} \leq_t \mathbf{A}$.

To begin the proof of this claim, we need another claim.

Claim: Suppose that $\mathbf{R}$ is a $(m, n+1)$-product, say $\mathbf{R} = \mathbf{F} \times \mathbf{G}$ where $m \leq n$ and $\mathbf{G}$ has a unique atom. If (e, f) is any pair of complements in $\mathbf{R}$ with e of height m and f of height $n+1$ then $e = (\top, \bot)$ and $f = (\bot, \top)$.

To prove the claim, note first that since $\mathbf{G}$ has a unique atom, then $e = (i, \bot)$ and $f = (i', \top)$ or else $e = (i, \top)$ and $f = (i', \bot)$. If $e = (i, \top)$ then the interval sublattice $I[\bot, e]$ in $\mathbf{R}$ has height not less than $n+1$. But this lattice has height m. Thus $e = (i', \bot)$. This means that the depth of e is greater than $n+1$ unless $i' = \top$. Thus in fact $e = (\top, \bot)$. Then f, the complement of e, must be $(\bot, \top)$.

Now using the second claim, it is easy to see that $\widehat{\mathbf{Q}}$ satisfies the properties we have formulated. Conversely, suppose that $\mathbf{Q}$ satisfies these properties. To show that $\mathbf{Q} \cong \widehat{\mathbf{Q}}$ we can suppose that $\mathbf{Q} = \mathbf{F} \times \mathbf{G}$ with $\mathrm{ht}(\mathbf{F}) = m$ and $\mathrm{ht}(\mathbf{G}) = n+1$. Obviously $\mathbf{C}_m \times \mathbf{G} \leq_t \mathbf{Q}$; therefore $\mathbf{G} \leq_t \mathbf{C}_1 \oplus \mathbf{B}$. This implies that $\mathbf{G}$ has a unique atom. Then by the claim, there is only one way to tightly embed $\mathbf{C}_m \times (\mathbf{C}_1 \oplus \mathbf{B})$ into $\mathbf{Q}$; namely we have that $\mathbf{C}_1 \oplus \mathbf{B} \leq_t \mathbf{G}$. It follows that $\mathbf{G} \cong \mathbf{C}_1 \oplus \mathbf{B}$. Now analogous arguments give that $\mathbf{F} \leq_t \mathbf{A}$ and $\mathbf{A} \leq_t \mathbf{F}$, so that $\mathbf{F} \cong \mathbf{A}$. This concludes our proof. •

Theorem 9.5. *The relation* $\{(\mathbf{A}, \mathbf{B}, \mathbf{U}) : \mathbf{U} \cong \mathbf{A} \times \mathbf{B}\}$ *is definable in* $\mathrm{Q{\small DLATT}}'$.

Proof. It suffices to show that $\{(\mathbf{A}, \mathbf{B}, \mathbf{A} \times \mathbf{B}) : \mathrm{ht}(\mathbf{A}) \leq \mathrm{ht}(\mathbf{B})\}$ is definable. Thus suppose that $m \leq n$ where $\mathrm{ht}(\mathbf{A}) = m$ and $\mathrm{ht}(\mathbf{B}) = n$. Let $\widehat{\mathbf{Q}} = \mathbf{A} \times (\mathbf{C}_1 \oplus \mathbf{B})$.

Put $\mathcal{M}$ equal to the set of all $\mathbf{L} \in \mathrm{Q{\small DLATT}}$ such that $\mathbf{L} \leq \widehat{\mathbf{Q}}$ and $\mathbf{L} \cong \mathbf{C}_1 \oplus \mathbf{X}$ for some $\mathbf{X}$. We claim that $\mathbf{A} \times \mathbf{B}$ is, up to isomorphism, the unique lattice $\mathbf{L}$ such that $\mathbf{C}_1 \oplus \mathbf{L}$ is a $\leq$-maximal member of $\mathcal{M}$ and $\mathbf{L}$ is an (m, n)-product. This claim, combined with Lemma 9.4, will establish the theorem.

To verify the claim, we need to show that $\widehat{\mathbf{P}} = \mathbf{C}_1 \oplus (\mathbf{A} \times \mathbf{B})$ is $\leq$-maximal in $\mathcal{M}$, and that whenever $\mathbf{C}_1 \oplus \mathbf{P}$ is $\leq$-maximal in $\mathcal{M}$ and not isomorphic with $\widehat{\mathbf{P}}$ then $\mathbf{P}$ fails to be an (m, n)-product.

Clearly, the $\leq$-maximal members of $\mathcal{M}$ are the same as the lattices isomorphic to $\mathbf{C}_1 \oplus \mathbf{R}$ for some $\mathbf{R}$ that is $\leq$-maximal in the class $\mathcal{M}' = \{q\!\uparrow : \bot \prec q \text{ in } \widehat{\mathbf{Q}}\}$.

So suppose that q is an atom in $\widehat{\mathbf{Q}}$. Then either $q = (\bot, p)$ where p is the unique atom in $\mathbf{C}_1 \oplus \mathbf{B}$, or else $q = (e, \bot)$ for some atom $e \in \mathbf{A}$. In the first case, $q\!\uparrow \cong \mathbf{A} \times \mathbf{B}$ and this lattice is an (m, n)-product.

In the second case, $q = (e, \bot)$, we have that $q\!\uparrow \cong \mathbf{A}_e \times (\mathbf{C}_1 \oplus \mathbf{B})$ is an $(m-1, n+1)$-product (and here $\mathbf{A}_e = e\!\uparrow < \mathbf{A}$).

To finish this proof, it will suffice to prove that $\mathbf{A}_e \times (\mathbf{C}_1 \oplus \mathbf{B})$ is never an (m, n)-product, and that

$$\mathbf{A} \times \mathbf{B} \not\leq \mathbf{A}_e \times (\mathbf{C}_1 \oplus \mathbf{B}).$$

If $\mathbf{A}\times\mathbf{B} \leq \mathbf{A}_e \times(\mathbf{C}_1\oplus\mathbf{B})$ then this is a tight embedding, from which we conclude that $\mathbf{A}_e \times (\mathbf{C}_1 \oplus \mathbf{B})$ is a (m, n)-product. This means that this lattice has a pair of complements (c, d) of heights (n, m) respectively. Since $\mathbf{C}_1 \oplus \mathbf{B}$ has a unique atom, we have $\{c, d\} = \{(i, \perp), (i', \top)\}$ for some i, i'. If $c = (i, \perp)$ then n is the height of i in $\mathbf{A}_e$. This is impossible because $\mathrm{ht}(\mathbf{A}_e) = m - 1 < n$. Thus $e = (i', \top)$. But then the height of e is not less than the height of $\mathbf{C}_1 \oplus \mathbf{B}$, i.e., $n \geq n + 1$. These contradictions show that $\mathbf{A} \times \mathbf{B} \not\leq \mathbf{A}_e \times (\mathbf{C}_1 \oplus \mathbf{B})$ and the latter lattice is not an (m, n)-product. •

10. Individual definability of the members of Qdlatt

We can now prove the principal result of this paper.

Theorem 10.1. *Every element of* Qdlatt *is first-order definable in the structure* Qdlatt$'$.

Proof. Let $\mathbf{A} \in$ Qdlatt, $\mathrm{ht}(\mathbf{A}) = n$. We can suppose that $n \geq 3$. (Certainly, all members of Qdlatt of height no greater than 2 are definable.) We can also suppose that $\mathbf{A} = P^\partial$, the lattice of down-sets of the ordered set P whose elements are $u_0, \ldots, u_{n-1}$.

Let

$$Q = \{u_{i,j} : 0 \leq i < n, 0 \leq j \leq 2n\} \cup \{v_i : 0 \leq i < n\} \cup \{w_i : 0 \leq i < n\},$$

a set of $n \cdot (2n + 3)$ elements where $u_{i,0} = u_i$ for $0 \leq i < n$ and so $P \subseteq Q$.

We introduce two orders on the set Q, giving two posets which we will call $Q, \widehat{Q}$.

In Q we have $u_{i,j} < u_{i,j'}$ when $0 \leq i < n$ and $0 \leq j < j' \leq 2n$; and we have $u_{i,j} < v_i$ and $u_{i,j} < w_i$ whenever $j \leq n + i$; and $v_i < u_{i,k}$ and $w_i < u_{i,k}$ whenever $k > n + i$. There are no other order relations in Q. This makes Q isomorphic to the disjoint (cardinal) sum of the ordered sets

$$Q_i = \{u_{i,0}, u_{i,1}, \ldots, u_{i,n+i}, v_i, w_i, u_{i,n+i+1}, \ldots, u_{i,2n}\};$$

and for $0 \leq i < n$, Q_i is order-isomorphic to the lattice $\mathbf{E}_{n+i,n-1-i}$.

Setting $\mathbf{B}_i = Q_i^\partial$ and $\mathbf{B} = \prod_{0\leq i<n} \mathbf{B}_i$, we have that $\mathbf{B}_i \cong \mathbf{E}_{n+i+1,n-i}$ and $\mathbf{B} \cong Q^\partial$.

The order just defined on Q will be denoted $\leq_0$. The second order on this set of points, giving the ordered set $\widehat{Q}$, will be denoted $\leq_1$. The order $\leq_1$ of $\widehat{Q}$ is taken to be the transitive closure of $\leq_0 \cup \leq$ where $\leq$ is the given order on P. This means that for $x, y \in Q$ we have that $x <_1 y$ iff $x <_0 y$ or else $x = u_{i,0}$ for some $i < n$, $y \in Q_{i'}$ for some $i' < n$, and we have $i \neq i'$ and $u_i < u_{i'}$ in P.

We define $\widehat{\mathbf{B}}$ to be the sublattice of $\mathbf{B}$ identified, in the natural isomorphism between $\mathbf{B}$ and Q^∂, with the sublattice $\widehat{Q}^\partial$ of Q^∂. Thus $\widehat{\mathbf{B}} \leq_t \mathbf{B}$.

Observe that $\mathrm{ht}(\mathbf{B}_i) = |Q_i| = 2n + 3$, that $\widehat{\mathbf{B}}$ is a tightly embedded sublattice of $\mathbf{B}$, and that $\mathrm{ht}(\widehat{\mathbf{B}}) = \mathrm{ht}(\mathbf{B}) = |Q| = n \cdot (2n + 3)$. The lattice $\mathbf{B}$ has the element π, identified in the natural isomorphism of $\mathbf{B}$ with Q^∂, to the element P (a down-set in Q). In fact, $\pi \in \widehat{B}$ (obviously, P is also a downset in $\widehat{Q}$) and $\pi\!\uparrow$ is the same lattice calculated either in $\widehat{\mathbf{B}}$ or in $\mathbf{B}$,

$$\pi\!\uparrow \cong \prod_{0 \le i < n} \mathbf{E}_{n+i,n-i} = \mathbf{B}'\,.$$

(The above formula is taken to be the definition of $\mathbf{B}'$.) Note that it follows from the work of previous sections that $\mathbf{B}$ and $\mathbf{B}'$ are definable elements of $\mathrm{Q}\textsc{dlatt}'$.

<u>Claim 1:</u> Suppose that $\mathbf{X} \in \{\mathbf{B}, \widehat{\mathbf{B}}\}$, $x \in X$, and the lattice $x\!\uparrow$ calculated in $\mathbf{X}$ is isomorphic to $\mathbf{B}'$. Then $x = \pi$.

To prove the claim, first we consider the case $\mathbf{X} = \widehat{\mathbf{B}}$. Since $x \in B$ it is an n-tuple, $x = \langle x_i : i < n\rangle$, with $x_i \in B_i$. Note that since $x\!\uparrow \cong \mathbf{B}'$, and $\widehat{\mathbf{B}}$ is a tight sublattice of $\mathbf{B}$, then x has the same height in $\widehat{\mathbf{B}}$ as in $\mathbf{B}$, and that height is $n \cdot (2n+3) - n \cdot (2n+2) = n$.

<u>Subclaim 1.1:</u> For all $i < n$, the element x_i has height at most n in $\mathbf{B}_i$, and so the lattice $x_i\!\uparrow$, calculated in $\mathbf{B}_i$, has a unique atom.

If $\mathrm{ht}(x_i) > n$, then calculated in $\mathbf{B}$, $\mathrm{ht}(x) > n$, but we know that this is false.

Continuing our proof of Claim 1, note that since $x\!\uparrow \cong \mathbf{B}'$, we have elements $y^i > x$ in $\widehat{\mathbf{B}}$, $i < n$, such that

$$y^i\!\uparrow \cong \mathbf{E}_{n+i,n-i} \ \text{ calculated in } \widehat{\mathbf{B}}\text{, for all } i\text{, and}$$
$$y^i \vee y^{i'} = \top \ \text{ for } i \neq i' \text{ while } \bigwedge_{0 \le i < n} y^i = x\,.$$

As elements of $\mathbf{B}$, we have that $y^i = \langle y^i_j : 0 \le j < n\rangle$. Since x_j has a unique cover in $\mathbf{B}_j$, then the displayed formulas imply that for all $j < n$, there is a unique $i = \varphi(j) < n$ such that $y^i_j = x_j$ while $y^{i'}_j = \top_{B_j}$ for all $i' \neq i$. Clearly, φ is surjective, and thus one-to-one; let τ be the inverse map to φ. Thus for all $i < n$, $y^i_j = \top_{B_j}$ for $j \neq \tau(i)$ while $y^i_{\tau(i)} = x_{\tau(i)}$. Thus the co-height of y^i calculated in $\mathbf{B}$ is the co-height of $x_{\tau(i)}$ calculated in $\mathbf{B}_i$—but this is the same as the co-height of y^i calculated in $\widehat{\mathbf{B}}$, i.e., it is $2n+2$. It follows that $x_{\tau(i)}$ is the unique atom in $\mathbf{B}_{\tau(i)}$ for all i. This means that $x = \pi$, as claimed.

The proof of Claim 1 in the case $\mathbf{X} = \mathbf{B}$ is the same, but easier at some points. We leave that proof to the reader.

<u>Claim 2:</u> Suppose that $\mathbf{X} \in \{\mathbf{B}, \widehat{\mathbf{B}}\}$, $x \in X$, the height of x in $\mathbf{X}$ is n, and $\mathbf{B}'$ is embeddable into the lattice $x\!\uparrow$, calculated in $\mathbf{X}$. Then $x = \pi$.

We simply sketch the proof. It is a modification of the proof of the Claim 1. Here, we have a necessarily tight embedding of $\mathbf{B}'$ in $x\!\uparrow$. This leads to the existence of the system of elements y^i as before, and to the same calculations showing that up to permutation, $y^0, \ldots, y^{n-1}$ are just the elements $\langle x_0, \top, \ldots, \top\rangle$, $\langle \top, x_1, \ldots, \top\rangle$, $\ldots$, $\langle \top, \ldots, \top, x_{n-1}\rangle$. It follows now that every x_i has height at most 1 in $\mathbf{B}_i$. Since $\mathrm{ht}(x) = n$ (in $\widehat{\mathbf{B}}$ and in $\mathbf{B}$), then it must be the case that $x = \pi$.

<u>Claim 3:</u> For all $\mathbf{R} \in \mathrm{Q}\textsc{dlatt}$, $\mathbf{R} \le_t \mathbf{A}$ iff $\mathbf{R} \oplus \mathbf{B}' \le_t \widehat{\mathbf{B}}$.

To see this, note first that (as we've seen) $\pi{\uparrow}$ in $\widehat{\mathbf{B}}$ is isomorphic to $\mathbf{B}'$, while $\pi{\downarrow}$ in $\widehat{\mathbf{B}}$ is canonically isomorphic with P^∂, i.e., with $\mathbf{A}$. Thus $\mathbf{A}\oplus\mathbf{B}' \leq_t \widehat{\mathbf{B}}$. On the other hand, suppose that $\mathbf{T}$ is a tight sublattice of $\widehat{\mathbf{B}}$ isomorphic to $\mathbf{R}\oplus\mathbf{B}'$, for some $\mathbf{R}$. Then there is an element $g \in \widehat{\mathbf{B}}$ such that $g{\uparrow}$ calculated in $\widehat{\mathbf{B}}$ has a tightly embedded copy of $\mathbf{B}'$ and $g{\downarrow}$ has a tightly embedded copy of $\mathbf{R}$. By Claim 2, $g = \pi$. Then $\mathbf{R} \leq_t g{\downarrow} \cong \mathbf{A}$.

Let us observe, at this point, that it is immediate from Claim 3 that $\mathbf{A}$ is, up to isomorphism, the unique member $\mathbf{U} \in$ QDLATT such that for all $\mathbf{R} \in$ QDLATT, $\mathbf{R} \leq_t \mathbf{U}$ iff $\mathbf{R}\oplus\mathbf{B}' \leq_t \widehat{\mathbf{B}}$. Since $\mathbf{B}'$ is definable in QDLATT$'$, what remains for us to do is simply to show that $\widehat{\mathbf{B}}$ is definable.

For that purpose, we consider some special elements of $\widehat{\mathbf{B}}$. Let p be any member of $\mathbf{A}$. Thus p is a down-set in P. Put

$$p_0 = p \cup \bigcup_{u_i \notin p} Q_i \quad \text{and } p_1 = \bigcup_{u_i \in p} Q_i \,.$$

It is easily seen that $p, p_0, p_1 \in \widehat{Q}^\partial$ and that $p_0 \cap p_1 = p$ and $p_0 \cup p_1 = Q$. Let q, q_0, q_1 be the elements of $\widehat{\mathbf{B}}$ corresponding to p, p_0, p_1 under the isomorphism of $\widehat{Q}^\partial$ with $\widehat{\mathbf{B}}$. Thus we have that $q \leq \pi$, $q_0 \geq \pi$, $q = q_0 \wedge q_1$ and $\top = q_0 \vee q_1$. Moreover,

$$q_0{\uparrow} \cong \prod_{u_i \in p} \mathbf{E}_{n+i,n-i} \,.$$

(This is true both in $\widehat{\mathbf{B}}$ and in $\mathbf{B}$ since $\pi \leq q_0$ and $\pi{\uparrow} \subseteq \widehat{B}$.) Finally, we have

$$(q_1 \vee \pi){\uparrow} \cong \prod_{u_i \notin p} \mathbf{E}_{n+i,n-i} \,.$$

We can capture this situation abstractly. For $p \in P^\partial$, put $\mathbf{B}'_{0,p} = \prod_{u_i \in p} \mathbf{E}_{n+i,n-i}$ and $\mathbf{B}'_{1,p} = \prod_{u_i \notin p} \mathbf{E}_{n+i,n-i}$. Note that each of the lattices $\mathbf{B}'_{0,p}$ and $\mathbf{B}'_{1,p}$ is a definable member of QDLATT.

For $p \in P^\partial$, define Φ_p to be the following property of a lattice $\mathbf{R} \in$ QDLATT: There are $\mathbf{S}, \mathbf{V} \in$ QDLATT such that

(1) $\mathbf{C}_n \oplus \mathbf{B}' \leq_t \mathbf{R} \leq_t \mathbf{B}$.
(2) $\mathbf{C}_k \oplus \mathbf{S} \leq_t \mathbf{R}$ where $k = |p|$.
(3) $\mathbf{S} \cong \mathbf{V} \times \mathbf{B}'_{0,p}$.
(4) $\mathbf{C}_\ell \oplus \mathbf{B}'_{1,p} \leq_t \mathbf{V}$ where $\ell = n - k$.

Claim 4: Suppose that $\mathbf{R}$ is a tightly embedded sublattice of $\mathbf{B}$ and $p \in P^\partial$. Then $\mathbf{R}$ satisfies Φ_p iff $\pi{\uparrow} \subseteq R$ and $\{q, q_0, q_1\} \subseteq R$ where q, q_0, q_1 are the three elements of $\widehat{\mathbf{B}}$ correlated with p as defined above.

To prove this, assume that $\mathbf{R}$ is a tightly embedded sublattice of $\mathbf{B}$. First, suppose that $\pi{\uparrow} \cup \{q, q_0, q_1\} \subseteq R$. Then $\mathbf{C}_n \oplus \mathbf{B}' \leq_t \mathbf{R} \leq_t \mathbf{B}$ obviously holds, since $\pi{\uparrow} \cong \mathbf{B}'$ and the height of π (in $\mathbf{B}$ or in $\mathbf{R}$) is n. Then in $\mathbf{R}$, $\mathrm{ht}(q) = k$ and $q{\uparrow}$ is isomorphic to the product of $q_0{\uparrow}$ and $q_1{\uparrow}$. We have that $q_0{\uparrow}$ is the same calculated in $\mathbf{R}$ as in $\mathbf{B}$, since $q_0 \geq \pi$. Thus $q{\uparrow}$ (in $\mathbf{R}$) is isomorphic to $\mathbf{V}\times\mathbf{B}'_{0,p}$ where $\mathbf{V} = q_1{\uparrow}$. We take $\mathbf{S} = q{\uparrow}$. Clearly $\mathbf{C}_k \oplus \mathbf{S} \leq_t \mathbf{R}$, since $\mathrm{ht}(q) = k$ (in either of $\mathbf{B}, \mathbf{R}$). Where

$z = q_1 \vee \pi$, we have that the height of the interval $I[q_1, z]$ is ℓ (in either of $\mathbf{B}, \mathbf{R}$). Moreover $z\!\uparrow \cong \mathbf{B}'_{1,p}$ as we observed earlier. It follows that $\mathbf{C}_\ell \oplus \mathbf{B}'_{1,p}$ is embedded in $\mathbf{V}$, and this embedding must be tight (the heights match). Thus indeed, $\mathbf{R}$ satisfies Φ_p.

For the other direction, assume that $\mathbf{R}$ satisfies Φ_p. From statement Φ_p (1) and Claim 2, $\pi\!\uparrow \subseteq R$. The statements Φ_p (2,3,4) yield that $\mathbf{R}$ has elements r, r_0, r_1, z such that $\mathrm{ht}(r) = k$, $r = r_0 \wedge r_1$, $r_0 \vee r_1 = \top$, $\mathbf{B}'_{0,p} \leq_t r_0\!\uparrow$ (calculated in $\mathbf{R}$), $r_1 \leq z$, $\mathrm{ht}(I[r_1, z]) = \ell$ $(= n - k)$ (in $\mathbf{R}$ and in $\mathbf{B}$), $\mathbf{B}'_{1,p} \leq_t z\!\uparrow$ (calculated in $\mathbf{R}$).

It follows easily that where $\lambda = z \wedge r_0$, then $\mathbf{B}'_{1,p} \times \mathbf{B}'_{0,p} \leq_t \lambda\!\uparrow$ and $\mathrm{ht}(\lambda) = n$ (in $\mathbf{R}$, and in $\mathbf{B}$). Since $\mathbf{B}'_{1,p} \times \mathbf{B}'_{0,p} \cong \mathbf{B}'$, then Claim 2 implies that $\lambda = \pi$.

Now the strict refinement property for direct products of distributive lattices, and the fact that the lattices $\mathbf{E}_{n+i,n-i}$ are all directly indecomposable and pairwise non-isomorphic, yields that $r_0 = q_0$ and $z = q_1 \vee \pi$. Thus, written as elements of $\mathbf{B}$, r_0 is the n-tuple $\langle r_{0,i} : i < n \rangle$ with value $r_{0,i} = u_{i,0}$ for $u_i \in p$ and value $r_{0,i} = \top$ for $u_i \notin p$. Since $r_1 \vee r_0 = \top$, it follows that $r_1 = \langle r_{1,i} : i < n \rangle$ where $r_{1,i} = \top$ for $u_i \in p$. Now $\mathrm{ht}(I[r_1, z]) = \ell$ and for all i with $u_i \notin p$, $r_{1,i} \leq z_i$ and $\bot_i \prec z_i$. I.e., $r_{1,i} = \top_{B_i}$ for $u_i \in p$ and

$$r_{1,i} \in \{\bot_i, z_i\}$$

for $u_i \notin p$. Since $\mathrm{ht}(I[r_1, z]) = \ell$, it follows that $r_{1,i} = \bot_i$ when $u_i \notin p$. Thus, finally, $r_1 = q_1$ and $r = r_0 \wedge r_1 = q_0 \wedge q_1 = q$.

This completes our proof of Claim 4.

Claim 5: Suppose that $\mathbf{R}$ is a tightly embedded sublattice of $\mathbf{B}$. Then $\mathbf{R}$ satisfies Φ_p for all $p \in P^\partial$ iff $\widehat{\mathbf{B}} \subseteq \mathbf{R}$.

Let $\mathbf{R}$ be a tightly embedded sublattice of $\mathbf{B}$. If $\widehat{\mathbf{B}} \subseteq \mathbf{R}$ then $\mathbf{R}$ satisfies Φ_p for all $p \in P^\partial$ by Claim 4.

On the other hand, suppose that $\mathbf{R}$ satisfies Φ_p for all $p \in P^\partial$. By Claim 4, we have then that $\pi\!\uparrow \subseteq R$ and for all $p \in P^\partial$, the element q^p $(= q_1$ for p as defined above) of $\mathbf{B}$ defined by $q^p_i = \top_i$ for $u_i \in p$ and $q^p_i = \bot_i$ for $u_i \notin p$, belongs to R. It is easy to check that every join-irreducible element of $\widehat{\mathbf{B}}$ is of the form $q^p \wedge y$ for some $y \geq \pi$. Thus all the join-irreducible members of $\widehat{\mathbf{B}}$ belong to R. It follows that $\mathbf{R} \supseteq \widehat{\mathbf{B}}$. This completes the proof of Claim 5.

Finally, from Claim 5, we have that $\widehat{\mathbf{B}}$ is the $\leq$-smallest $\mathbf{R} \in \textsc{Qdlatt}$ such that $\mathbf{R} \leq_t \mathbf{B}$ and $\mathbf{R}$ satisfies Φ_p for all $p \in P^\partial$. (In fact, $\widehat{\mathbf{B}}$ is the smallest tightly embedded subalgebra of $\mathbf{B}$ that satisfies the abstract property Φ_p for all $p \in P^\partial$.) •

Corollary 10.2. *The ordered set $\langle \mathcal{D}, \leq \rangle$ has only two automorphisms: the identity map and the map $\ell \mapsto \ell^{\mathrm{opp}}$. Every element ℓ in $\mathcal{D}$ is definable in this ordered set up to the two automorphisms, i.e., the set $\{\ell, \ell^{\mathrm{opp}}\}$ is definable.*

Proof. It follows from Theorem 10.1 that every element of $\mathcal{D}$ is definable in the pointed ordered set $\langle \mathcal{D}, \leq, e_1 \rangle$. Thus the identity is the only automorphism of $\langle \mathcal{D}, \leq \rangle$ that fixes e_1. Now e_0 and $e_1 = e_0^{\mathrm{opp}}$ are the only covers of c_3 (the isomorphism type of the four-element chain) in $\langle \mathcal{D}, \leq \rangle$ different from c_4. The elements c_3 and

c_4 are definable, so $\{e_0, e_1\}$ is a definable set in $\langle \mathcal{D}, \leq \rangle$. Suppose that σ is an automorphism different from the identity. Then $\sigma(e_1) \neq e_1$, forcing $\sigma(e_1) = e_0$ since $\{e_0, e_1\}$ is invariant under σ. Now $\mathrm{opp} \circ \sigma$ is an automorphism fixing e_1. Thus $\mathrm{opp} \circ \sigma = \mathrm{id}$, and $\sigma = \mathrm{opp} \circ \mathrm{opp} \circ \sigma = \mathrm{opp}$. •

11. Universal classes of distributive lattices

A class K of distributive lattices is called *universal* if it can be axiomatized by a set of universal sentences (universal closures of quantifier-free formulas). Equivalently, K is universal iff it is closed under sublattices and contains every lattice $\mathbf{L}$ such that all finite sublattices of $\mathbf{L}$ belong to K. Clearly, the family of all universal classes of distributive lattices constitutes a lattice with respect to inclusion (if we neglect the obstacle that it is a set of proper classes); and this lattice is isomorphic to the lattice of order-ideals of $\langle \textsc{Qdlatt}, \leq \rangle$.

For a class K of distributive lattices, denote by K^{opp} the class of the lattices opposite to the lattices of K. The mapping $K \mapsto K^{\mathrm{opp}}$ is clearly an automorphism of the lattice of universal classes of distributive lattices.

Theorem 11.1. *The lattice of universal classes of distributive lattices has only two automorphisms: the identity and the $K \mapsto K^{\mathrm{opp}}$. The set of all finitely based and also the set of all finitely generated universal classes of distributive lattices are definable subsets of this lattice, and every element of the two subsets is an element definable up to the two automorphisms of this lattice.*

Proof. As we mentioned above, the lattice of universal classes of distributive lattices is isomorphic to the lattice $\mathbf{L}$ of order-ideals of the quasi-ordered set $\langle \textsc{Qdlatt}, \leq \rangle$. The members of $\mathbf{L}$ are the subsets $K \subseteq \textsc{Qdlatt}$ such that $\mathbf{A} \leq \mathbf{B} \in K$ implies $\mathbf{A} \in K$. Under the isomorphism between these lattices, the finitely generated order-ideals are carried onto the finitely generated universal classes, and the set-complements of the finitely generated order-filters are carried onto the finitely axiomatizable universal classes.

Thus let I be an order-ideal in $\textsc{Qdlatt}$ that is either finitely generated or the complement of a finitely generated order-filter. We need to show that $\{I, I^{\mathrm{opp}}\}$ is first-order definable in the lattice $\mathbf{L}$. There are finitely many lattices $\mathbf{A}_1, \dots, \mathbf{A}_n$ in $\textsc{Qdlatt}$ so that either we have

$$I = \{\mathbf{B} \in \textsc{Qdlatt} : \mathbf{B} \leq \mathbf{A}_i \text{ for some } 1 \leq i \leq n\}$$

or we have

$$I = \{\mathbf{B} \in \textsc{Qdlatt} : \text{ for all } i \text{ with } 1 \leq i \leq n \;\; \mathbf{B} \not\geq \mathbf{A}_i\}\,.$$

For $\mathbf{A} \in \textsc{Qdlatt}$ put $\mathbf{A}{\downarrow} = \{\mathbf{B} \in \textsc{Qdlatt} : \mathbf{B} \leq \mathbf{A}\}$. The set of strictly join-irreducible members of $\mathbf{L}$, definable in $\mathbf{L}$, is precisely the set of order-ideals of $\textsc{Qdlatt}$ of the form $\mathbf{A}{\downarrow}$ (for $\mathbf{A} \in \textsc{Qdlatt}$). Thus Theorem 10.1 implies that each of $\mathbf{A}_1{\downarrow}, \dots, \mathbf{A}_n{\downarrow}$ is a definable member of the pointed lattice $(\mathbf{L}, \mathbf{E}_1{\downarrow})$. Thus for $1 \leq i \leq n$ there is a first-order lattice-theoretic formula $\varphi_i(x, y)$ so that $\mathbf{A}_i{\downarrow}$ is the unique member x of $\mathbf{L}$ such that $\mathbf{L} \models \varphi(x, \mathbf{E}_1{\downarrow})$. Also, there is a formula $\varepsilon(x)$ so that $\mathbf{E}_1\!\downarrow, \mathbf{E}_1^{\mathrm{opp}}\!\downarrow$ are the unique elements of $\mathbf{L}$ that satisfy $\varepsilon(x)$. (Because the set $\{\mathbf{E}_1, \mathbf{E}_1^{\mathrm{opp}}\}$ is definable in $\textsc{Qdlatt}'$; see the last paragraph of section 2.)

Define $\Phi(x)$ to be the formula

$$(\exists y)(\exists x_1, \ldots, x_n)[\varepsilon(y) \wedge \bigwedge_{1\le i\le n} \varphi(x_i, y) \wedge x = x_1 + \cdots + x_n]\,;$$

and $\Psi(x)$ to be the formula

$$\begin{aligned}(\exists y)(\exists x_1, \ldots, x_n)[\varepsilon(y) \wedge \bigwedge_{1\le i\le n} \varphi(x_i, y) \quad \wedge \\ (\forall z)[z \le x \leftrightarrow \bigwedge_{1\le i\le n} x_i \not\le z]\,] \quad .\end{aligned}$$

In the first formula, $+$ is the symbol for the lattice join operation in $\mathbf{L}$.

We claim that for $x \in \mathbf{L}$, $\mathbf{L} \models \Phi(x)$ iff $x = I$ or $x = I^{\mathrm{opp}}$ where I is the order-ideal generated by $\mathbf{A}_1, \ldots, \mathbf{A}_n$; and $\mathbf{L} \models \Psi(x)$ iff $x = J$ or $x = J^{\mathrm{opp}}$ where J is the largest order-ideal containing none of $\mathbf{A}_1, \ldots, \mathbf{A}_n$.

We shall prove just the claim for $\Psi(x)$ and J. Suppose first that $U \in \mathbf{L}$ and $\mathbf{L} \models \Psi(U)$. Let Y and $X_1, \ldots, X_n$ be the elements of $\mathbf{L}$ that witness the satisfaction of $\Psi(U)$. Then $\mathbf{L} \models \varepsilon(Y)$ and $\mathbf{L} \models \varphi(X_i, Y)$ for $i = 1, \ldots, n$. It follows that $Y = \mathbf{E}_1\!\downarrow$ or $Y = \mathbf{E}_1^{\mathrm{opp}}\!\downarrow$. If $Y = \mathbf{E}_1\!\downarrow$ then it follows that $X_i = \mathbf{A}_i\!\downarrow$ for $i = 1, \ldots, n$. In this case, the fact that $\mathbf{L} \models \Psi(U)$ tells us that U is the largest member of $\mathbf{L}$ that fails to intersect $\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$, i.e., $U = J$. In the case that $Y = \mathbf{E}_1^{\mathrm{opp}}$, consider U^{opp} $(= \{\mathbf{A}^{\mathrm{opp}} : \mathbf{A} \in U\})$. Since opp is an automorphism of $\langle \textsc{Qdlatt}, \le\rangle$, it induces an automorphism of $\mathbf{L}$. It follows that $\mathbf{L} \models \Psi(U^{\mathrm{opp}})$ with witnesses $Y^{\mathrm{opp}} = \mathbf{E}_1\!\downarrow$ and X_i^{opp}. This puts us in the first case, and we can conclude that $U^{\mathrm{opp}} = J$. So it follows that $U = J^{\mathrm{opp}}$ in this case. Since it is more or less obvious that $\mathbf{L} \models \Psi(J)$ and $\mathbf{L} \models \Psi(J^{\mathrm{opp}})$, we regard the proof of Theorem 11.1 as regards definability to be finished.

It remains to show that $U \mapsto U^{\mathrm{opp}}$ is the only non-identity automorphism of $\mathbf{L}$. Here is a proof. Let σ be any automorphism of $\mathbf{L}$. Since $\{\mathbf{E}_1\!\downarrow, \mathbf{E}_1^{\mathrm{opp}}\!\downarrow\}$ is a definable subset of $\mathbf{L}$ then $\sigma(\mathbf{E}_1\!\downarrow)$ belongs to this set. Thus if $\sigma(\mathbf{E}_1\!\downarrow) \ne \mathbf{E}_1\!\downarrow$ then $\tau(\mathbf{E}_1\!\downarrow) = \mathbf{E}_1\!\downarrow$ where τ is the automorphism $U \mapsto \sigma(U)^{\mathrm{opp}}$. We now show that any automorphism which fixes the element $\mathbf{E}_1\!\downarrow$ must be the identity. It will follow that σ is the identity, or σ followed by the map 'opposite' is the identity; so that σ is the identity or the map $U \mapsto U^{\mathrm{opp}}$.

So finally, suppose that σ is an automorphism of $\mathbf{L}$ and that $\sigma(\mathbf{E}_1\!\downarrow) = \mathbf{E}_1\!\downarrow$. For every $\mathbf{A} \in \textsc{Qdlatt}$ there is, as we noted above, a lattice-theoretic formula $\varphi(x, y)$ such that $\mathbf{A}\!\downarrow$ is the unique element $X \in \mathbf{L}$ for which $\mathbf{L} \models \varphi(X, \mathbf{E}_1\!\downarrow)$. Since $\mathbf{L} \models \varphi(\mathbf{A}\!\downarrow, \mathbf{E}_1\!\downarrow)$ then $\mathbf{L} \models \varphi(\sigma(\mathbf{A}\!\downarrow), \sigma(\mathbf{E}_1\!\downarrow))$; but since σ fixes $\mathbf{E}_1\!\downarrow$ then $\mathbf{L} \models \varphi(\sigma(A\!\downarrow), \mathbf{E}_1\!\downarrow)$, and $\sigma(A\!\downarrow) = A\!\downarrow$ is forced. Thus the fixed points of σ include all the $A\!\downarrow$ and, consequently, every point of $\mathbf{L}$ is fixed by σ, as every member of $\mathbf{L}$ is the join in $\mathbf{L}$ of some subset of the family of members of the form $A\!\downarrow$. •

We remark that the lattice of universal classes of distributive lattices is uncountable. This follows from [2] where we have constructed an infinite antichain in the ordered set $\langle \mathcal{D}, \leq \rangle$.

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Department of Mathematics, Charles University, Prague, Czech Republic
E-mail address: jezek@karlin.mff.cuni.cz

Department of Mathematics, Vanderbilt University, Nashville, TN 37235, USA
E-mail address: ralph.mckenzie@vanderbilt.edu