FINITELY GENERATED COMMUTATIVE DIVISION SEMIRINGS

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Abstract. One-generated commutative division semirings are found.

The aim of this (partially expository) note is to find all one-generated (commutative) division semirings (see Theorem 8.5). In particular, all such semirings turn out to be finite. To achieve this goal, we have to correct some results from [1] (especially Proposition 12.1 of [1]) and to complete some results from [2]. Anyway, all the presented results are fairly basic and (with two exceptions) we shall not attribute them to any particular source.

1. Introduction

A *semiring* is an algebraic structure with two associative binary operations (usually denoted as addition and multiplication) such that the addition is commutative and the multiplication distributes over the addition from either side. If the multiplication is commutative, the semiring is called so. In the sequel, we consider only commutative semirings.

A semiring S is called

- *congruence-simple* if S has just two congruence relations;
- *ideal-simple* if S is non-trivial and $I = S$ whenever I is an ideal of S containing at least two elements;
- a *division semiring* if S is non-trivial and contains an element w such that $S \setminus \{w\} \subseteq Sa$ for every $a \in S \setminus \{w\}$;
- a *semifield* if S is non-trivial and contains a multiplicatively absorbing element w such that $S \setminus \{w\}$ is a subgroup of the multiplicative semigroup of S;
- a *parasemifield* if the multiplicative semigroup of S is a non-trivial group.

We denote by $\mathbf{N}$ the semiring of positive integers, by $\mathbf{N}_0$ the semiring of non-negative integers, by $\mathbf{Z}$ the ring of integers, by $\mathbf{Q}$ the field of rational numbers, by $\mathbf{Q}^+$ the parasemifield of positive rational numbers, by $\mathbf{Q}_0^+$ the semifield of non-negative rational numbers, and by $\mathbf{R}$ the field of real numbers. Put $\mathbf{R}^+ = \{a \in \mathbf{R} : a > 0\}$ and $\mathbf{R}^- = \{a \in \mathbf{R} : a < 0\}$.

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Notice that all semifields and parasemifields are ideal-simple division semirings. On the other hand, zero multiplication rings of finite odd prime order are both congruence- and ideal-simple, but they are not division semirings. Observe that every division semiring has at most two ideals.

For a semiring S, let $\mathbf{Ida}(S) = \{a \in S : a + a = a\}$. Clearly, $\mathbf{Ida}(S)$ is either empty or an ideal of S. The semiring S is called

- *additively idempotent* if $\mathbf{Ida}(S) = S$;
- *almost additively idempotent* if the set $S \setminus \mathbf{Ida}(S)$ has at most one element.

1.1. Lemma. *Let S be an almost additively idempotent semiring and $S \setminus \mathbf{Ida}(S) = \{w\}$. Put $s = w + w$. Then:*

(i) *$s \in \mathbf{Ida}(S)$.*
(ii) *$wa = sa$ for every $a \in \mathbf{Ida}(S)$.*
(iii) *Either $w^2 = w$ and $s^2 = s$ or else $w^2 = s^2$.*

Proof. It is easy. □

1.2. Lemma. *Let P be a parasemifield. Put $K = \{a \in P : a + 1_P \neq 1_P\}$ and $L = \{a \in P : a + 1_P = 1_P\}$. Then:*

(i) *$K \cup L = P$ and $K \cap L = \emptyset$.*
(ii) *$K \neq \emptyset$.*
(iii) *If $a \in L$ and $a \neq 1_P$, then $a^{-1} \in K$.*
(iv) *$L + L \subseteq L$ and $LL \subseteq L$.*
(v) *If $L \neq \emptyset$, then L is a sebsemiring of P.*
(vi) *$K + L \subseteq K$.*
(vii) *P is additively idempotent if and only if $1_P \in L$.*
(viii) *If P is additively idempotent, then $K + P \subseteq K$.*
(ix) *If P is additively cancellative, then $L = \emptyset$.*

Proof. It is easy. □

1.3. Lemma. *Let P be a parasemifield and $e \in P$. Put $K_e = Ke$ and $L_e = Le$, where K and L are as in 1.2. Then:*

(i) *$K_e = \{a \in P : a + e \neq e\}$ and $L_e = \{a \in P : a + e = e\}$.*
(ii) *$K_e \cup L_e = P$ and $K_e \cap L_e = \emptyset$.*
(iii) *$K_e \neq \emptyset$.*
(iv) *If $a \in L_e$ and $a \neq e$, then $a^{-1}e^2 \in K_e$.*
(v) *$L_e + L_e \subseteq L_e$ and $L_eL_e \subseteq L_ee = L_{e^2}$.*
(vi) *$K_e + L_e \subseteq K_e$.*
(vii) *P is additively idempotent if and only if $e \in L_e$.*
(viii) *If P is additively idempotent, then $K_e + P \subseteq K_e$.*
(ix) *If P is additively cancellative, then $L_e = \emptyset$.*

Proof. It follows from 1.2. □

Denote by $\mathbf{P}$ the variety of universal algebras with two binary operations (addition and multiplication) and one unary operation x^{-1}, determined by the equations of commutative semirings and the equations of (multiplicatively denoted) commutative groups. Clearly, there is a one-to-one correspondence between parasemifields and the non-trivial algebras from $\mathbf{P}$. In this paper we prefer to consider parasemifields as special semirings, rather than elements of $\mathbf{P}$. However, there could be a confusion if we need to speak about generating subsets of parasemifields. We say that a parasemifield P (or any semiring) is generated by a subset X as a semiring if P is the least subsemiring of P containing X. We say that a parasemifield P is generated by a subset X as a parasemifield if P is the least subparasemifield of P containing X.

Similarly, we need to distinguish between subsets of a ring generating the given ring as a subsemiring or as a subring.

1.4. Lemma. *Let P be a parasemifield. Then P is not one-generated as a semiring.*

Proof. Since $\mathbf{P}$ is a variety, there exists a one-generated free object F in $\mathbf{P}$. It is easy to see that F is isomorphic to the parasemifield $\mathbf{Q}^+$ (considered as an element of $\mathbf{P}$). Also, it is easy to see that $\mathbf{Q}^+$ is congruence-simple. From this it follows that $\mathbf{Q}^+$ is, up to isomorphism, the only non-trivial one-generated algebra in $\mathbf{P}$. Of course, $\mathbf{Q}^+$ is one-generated as a parasemifield. On the other hand, it is easy to see that it is not one-generated as a semiring. □

The following folklore type result is usually attributed to I. Kaplansky.

1.5. Lemma. *Let A be an infinite field. Then A is not finitely generated as a semiring.*

2. Auxiliary results on commutative semigroups

In this section let S be a non-trivial commutative semigroup (denoted multiplicatively), containing an element w such that $T = S \setminus \{w\} \subseteq Sa$ for every $a \in T$. (Clearly, $T \subseteq SS$.)

2.1. Lemma. *If $w = 1_S \in TT$, then S is a group.*

Proof. We have $1_S = ab$ for some $a, b \in T$. If $c \in T \setminus \{a\}$, then $a = cd$ for some $c, d \in T$ and $1_S = cdb$. Thus every element of S has an inverse in S, which means that S is a group. □

2.2. Lemma. *If $w = 1_S \notin TT$, then T is a subgroup of S.*

Proof. The result is clear for $|T| = 1$. If a, b are two distinct elements of T, then $ac = b$ and $bd = a$ for some $c, d \in T$; we get $acd = a$ and then obviuosly $cd = 1_T$; now it is clear that T is a subgroup of S. □

2.3. Lemma. *If $w \neq 1_S$ and $wa = a$ for all $a \in T$, then $w^2 = 1_T \in T$ and T is a subgroup of S.*

Proof. Since $w \neq 1_S$, we have $w^2 \in T$ and then $w^2 = 1_T$. Furthermore, $bc = wbc$ for all $b, c \in T$ and it follows that $bc \in T$. Now it is easy to see that T is a subgroup of S. □

2.4. Lemma. *If $wa_0 \neq a_0$ for at least one $a_0 \in T$, then $1_T \in T$.*

Proof. We have $a_0 = a_0b_0$ for some $b_0 \in T$. For every $c \in T$ there is a $d \in S$ with $c = a_0d$ and then $cb_0 = c$. Thus $b_0 = 1_T \in T$. □

2.5. Lemma. *If $wa_0 \neq a_0$ for at least one $a_0 \in T$ and $w1_T = a_1 \in T$, then $a_1 \neq 1_T$, $wa = a_1a$ for every $a \in T$, $w^2 = a_1^2$, $SS \subseteq T$ and T is a subgroup of S.*

Proof. We have $wa = w1_Ta = a_1a$ for every $a \in T$. Since $wa_0 \neq a_0$, we have $a_1 \neq 1_T$. If $b, c \in T$, then $bc = bc1_T$ implies $bc \in T$ and it follows that $ST \subseteq T$.

For every $a \in T$ there is a $d \in S$ with $ad = 1_T$. If $d = w$, then $1_T = aw = a_1a$ and we see that every element of T is invertible. Thus T is a group. Finally, $a_1^2 \neq a_1$ and $w^21_T = wa_1 = a_1^2$. Thus $w^2 = a_1^2$. □

2.6. Lemma. *If $wa_0 \neq a_0$ for at least one $a_0 \in T$, $w1_T = w$ and $1_T \in Sw$, then S is a group.*

Proof. We have $1_T = 1_S$ and the rest is clear. □

2.7. Lemma. *If $wa_0 \neq a_0$ for at least one $a_0 \in T$, $w_1T = w$ and $1_T \notin Sw$, then $Sw = \{w\}$ and T is a subgroup of S.*

Proof. We have $1_T = 1_S$ and T is the set of invertible elements of S. Then, of course, T is a subgroup of S. Since w is not invertible, we have $Sw = \{w\}$. □

2.8. Proposition. *Let S be a non-trivial commutative semigroup and $w \in S$ be an element such that $T = S \setminus \{w\} \subseteq Sa$ for every $a \in T$. Then either S is a group, or else T is a subgroup of S and at least one of the following three cases takes place:*

(1) $w = 1_S$;
(2) $w1_T = e \in T$, $w^2 = e^2$ *and* $wa = ea$ *for all* $a \in T$;
(3) $wS = \{w\}$.

Proof. Combine the preceding seven lemmas. □

2.9. Remark. If S is either a group or the two-element semilattice, then for an arbitrary element $w \in S$ the pair S, w serves as an example for the above investigated situation; in the semilattice case, with one choice of w we get the case 2.8(1) and with the other one the case 2.8(3). If S is neither a group nor the two-element semilattice, then the element w is unique and only one of the three cases 2.8(1),(2),(3) can take place.

3. Division semirings – classification

Let S be a division semiring and and let $w \in S$ be such that $T = S \setminus \{w\} \subseteq Sa$ for every $a \in T$. If follows from 2.8 that the pair (S, w) belongs to exactly one of the following four types:

(I) S is a parasemifield;
(II) T is a subgroup of $S(\cdot)$ and $w = 1_S$;
(III) T is a subgroup of $S(\cdot)$, $w1_T = e \in T$, $w^2 = e^2$ and $wa = ea$ for all $a \in T$;
(IV) T is a subgroup of $S(\cdot)$ and w is a multiplicatively absorbing element of S.

We say that S is of type (X) if there exists an element $w \in S$ such that the pair (S, w) is of type (X). Clearly, the type of a division semiring is uniquely determined, with just four exceptions: the two-element division semirings Z_2, Z_5, Z_6, Z_8 (see 4.1) are of type (II) and of type (IV) at the same time.

If S is a parasemifield, then S is infinite and w can be any element of S. If S is not a parasemifield, then the element w is uniquely determined by S together with the specification of the type of S; and if $|S| \geq 3$, it is uniquely determined by S.

3.1. Example. Let S be a zero multiplication ring of finite prime order. Then S is both congruence- and ideal-simple, but S is not a division semiring.

3.2. Example. Let $S = \{n\sqrt{2} - m : n, m \in \mathbf{N}_0, \quad n + m \geq 1\}$. Define operations $\oplus$ and $\odot$ on S by $a \oplus b = \min(a, b)$ and $a \odot b = a + b$. Then $S = S(\oplus, \odot)$ is an additively idempotent congruence-simple semiring that is not ideal-simple and that is not a division semiring.

3.3. Example. The product $S = \mathbf{Q}^+ \times \mathbf{Q}^+$ is a parasemifield, and hence S is an ideal-simple division semiring. Of course, S not congruence-simple.

3.4. Example. Let G be a commutative group (denoted multiplicatively), $o \notin G$ and $S = G \cup \{o\}$. Put $x + y = o$ for all $x, y \in S$ and extend the multiplication of G by $xo = ox = o$ for all $x \in S$. Then S becomes a division semiring (moreover, a semifield) and o is the only additive idempotent of S. If G is non-trivial, then S is not congruence-simple.

3.5. Example. Let m be a non-negative integer. Put $S = \mathbf{Z} \cup \{w\}$ where w is an element not belonging to $\mathbf{Z}$ and define two binary commutative operations $\oplus$ and $\odot$ on S as follows: $a \odot b = a + b$ for all $a, b \in \mathbf{Z}$; $w \odot x = x$ for all $x \in S$; $a \oplus b = \min(a, b)$ for all $a, b \in \mathbf{Z}$; $w \oplus a = \min(0, a)$ for all $a \in \mathbf{Z}$ with $a < m$; $w \oplus a = w$ for all $a \in \mathbf{Z}$ with $a \geq m$; finally, we define the element $w \oplus w$ to be either 0 or w. We obtain two division semirings $S = S(\oplus, \odot)$ (they differ only by the value of $w \oplus w$). These division semirings are neither congruence- nor ideal-simple; they are almost additively idempotent; only that one with $w \oplus w = w$ is additively idempotent.

4. A few constructions

4.1. Construction. The following eight semirings $Z_1, \dots, Z_8$ are (up to isomorphism) all two-element semirings:

Z_1:

+	0	1
0	0	0
1	0	0

·	0	1
0	0	0
1	0	0

Z_2:

+	0	1
0	0	0
1	0	0

·	0	1
0	0	0
1	0	1

Z_3:

+	0	1
0	0	0
1	0	1

·	0	1
0	0	0
1	0	0

Z_4:

+	0	1
0	0	0
1	0	1

·	0	1
0	1	1
1	1	1

Z_5:

+	0	1
0	0	0
1	0	1

·	0	1
0	0	1
1	1	1

Z_6:

+	0	1
0	0	0
1	0	1

·	0	1
0	0	0
1	0	1

Z_7:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	0

Z_8:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

All of them are congruence- and ideal-simple division semirings.

4.2. Construction. Let P be a parasemifield and let A be a subset of P such that $A + P \subseteq A$, $B + B \subseteq B$ and $1_P + b = 1_P$ for all $b \in B$, where $B = P \setminus A$.

4.2.1. Lemma.

(i) *If $B \neq \emptyset$, then B is a subsemiring of P.*
(ii) *If $b \in B$ and $b \neq 1_P$, then $b^{-1} \in A$.*
(iii) *A is non-empty.*
(iv) *If $1_P \in B$, then P is additively idempotent.*

Proof. (i) Let $b, c \in B$. We have $bc + b + c = b(c + 1_P) + c = b + c \in B$ and hence $bc \notin A$.

(ii) If $b^{-1} \in B$ then $b^{-1} = b^{-1}(1_P + b) = b^{-1} + 1_P = 1_P$, so that $b = 1_P$, a contradiction.

(iii) follows from (ii) and (iv) is evident. □

Let $w \notin P$ and $S = P \cup \{w\}$. Define addition and multiplication on S (extending the operations on P) by $w = 1_S$ (multiplicatively neutral in S), $w + a = a + w = 1_P + a$ for every $a \in A$ and $w + b = b + w = w$ for every $b \in B$. It remains to define the element $w + w = 2w$. We have two options.

(1) Assume that P is additively idempotent and put $2w = w$. In this case, S will be denoted by $Z(P, A, 1)$. It is easy to check that $S = Z(P, A, 1)$ is an additively idempotent division semiring, (S, w) is of type (II), P is a subparasemifield of S and P is an ideal of S.

(2) With P arbitrary, put $2w = 1_P + 1_P$. In this case, S will be denoted by $Z(P, A, 2)$. It is easy to check that $S = Z(P, A, 2)$ is a division semiring, (S, w) is of type (II), P is a subparasemifield of S, P is an ideal of S and S is not additively idempotent.

4.2.2. Lemma. *Let $S = Z(P, A, 1)$.*

(i) *S and P are the only ideals of S.*
(ii) *The semiring S is not ideal-simple.*

Proof. It is obvious. □

4.2.3. Lemma. *Let $S = Z(P, A, 1)$.*

(i) *The equivalence $\rho = id_S \cup \{(w, 1_P), (1_P, w)\}$ is a congruence of the semiring S and $S/\rho \simeq P$.*
(ii) *The semiring S is not congruence-simple.*

Proof. It is easy. □

4.2.4. Lemma. *Let $S = Z(P, A, 1)$ and let r be a congruence of the semiring S such that $r \restriction P = id_P$. Then either $r = id_S$ or $r = \rho$ (see 4.2.3).*

Proof. If $r \neq \mathrm{id}_S$, then $(w, e) \in r$ for some $e \in P$. Now, $(c, ce) = (cw, ce) \in r$ for every $c \in P$, and hence $c = ce$ and $e = 1_P$. Thus $r = \rho$. □

4.2.5. Lemma. *Let $S = Z(P, A, 1)$ where $B \neq \emptyset$ and r be a congruence of the semiring S such that $P \times P \subseteq r$. Then $r = S \times S$.*

Proof. There are $a \in A$ and $b \in B$ with $(a, b) \in r$. Then $(1_P + a, w) = (a + w, b + w) \in r$ and $r = S \times S$. □

4.2.6. Lemma. *Let $S = Z(P, A, 1)$ where $B = \emptyset$.*

(i) *$\eta = (P \times P) \cup \{(w, w)\}$ is a congruence of S and $S/\eta \simeq Z_6$.*
(ii) *If r is a congruence of S with $P \times P \subseteq r$, then either $r = \eta$ or $r = S \times S$.*

Proof. It is easy. □

4.2.7. Proposition. *Let $S = Z(P, A, 1)$ and assume that the parasemifield P is congruence-simple. Then the semiring S is subdirectly irreducible and:*

(i) *If $B \neq \emptyset$, then id_S, ρ and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq S \times S$ and $S/\rho \simeq P$.*

(ii) *If $B = \emptyset$, then id_S, ρ, η and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq \eta \subseteq S \times S$, $S/\rho \simeq P$ and $S/\eta \simeq Z_6$.*

Proof. Combine the previous four lemmas. □

4.2.8. Proposition. *Let $S = Z(P, A, 1)$.*

(i) *The semiring S is finitely generated if and only if P is finitely generated (as a semiring).*
(ii) *S is not one-generated.*

Proof. It is easy. □

4.2.9. Lemma. *Let $S = Z(P, A, 2)$.*

(i) *S and P are the only ideals of S.*
(ii) *The semiring S is not ideal-simple.*

Proof. It is obvious. □

4.2.10. Lemma. *Let $S = Z(P, A, 2)$.*

(i) *The equivalence $\rho = id_S \cup \{(w, 1_P), (1_P, w)\}$ is a congruence of the semiring S and $S/\rho \simeq P$.*
(ii) *The semiring S is not congruence-simple.*

Proof. It is easy. □

4.2.11. Proposition. *Let $S = Z(P, A, 2)$ and assume that the parasemifield P is congruence-simple. Then the semiring S is subdirectly irreducible and:*

(i) *If $B \neq \emptyset$, then id_S, ρ and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq S \times S$ and $S/\rho \simeq P$.*
(ii) *If $B = \emptyset$, then id_S, ρ, η and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq \eta \subseteq S \times S$, $S/\rho \simeq P$ and $S/\eta \simeq Z_2$.*

Proof. It is similar to the proof of 4.2.7. □

4.2.12. Proposition. *Let $S = Z(P, A, 2)$ and let R be the subsemiring of S generated by the element w.*

(i) *If $1_P \in A$, then $R = \{w, 2_P, 3_P, 4_P, \dots\}$.*
(ii) *If P is additively idempotent (e.g., $1_P \in B$), then $R = \{w, 1_P\}$.*
(iii) *If P is not additively idempotent, then $1_P \notin R$.*

Proof. (i) and (ii) are easy. In order to prove (iii), it is sufficient to prove that for any $n \geq 2$, the element n_P (the sum of n copies of 1_P) is different from 1_P. This is clear for $n = 2$. Let $n \geq 3$ and suppose that $n_P = 1_P$. Then $n_P + (n-2)_P = 1_P + (n-2)_P$, i.e., $a + a = a$ where $a = 1_P + (n-2)_P$. We see that P contains an additively idempotent element. But then all elements of P are additively idempotent, a contradiction. □

4.2.13. Proposition. *Let $S = Z(P, A, 2)$.*

(i) *The semiring S is finitely generated if and only if P is finitely generated* (*as a semiring*).
(ii) *S is not one-generated.*

Proof. It is easy. □

4.2.14. Lemma. *Let $S = Z(P, A, 2)$. The semiring S is almost additively idempotent if and only if P is additively idempotent.*

Proof. It is obvious. □

4.3. Construction. Let P be a parasemifield, $e \in P$, and let A be a subset of P such that $A + P \subseteq A$, $B + B \subseteq B$ and $e + b = e$ for all $b \in B$, where $B = P \setminus A$.

4.3.1. Lemma.

(i) *$BB \subseteq Be$.*
(ii) *If $b \in B$ and $b \neq e$, then $b^{-1}e^2 \in A$.*
(iii) *A is non-empty.*
(iv) *If $e \in B$, then P is additively idempotent.*

Proof. (i) Let $b, c \in B$. We have $bc = ae$ for some $a \in P$. Suppose that $a \in A$. Then $(b + c)e = b(c + e) + ce = ae + be + ce = (a + b + c)e$, so that $b + c = a + b + c \in A \cap B$, a contradiction. Thus $a \in B$.

(ii) If $b^{-1}e^2 \in B$ then $b^{-1}e^2 = b^{-1}e(e + b) = b^{-1}e^2 + e = e$, so that $b = e$.

(iii) follows from (ii) and (iv) is evident. □

Let $w \notin P$ and $S = P \cup \{w\}$. Define addition and multiplication on S (extending the operations on P) by $w^2 = e^2$, $wc = cw = ec$ for every $c \in P$, $w + a = a + w = e + a$ for every $a \in A$ and $w + b = b + w = w$ for every $b \in B$. It remains to define the element $2w$. We have two options.

(1) Assume that P is additively idempotent and put $2w = w$. In this case, S will be denoted by $Z(P, A, e, 1)$. It is easy to check that $S = Z(P, A, e, 1)$ is an additively idempotent division semiring, (S, w) is of type (III), P is a subparasemifield of S and P is an ideal of S.

(2) With P arbitrary, put $2w = 2e$. In this case, S will be denoted by $Z(P, A, e, 2)$. It is easy to check that $S = Z(P, A, e, 2)$ is a division semiring, (S, w) is of type (III), P is a subparasemifield of S, P is an ideal of S and S is not additively idempotent.

4.3.2. Lemma. *Let $S = Z(P, A, e, 1)$.*

(i) *S and P are the only ideals of S.*
(ii) *The semiring S is not ideal-simple.*

Proof. It is obvious. □

4.3.3. Lemma. *Let $S = Z(P, A, e, 1)$.*

(i) *The equivalence $\rho = id_S \cup \{(w, e), (e, w)\}$ is a congruence of the semiring S and $S/\rho \simeq P$.*
(ii) *The semiring S is not congruence-simple.*

Proof. It is easy. □

4.3.4. Proposition. *Let $S = Z(P, A, e, 1)$ and assume that the parasemifield P is congruence-simple. Then the semiring S is subdirectly irreducible and:*

(i) *If $B \neq \emptyset$, then id_S, ρ and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq S \times S$ and $S/\rho \simeq P$.*
(ii) *If $B = \emptyset$, then id_S, ρ, $\eta = (P \times P) \cup \{(w, w)\}$ and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq \eta \subseteq S \times S$, $S/\rho \simeq P$ and $S/\eta \simeq Z_3$.*

Proof. It is similar to that of 4.2.7 or 4.2.11. □

4.3.5. Lemma. *Let $S = Z(P, A, e, 1)$; denote by R the subsemiring of S generated by the element w and by R_1 the subsemiring of P generated by e. Then $R \subseteq R_1 \cup \{w\}$.*

Proof. It is easy. □

4.3.6. Proposition. *Let $S = Z(P, A, e, 1)$.*

(i) *The semiring S is finitely generated if and only if P is finitely generated (as a semiring).*
(ii) *S is not one-generated.*

Proof. (i) is easy and (ii) follows from 1.4. □

4.3.7. Lemma. *Let $S = Z(P, A, e, 2)$.*

(i) *S and P are the only ideals of S.*
(ii) *The semiring S is not ideal-simple.*

Proof. It is obvious. □

4.3.8. Lemma. *Let $S = Z(P, A, e, 2)$.*

(i) *The equivalence $\rho = id_S \cup \{(w, e), (e, w)\}$ is a congruence of the semiring S and $S/\rho \simeq P$.*
(ii) *The semiring S is not congruence-simple.*

Proof. It is easy. □

4.3.9. Proposition. *Let $S = Z(P, A, e, 2)$ and assume that the parasemifield P is congruence-simple. Then the semiring S is subdirectly irreducible and:*

(i) *If $B \neq \emptyset$, then id_S, ρ and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq S \times S$ and $S/\rho \simeq P$.*
(ii) *If $B = \emptyset$, then id_S, ρ, $\eta = (P \times P) \cup \{(w, w)\}$ and $S \times S$ are the only congruences of S; we have $id_S \subseteq \rho \subseteq \eta \subseteq S \times S$, $S/\rho \simeq P$ and $S/\eta \simeq Z_1$.*

Proof. It is similar to that of 4.3.4. □

4.3.10. Lemma. *Let $S = Z(P, A, e, 2)$; denote by R the subsemiring of S generated by the element w and by R_1 the subsemiring of P generated by e. Then $R \subseteq R_1 \cup \{w\}$.*

Proof. It is easy. □

4.3.11. Proposition. *Let $S = Z(P, A, e, 2)$.*

(i) *The semiring S is finitely generated if and only if P is finitely generated (as a semiring).*
(ii) *S is not one-generated.*

Proof. (i) is easy and (ii) follows from 1.4. □

4.3.12. Lemma. *Let $S = Z(P, A, e, 2)$. The semiring S is almost additively idempotent if and only if P is additively idempotent.*

Proof. It is obvious. □

4.4. Construction. Let P be a parasemifield, $0 \notin P$, and put $S = P \cup \{0\}$. Define addition and multiplication on S (extending the operations on P) by $v0 = 0v = 0$ and $v + 0 = 0 + v = v$ for all $v \in S$ (so that 0 is additively neutral and multiplicatively absorbing in S). We denote S constructed in this way by $Z(P, 0)$. One can easily check that $S = Z(P, 0)$ is an ideal-simple division semiring, $(S, 0)$ is of type (IV) and P is a subparasemifield of S. Of course, S is a semifield and $1_P = 1_S$.

4.4.1. Lemma. *Let $S = Z(P, 0)$.*

(i) *If r is a congruence of P, then $r \cup \{(0, 0)\}$ is a congruence of S.*
(ii) *$\eta = (P \times P) \cup \{(0, 0)\}$ is a congruence of S and $S/\eta \simeq Z_5$.*

Proof. It is obvious. □

4.4.2. Lemma. *$Z(P, 0)$ is not congruence-simple.*

Proof. It follows from 4.4.1. □

4.4.3. Proposition. *Let $S = Z(P, 0)$ and assume that P is congruence-simple. Then S is subdirectly irreducible and id_S, η, $S \times S$ are the only congruences of S.*

Proof. It is easy. □

4.4.4. Lemma. *$Z(P, 0)$ is additively idempotent if and only P is additively idempotent.*

Proof. It is obvious. □

4.4.5. Proposition. *Let $S = Z(P, 0)$.*

(i) *S is finitely generated if and only if P is finitely generated.*
(ii) *S is neither one- nor two-generated.*

Proof. (i) is easy and (ii) follows from 1.4. □

4.5. Construction. Let G be a commutative group (denoted multiplicatively), $o \notin G$ and $S = G \cup \{o\}$. Define addition and multiplication on S (extending the multiplication on G) by $xo = ox = x$ and $x + y = o$ for all $x, y \in S$. We denote S constructed in this way by $U(G)$. One can easily check that $S = U(G)$ is a division semiring and (S, o) is of type (IV); if $|G| = 1$ then $S \simeq Z_2$. Of course, S is a semifield.

4.5.1. Lemma. *Let $S = U(G)$.*

(i) $\eta = (G \times G) \cup \{(o, o)\}$ *is a congruence of S and $S/\eta \simeq Z_2$.*
(ii) *If $|G| \geq 2$, then S is not congruence-simple.*

Proof. It is easy. □

4.5.2. Proposition. *Let $S = U(G)$ where G is a finite group of prime order. Then id_S, η and $S \times S$ are the only congruences of S.*

Proof. It is easy. □

4.5.3. Proposition. *Let $S = U(G)$.*

(i) *S is finitely generated if and only if the group G is finitely generated.*
(ii) *S is one-generated if and only if G is a finite cyclic group.*

Proof. It is easy. □

4.5.4. Lemma. *let $S = U(G)$.*

(i) *S is not additively idempotent.*
(ii) *S is almost additively idempotent if and only if $|G| = 1$. (Then $S \simeq Z_2$.)*

Proof. It is obvious. □

4.6. Construction. Let G be a commutative group (denoted multiplicatively), $o \notin G$ and $S = G \cup \{o\}$. Define addition and multiplication on S (extending the multiplication on G) by $xo = ox = o$, $x+y = o$ and $x+x = x$ for all $x, y \in S$ with $x \neq y$. We denote S constructed in this way by $V(G)$. One can easily check that $S = V(G)$ is an additively idempotent division semiring and (S, o) is of type (IV); if $|G| = 1$ then $S \simeq Z_6$. Of course, S is a semifield.

4.6.1. Proposition. *$V(G)$ is congruence-simple.*

Proof. It is easy. □

4.6.2. Proposition. *Let $S = V(G)$.*

(i) *S is finitely generated if and only if the group G is finitely generated.*
(ii) *S is one-generated if and only if G is a non-trivial finite cyclic group.*

Proof. It is easy. □

4.7. Construction. Let P be a parasemifield, $o \notin P$, and put $S = P \cup \{o\}$. Define addition and multiplication on S (extending the operations on P) by $vo = ov = v + o = o + v = o$ for all $v \in S$ (so that o is a bi-absorbing element). We denote S constructed in this way by $U(P)$. One can easily check that $S = U(P)$ is a division semiring, (S, o) is of type (IV) and S is a semifield.

4.7.1. Lemma. *Let $S = U(P)$.*

(i) *If r is a congruence of P, then $r \cup \{(o, o)\}$ is a congruence of S.*
(ii) *$\eta = (P \times P) \cup \{(o, o)\}$ is a congruence of S and $S/\eta \simeq Z_6$.*

Proof. It is easy. □

4.7.2. Lemma. *$U(P)$ is not congruence-simple.*

Proof. It follows from 4.7.1. □

4.7.3. Proposition. *Let $S = U(P)$ and assume that P is congruence-simple. Then S is subdirectly irreducible and id_S, η, $S \times S$ are the only congruences of S.*

Proof. It is easy. □

4.7.4. Lemma. *$U(P)$ is additively idempotent if and only P is additively idempotent.*

Proof. It is obvious. □

4.7.5. Proposition. *Let $S = U(P)$.*

(i) *S is finitely generated if and only if P is finitely generated.*
(ii) *S is neither one- nor two-generated.*

Proof. (i) is easy and (ii) follows from 1.4. □

4.8. Construction. Let P be a parasemifield and let $T(\cdot)$ be a commutative group such that $P(\cdot)$ is a proper subgroup of $T(\cdot)$. Let $o \notin T$ and $S = T \cup \{o\}$. Define addition and multiplication on S (extending the operations on P and the multiplication on T) by

$vo = ov = v + o = o + v = o$ for every $v \in S$,
$a + b = o$ for all $a, b \in T$ such that $a^{-1}b \notin P$,
$a + b = (1_T + a^{-1}b)a \in T$ for all $a, b \in T$ such that $a^{-1}b \in P$.

Observe that if $a^{-1}b \in P$ then $b^{-1}a \in P$, $(1_T + a^{-1}b)b^{-1}a = b^{-1}a + 1_T$ and $a + b = (1_T + a^{-1}b)a = (1_T + b^{-1}a)b$. It is easy to check that S is a divisible semiring. It will be denoted by $V(P, T(\cdot))$. Clearly, (S, o) is of type (IV), S is a semifield and P is a subparasemifield of S.

4.8.1. Lemma. *Let $S = V(P, T(\cdot))$. Define a relation σ on S by $(x, y) \in \sigma$ if and only if either $x = y$ or else $x, y \in T$ and $x^{-1}y \in P$. Then σ is a congruence of the semiring S and $S/\sigma \simeq V(T(\cdot)/P)$.*

Proof. It is easy. □

4.8.2. Lemma. *The semiring $V(P,T(\cdot))$ is not congruence-simple.*

Proof. Use 4.8.1. □

4.8.3. Remark. Let $S = V(P,T(\cdot))$.

(i) For every congruence α of the parasemifield P we can construct a congruence $\beta = \beta(\alpha)$ such that $\alpha = \beta \cap (P \times P)$ as follows. Put $R = \{a \in P : (a, 1_S) \in \alpha\}$, so that R is a subgroup of $P(\cdot)$. Now, put $\beta = \alpha_1 \cup \{(o,o)\}$ where $\alpha_1 = \{(a,b) : a,b \in T, \quad a^{-1}b \in R\}$. Clearly, β is a congruence of the multiplicative semigroup $S(\cdot)$. If $(a,b) \in \beta$ where $a,b \in T$ and $c \in T$ is an element such that $a^{-1}c \notin P$, then $b^{-1}c \notin P$, since $a^{-1}b \in P$, and we have $(a+c, b+c) = (o,o) \in \beta$. If $b^{-1}c \notin P$, the proof is by symmetry. Finally, if $a^{-1}c \in P$ and $b^{-1}c \in P$ then $(c^{-1}a, c^{-1}b) \in \alpha$, $(1_S + c^{-1}a, 1_S + c^{-1}b) \in \alpha$ and $(a+c, b+c) = ((1_S + c^{-1}a)c, (1_S + c^{-1}b)c) \in \beta$. It follows that β is a congruence of the semiring S. Clearly, $\alpha = \beta \cap (P \times P)$.

(ii) Let us prove that every congruence β of S other than $S \times S$ can be obtained as $\beta(\alpha)$ for some congruence α of P. Clearly, $\beta = \beta_1 \cup \{(o,o)\}$ where $\beta_1 = \beta \cap (T \times T)$ is a congruence of the group $T(\cdot)$. Put $\alpha = \beta \cap (P \times P)$. Clearly, α is a congruence of the parasemifield P. Put $R = \{a \in T : (a, 1_S) \in \beta\}$. Then R is a subgroup pf $T(\cdot)$ and, if $a \in R \setminus P$, then $a + 1_S = o$ from which we get $(o, 1_S + 1_S) \in \beta$, a contradiction. Thus $R \subseteq P$ and consequently $\beta = \beta(\alpha)$.

(iii) It follows that the congruence lattice of S is isomorphic to the congruence lattice of P with a new top element added. In particular, S is subdirectly irreducible if and only if P is. If P is congruence-simple, then id_S, σ (see 4.8.1) and $S \times S$ are the only congruences of the semiring S.

4.8.4. Lemma. *The semiring $V(P,T(\cdot))$ is additively idempotent if and only if P is additively idempotent.*

Proof. It is easy. □

4.8.5. Lemma. *Let $S = V(P,T(\cdot))$ and let M be a generating subset of the semiring S. Then the set $N = M \cap T$ is non-empty and generates S, as well.*

Proof. N is non-empty, since $S \neq \{o\}$. Denote by S_1 the subsemiring of S generated by N. If $o \notin S_1$, then $S_1 = T$ and $o = 1_S + a \in S_1$ for some $a \in T \setminus P$, a contradiction. Thus $o \in S_1$ and $S_1 = S$. □

4.8.6. Lemma. *Let $S = V(P,T(\cdot))$ and let $N \subseteq T$ be a generating subset of S. Then the factor-group $T(\cdot)/P$ is generated by the set $\{aP : a \in N\}$ of cosets as a semigroup.*

Proof. Let $b \in T$. Then $b = b_1 + \dots + b_n$ for some elements $b_1, \dots, b_n$ ($n \geq 1$) belonging to the subsemigroup A of $T(\cdot)$ generated by N. For every $i = 1, \dots, n$ we have $b_i = b_1 c_i$ for some $c_i \in P$, and so $b = b_1 c$ where $c = c_1 + \dots + c_n \in P$. Then $bP = b_1 P$ and the rest is clear. □

4.8.7. Lemma. *Let $S = V(P, T(\cdot))$ and let N be a subset of T such that the factor-group $T(\cdot)/P$ is generated by $\{aP : a \in N\}$ as a semigroup. If A is the subsemigroup of $T(\cdot)$ generated by N, then $T = AP$.*

Proof. It is easy. □

4.8.8. Lemma. *Let $S = V(P, T(\cdot))$ and let $N \subseteq T$ be a generating subset of S. Denote by A the subsemigroup of $T(\cdot)$ generated by N. Then:*

(i) *$B = AA^{-1}$ is a subgroup of $T(\cdot)$ and B is generated by $N \cup N^{-1}$ as a semigroup.*

(ii) *P is generated by the subgroup $C = B \cap P$ of B as a semiring.*

Proof. (i) is obvious.

(ii) Let $a \in P$. We have $a = a_1 + \cdots + a_n$ for some $n \geq 1$ and elements $a_i \in A$. For every i we have $a_i = b_i a_1$ for some $b_i \in C$, so that $a = ba_1$ where $b = b_1 + \cdots + b_n$. Of course, $a, b \in P$, and so $a_1 = ab^{-1} \in A \cap P = C$. Consequently, the elements $a_i = b_i a_1$ belong to C. □

4.8.9. Proposition. *$S = V(P, T(\cdot))$ is a finitely generated semiring if and only if P is a finitely generated semiring and $T(\cdot)/P$ is a finitely generated group.*

Proof. The direct implication follows from 4.8.5, 4.8.6 and 4.8.8, taking into account the following two well-known facts: any subgroup of a finitely generated commutative group is finitely generated; if a commutative group is finitely generated, then it is finitely generated as a semigroup. The converse follows from 4.8.7. □

4.8.10. Proposition. *$V(P, T(\cdot))$ is not a one-generated semiring.*

Proof. Put $S = V(P, T(\cdot))$ and suppose that S is generated by a single element s. Clearly, $s \in T$ and $s \notin P$. According to 4.8.6, the factor-group $T(\cdot)/P$ is a (non-trivial) finite cyclic group, and so $T(\cdot)/P \simeq \mathbf{Z}_m(+)$ for some $m \geq 2$. It follows that $a^m \in P$ for every $a \in T$.

Take $a \in P$. We have $a = l_1 s^{k_1} + \cdots + l_n s^{k_n}$ for some $n \geq 1$, $l_i \geq 1$, $1 \leq k_1 < k_2 < \cdots < k_n$. Since $s^{k_1} + s^{k_i} \neq o$, $s^{k_i - k_1} \in P$ and m divides $k_i - k_1$. Furthermore, $as^{-k_1} = l_1 1_S + l_2 s^{k_2 - k_1} + \cdots + l_n s^{k_n - k_1} \in P$, $s^{k_1} \in P$ and m divides k_1. Consequently, m divides all the numbers $k_1, \ldots, k_n$ and we conclude that the semiring P is generated by the element s^m, a contradiction with 1.4. □

4.9. Construction. Let A be a subsemigroup of the additive group $\mathbf{R}(+)$ of real numbers such that $A \cap \mathbf{R}^+ \neq \emptyset \neq A \cap \mathbf{R}^-$. Define operations $\oplus$ and $\odot$ on A by $a \oplus b = \min(a, b)$ and $a \odot b = a + b$. It is easy to check that with respect to these operations, the set A becomes an additively idempotent semiring. This semiring will be denoted by $W(A)$. According to Lemma 5.1.1 of [1], $W(A)$ is congruence-simple.

4.9.1. Lemma. *The following conditions are equivalent:*

(i) *$W(A)$ is ideal-simple;*
(ii) *$W(A)$ is a division semiring;*
(iii) *$W(A)$ is a parasemifield;*
(iv) *A is a subgroup of $\mathbf{R}(+)$.*

Proof. It is easy. □

4.9.2. Lemma.
(i) *$W(A)$ is a finitely generated semiring if and only if A is a finitely generated semigroup.*
(ii) *$W(A)$ is not one-generated.*

Proof. It is easy. □

5. Division semirings of type (II)

In this section let S be a division semiring that is of type (II) with respect to an element w. That is, $w = 1_S \in S$ and $T = S \setminus \{1_S\}$ is a subgroup of the multiplicative semigroup $S(\cdot)$.

5.1. Lemma. *If $|T| = 1$, then S is isomorphic to one of the semirings Z_2, Z_5, Z_6, Z_8.*

Proof. See 4.1. □

5.2. Lemma. *If $|T| \geq 2$, then T is a subparasemifield of S.*

Proof. $T(\cdot)$ is a non-trivial group. If $a, b \in T$ are such that $a + b = 1_S$, then $1_S = a + b = a1_T + b1_T = (a + b)1_T = 1_S 1_T = 1_T$, a contradiction. Thus $T + T \subseteq T$ and T is a parasemifield. □

5.3. Lemma. *If $a \in T$ is such that $1_S + a \in T$, then $1_S + a = 1_T + a$.*

Proof. $1_S + a = (1_S + a)1_T = 1_S 1_T + a1_T = 1_T + a$. □

5.4. Lemma. *If $a \in T$ is such that $1_S + a = 1_S$, then $1_T + a = 1_T$.*

Proof. We have $1_T = 1_T 1_S = 1_T(1_S + a) = 1_T 1_S + 1_T a = 1_T + a$. □

5.5. Lemma.
(i) *If $1_S + 1_S = 1_S$, then S is additively idempotent.*
(ii) *If $1_S + 1_S \in T$, then $1_S + 1_S = 1_T + 1_T$.*

Proof. (i) is obvious. If $1_S + 1_S = a \in T$, then $ab = b + b$ for every $b \in T$. In particular, $a = a1_T = 1_T + 1_T$. □

Put $A = \{a \in T : 1_S + a = 1_T + a\}$ and $B = \{b \in T : 1_S + b = 1_S\}$.

5.6. Lemma.
(i) *$A \cup B = T$ and $A \cap B = \emptyset$.*
(ii) *$A + T \subseteq A$.*
(iii) *$B + B \subseteq B$.*
(iv) *$1_T + b = 1_T$ for every $b \in B$.*

Proof. Use 5.3 and 5.4. □

5.7. Proposition. *Precisely one of the following four cases takes place:*

(1) *S is isomorphic to either Z_5 or Z_6 and is additively idempotent;*
(2) *S is isomorphic to either Z_2 or Z_8 and S is not additively idempotent (but is almost additively idempotent);*
(3) *T is a subparasemifield of S, $S \simeq Z(T, A, 1)$ and S is additively idempotent;*
(4) *T is a subparasemifield of S, $S \simeq Z(T, A, 2)$ and S is not additively idempotent; it is almost additively idempotent if and only it T is idempotent.*

Proof. Combine 5.5, 5.6 and 4.2. □

5.8. Corollary. *The following conditions are equivalent:*

(i) *S is congruence-simple;*
(ii) *S is ideal-simple;*
(iii) $|S| = 2$.

5.9. Corollary. *S is a one-generated semiring if and only if it is isomorphic to either Z_2 or Z_8.*

6. Division semirings of type (III)

In this section let S be a division semiring that is of type (III) with respect to an element w. That is, $w \in S$, $T = S \setminus \{w\}$ is a subgroup of the multiplicative semigroup $S(\cdot)$, $w1_T = e \in T$, $w^2 = e^2$ and $wa = ea$ for every $a \in T$.

6.1. Lemma. *If $|T| = 1$, then S is isomorphic to one of the semirings Z_1, Z_3, Z_4, Z_7.*

Proof. See 4.1. □

6.2. Lemma. *If $|T| \geq 2$, then T is a subparasemifield of S.*

Proof. It remains to show that $T+T \subseteq T$. If $a, b \in T$ are such that $a+b = w$, then $w = a + b = a1_T + b1_T = (a + b)1_T = w1_T = e$, a contradiction. □

6.3. Lemma. *If $a \in T$ is such that $w + a \in T$, then $w + a = e + a$.*

Proof. $w + a = (w + a)1_T = w1_T + a1_T = e + a$. □

6.4. Lemma. *If $b \in T$ is such that $w + b = w$, then $e + b = e$.*

Proof. We have $e = w1_T = (w + b)1_T = w1_T + b1_T = e + b$. □

6.5. Lemma. *If $w + w \in T$, then $w + w = e + e$.*

Proof. We have $w + w = (w + w)1_T = w1_T + w1_T = e + e$. □

6.6. Lemma. *If $w + w = w$, then S is additively idempotent.*

Proof. We have $e = w1_T = (w+w)1_T = w1_T + w1_T = e+e$. Consequently, $a = a+a$ for all $a \in T$. □

Put $A = \{a \in T : w+a = e+a\}$ and $B = \{b \in T : w+b = w\}$.

6.7. Lemma.

(i) $A \cup B = T$ *and* $A \cap B = \emptyset$.
(ii) $A + T \subseteq A$.
(iii) $B + B \subseteq B$.
(iv) $e + b = e$ *for every* $b \in B$.

Proof. Use 6.3 and 6.4. □

6.8. Proposition. *Precisely one of the following four cases takes place:*

(1) *S is isomorphic to either Z_3 or Z_4 and is additively idempotent;*
(2) *S is isomorphic to either Z_1 or Z_7 and S is not additively idempotent (but is almost additively idempotent);*
(3) *T is a subparasemifield of S, $S \simeq Z(T, A, e, 1)$ and S is additively idempotent;*
(4) *T is a subparasemifield of S, $S \simeq Z(T, A, e, 2)$ and S is not additively idempotent; it is almost additively idempotent if and only it T is idempotent.*

Proof. Combine 6.4, 6.5, 6.7 and 4.3. □

6.9. Corollary. *The following conditions are equivalent:*

(i) *S is congruence-simple;*
(ii) *S is ideal-simple;*
(iii) $|S| = 2$.

6.10. Corollary. *S is a one-generated semiring if and only if it is isomorphic to one of Z_1, Z_3, Z_4, Z_7.*

7. Division semirings of type (IV)

In this section let S be a division semiring that is of type (IV) with respect to an element w. That is, w is a multiplicatively absorbing element and $T = S \setminus \{w\}$ is a subgroup of the multiplicative semigroup $S(\cdot)$. Thus S is a semifield and S is ideal-simple.

7.1. Lemma. *If $|T| = 1$, then S is isomorphic to one of Z_2, Z_5, Z_6, Z_8.*

Proof. See 4.1. □

7.2. Lemma. $1_T = 1_S$ *is multiplicatively neutral in S.*

Proof. It is obvious. □

7.3. Lemma. *Either $w = o_S$ is additively absorbing in S or $w = 0_S$ is additively neutral in S.*

Proof. We have $w + w = (1_S + 1_S)w = w$. If $1_S + w = w$, then $w = aw = a(1_S + w) = a1_S + aw = a + w$ for every $a \in T$, and so $w = o_S$. On the other hand, if $1_S + w \neq w$, then $1_S = (1_S + w)^{-1}(1_S + w) = (1_S + w)^{-1} + w$. From this, $a = a(1_S + w)^{-1} + aw = a(1_S + w)^{-1} + w$ and $a + w = a(1_S + w)^{-1} + w + w = a(1_S + w)^{-1} + w = a$ for every $a \in T$. Thus $w = 0_S$. □

7.4. Lemma. *If $w = 0_S$, then either S is a field, or $S \simeq Z_5$, or T is a subparasefimield of S and $S \simeq Z(T, 0)$.*

Proof. If $|S| = 2$, then S is isomorphic either to Z_5 or to the two-element field Z_8. Let $|S| \geq 3$. Consider first the case when $a + b = 0_S$ for some $a, b \in T$. Then $c + ca^{-1}b = ca^{-1}(a + b) = ca^{-1}0_S = 0_S$ for every $c \in T$ and it follows that $S(+)$ is a group. Then, obviously, S is a field. The remaining case is when $T + T \subseteq T$. Then, clearly, T is a subparasemifield and $S \simeq Z(T, 0)$. □

In the next seven lemmas assume that $|T| \geq 2$ and $w = o_S = o$ is bi-absorbing.

7.5. Lemma. *If $T + a = \{o\}$ for at least one $a \in T$, then $S + S = \{o\}$ and $S \simeq U(T(\cdot))$.*

Proof. We have $T + ab = (T + a)b = \{o\}$ for every $b \in T$. Thus $T + T = \{o\}$ and $S + S = \{o\}$. The rest is clear. □

Now, assume that $T + a \neq \{o\}$ for every $a \in T$. Put $A_x = \{y \in S : x + y = o\}$ for every $x \in S$.

7.6. Lemma.

(i) *$o \in A_x$ and $S + A_x \subseteq A_x$.*
(ii) *$A_x \subseteq A_{x+y}$ for all $x, y \in S$.*
(iii) *$A_o = S$.*
(iv) *$A_a \neq S$ for every $a \in T$.*
(v) *$aA_b = bA_a$ for all $a, b \in T$.*
(vi) *$A_b = a^{-1}bA_a$ for all $a, b \in T$.*
(vii) *$A_a = aA_{1_S}$ for every $a \in T$.*

Proof. It is easy. □

7.7. Lemma.

(i) *$P + P \subseteq P$ and $PP \subseteq P$ (i.e., P is a subsemiring of S).*
(ii) *$P(\cdot)$ is a subgroup of $S(\cdot)$.*
(iii) *If $a, b \in T$, then $a + b \neq o$ if and only if $a^{-1}b \in P$.*

Proof. (i) If $a, b \in P$, then $1_S + a \neq o$ and $1_S + b \neq o$. Consequently, $1_S + a + b + ab = (1_S + a)(1_S + b) \neq o$. But then $1_S + a + b \neq o$, $1_S + ab \neq o$ and it follows that $a + b \in P$ and $ab \in P$.

(ii) If $1_S + a \neq o$, then $a^{-1} + 1_S \neq o$.

(iii) We have $a + b \neq o$ if and only if $1_S + a^{-1}b \neq o$. □

7.8. Remark. We have $A_{1_S} = T \setminus P$ and P is a subgroup of $T(\cdot)$. Now it is easy to see that $1_S \notin A_{1_S}$ and $P = \{a \in T : A_a = aA_{1_S} = A_{1_S}$.

7.9. Lemma. *Let $a, b \in T$ be such that $a+b \neq o$ (equivalently, $a^{-1}b \in P$). Then $1_S+a^{-1}b \in P$, $1_S+b^{-1}a \in P$ and $a+b = a(1_S+a^{-1}b) = b(1_S+b^{-1}a)$.*

Proof. It is easy (use 7.7). □

7.10. Lemma. *If $|P| = 1$, then $P = \{1_S\}$ and $S \simeq V(T(\cdot))$.*

Proof. Combine 7.7 and 7.9. □

7.11. Lemma. *If $P = T$, then $S \simeq U(P)$.*

Proof. It is easy. □

7.12. Proposition. *Let S be a division semiring of type (IV) with respect to w and $T = S \setminus \{w\}$. Then one of the following cases takes place:*

(1) *S is a field or S is isomorphic to one of Z_2, Z_5, Z_6;*
(2) *T is a subparasemifield of S and $S \simeq Z(T, 0)$ (then S is additively idempotent if and only if T is);*
(3) *$|T| \geq 2$ and $S \simeq U(T(\cdot))$ (then S is not additively idempotent);*
(4) *$|T| \geq 2$, $w = o_S$ is bi-absorbing, $1_S + a = o_S$ for every $a \in T \setminus \{1_S\}$ and $S \simeq V(T(\cdot))$ (then S is additively idempotent);*
(5) *$w = o_S$ is bi-absorbing, T is a subparasemifield of S and $S \simeq U(T)$ (then S is additively idempotent if and only if T is);*
(6) *$w = o_S$ is bi-absorbing, $P = \{a \in T : 1_S + a \neq o_S\}$ is a subparasemifield of S, $P \neq T$, and $S \simeq V(P, T(\cdot))$ (then S is additively idempotent if and only if P is).*

Proof. Combine 7.1, 7.3, 7.4, 7.5, 7.10 and 7.11. □

7.13. Corollary. *S is congruence-simple if and only if either S is a field or $|S| = 2$ or $S \simeq V(G(\cdot))$ for a commutative group $G(\cdot)$.*

7.14. Corollary. *S is one-generated if and only if one of the following three cases takes place:*

(1) *$|S| = 2$;*
(2) *S is a finite field;*
(3) *$S \simeq V(G(\cdot))$ for a non-trivial finite cyclic group $G(\cdot)$;*
(4) *$S \simeq U(G(\cdot))$ for a non-trivial finite cyclic group $G(\cdot)$.*

Proof. Easy, using 1.5. □

8. Summary

8.1. Theorem. *Division semirings are commutative semirings of (exactly) one of the following twelve types:*

(1) *The two-element semirings* $Z_1, \dots, Z_7$ (*see* 4.1)*;*
(2) *Fields;*
(3) *Parasemifields;*
(4) *The semifields* $U(G)$*, where* G *is a non-trivial commutative group* (*see* 4.5)*;*
(5) *The semifields* $V(G)$*, where* G *is a non-trivial commutative group* (*see* 4.6)*;*
(6) *The semifields* $U(P)$*, where* P *is a parasemifield* (*see* 4.7)*;*
(7) *The semifields* $Z(P, 0)$*, where* P *is a parasemifield* (*see* 4.4)*;*
(8) *The semifields* $V(P, T(\cdot))$*, where* P *is a parasemifield and the multiplicative group* $P(\cdot)$ *is a proper subgroup of a commutative group* $T(\cdot)$ (*see* 4.8)*;*
(9) *The semirings* $Z(P, A, 1)$*, where* P *is an additively idempotent parasemifield and* A *is a non-empty subset of* P *such that* $A + P \subseteq A$*,* $(P \setminus A) + (P \setminus A) \subseteq P \setminus A$ *and* $1_P + x = 1_P$ *for every* $x \in P \setminus A$ (*see* 4.2)*;*
(10) *The semirings* $Z(P, A, 2)$*, where* P *is a parasemifield and* A *is as in* (9) (*see* 4.2)*;*
(11) *The semirings* $Z(P, A, e, 1)$*, where* P *is an additively idempotent parasemifield,* $e \in P$ *and* A *is a non-empty subset of* P *such that* $A + P \subseteq A$*,* $(P \setminus A) + (P \setminus A) \subseteq P \setminus A$ *and* $e + x = e$ *for every* $x \in P \setminus A$ (*see* 4.3)*;*
(12) *The semirings* $Z(P, A, e, 2)$*, where* P *is a parasemifield,* $e \in P$ *and* A *is as in* (11) (*see* 4.3)*;*

Proof. Combine 5.7, 6.8, 7.12. □

8.2. Remark. The semirings Z_3, Z_4, Z_5, Z_6, $V(G)$, $Z(P, A, 1)$ and $Z(P, A, e, 1)$ are additively idempotent. The semirings $U(P)$, $Z(P, 0)$ and $V(P, T(\cdot))$ are additively idempotent if and only if the parasemifield P is. The semirings $Z(P, A, 2)$ and $Z(P, A, e, 2)$ are almost additively idempotent if and only if P is additively idempotent. The semirings $U(P)$ contain just one additively idempotent element.

8.3. Remark. The semirings $Z_1, \dots, Z_7$ are finite, and hence finitely generated. A field is a finitely generated semiring if and only if it is finite. The semirings $U(G)$ and $V(G)$ are finitely generated if and only if the group G is finitely generated. The semirings $U(P)$, $Z(P, 0)$, $Z(P, A, 1)$, $Z(P, A, 2)$, $Z(P, A, e, 1)$, $Z(P, A, e, 2)$ are finitely generated if and only if the parasemifield P is finitely generated. The semirings $V(P, T(\cdot))$ are finitely generated if and only if P is finitely generated and the factor-group $T(\cdot)/P$ is finitely generated.

8.4. Remark. Taking into account 8.2 and 8.3, we conclude that the following two statements are equivalent:

(A) A parasemifield is additively idempotent, provided that it is a finitely generated semiring.
(B) A finitely generated division semiring is either almost additively idempotent or it is a finite field or a copy of the semifield $U(G)$ for a non-trivial finitely generated commutative group G.

8.5. Theorem. *One-generated division semirings are just (copies of) the two-element semirings Z_1, Z_2, Z_3, Z_4, Z_7, finite fields and the semifields $U(G)$ and $V(G)$, where G is a non-trivial finite cyclic group. In particular, all such semirings are finite.*

Proof. Combine 4.1, 4.2.8, 4.2.13, 4.3.6, 4.3.11, 4.4.5, 4.5.3, 4.6.2, 4.7.5, 4.8.9 and 4.8.10. □

8.6. Remark. Division semirings containing an additively neutral element are just the following ones:

(1) The two-element semirings $Z_3, \dots, Z_7$;
(2) Fields;
(3) The semifields $Z(P, 0)$.

8.7. Remark. Division semirings containing a multiplicatively neutral element are just the following ones:

(1) The two-element semirings Z_2, Z_5, Z_6;
(2) Fields;
(3) The semifields $U(G)$;
(4) The semifields $V(G)$;
(5) The semifields $U(P)$;
(6) The semifields $Z(P, 0)$;
(7) The semifields $V(P, T(\cdot))$;
(8) The semirings $Z(P, A, 1)$;
(9) The semirings $Z(P, A, 2)$.

8.8. Remark. Division semirings containing an additively absorbing element are just the following ones:

(1) The two-element semirings $Z_1, \dots, Z_6$;
(2) The semifields $U(G)$;
(3) The semifields $V(G)$;
(4) The semifields $U(P)$;
(5) The semifields $V(P, T(\cdot))$.

8.9. Remark. Division semirings containing a multiplicatively absorbing element are just the following ones:

(1) The two-element semirings $Z_1, \dots, Z_7$;
(2) Fields;
(3) The semifields $U(G)$;

(4) The semifields $V(G)$;
(5) The semifields $U(P)$;
(6) The semifields $Z(P, 0)$;
(7) The semifields $V(P, T(\cdot))$.

Notice that (except for Z_1, Z_3, Z_4 and Z_7) all these semirings have a multiplicatively neutral element. Furthermore, except for Z_7, fields and the semifields $Z(P, 0)$, the other semirings have an additively absorbing element.

8.10. Remark. All division semirings have at most two ideals. The ideal-simple ones among them are just the following semirings:

(1) The two-element semirings $Z_1, \ldots, Z_7$;
(2) Fields;
(3) Parasemifields (these are ideal-free);
(4) The semifields 8.1(4),...,(8).

8.11. Remark. Congruence-simple division semirings are just the following ones:

(1) The two-element semirings $Z_1, \ldots, Z_7$;
(2) Fields;
(3) Congruence-simple parasemifields (see 8.19);
(4) The semifields $V(G)$, where G is a non-trivial commutative group.

8.12. Remark. Finite division semirings are just the following ones:

(1) The two-element semirings $Z_1, \ldots, Z_7$;
(2) Finite fields;
(3) The semifields $U(G)$, where G is a non-trivial finite commutative group;
(4) The semifields $V(G)$, where G is a non-trivial finite commutative group.

Notice that every finite division semiring is ideal-simple.

8.13. Remark. Let S be a non-trivial semiring that is a division semiring with respect to two different elements of S. According to 2.9, S is either a parasemifield or a two-element semiring isomorphic to one of the semirings Z_2, Z_5, Z_6, Z_8.

8.14. Theorem. *Ideal-simple commutative semirings are just the semirings of one of the following five types:*

(1) *The two-element semirings* $Z_2, \ldots, Z_6$*;*
(2) *Fields;*
(3) *Zero multiplication rings of finite prime order;*
(4) *Parasemifields* (*these are ideal-free*)*;*
(5) *Proper semifields* (*i.e., semifields that are not fields*).

Proof. Let S be an ideal-simple commutative semiring with at least three elements. If S is a ring, then either (2) or (3) takes place. Let S be neither a ring nor a parasemifield. The multiplicative semigroup $S(\cdot)$ is not a group, and hence it is not a division semigroup. Consequently, the set $A = \{a \in$

$S : Sa \neq S\}$ is non-empty. Since S is ideal-simple, there exists an element $w \in S$ such that $Sa = \{w\}$ for every $a \in A$. Of course, w is additively idempotent and multiplicatively absorbing and we see that A is an ideal of S. If $A = \{w\}$, then $Sx = S$ for every $x \in S \setminus \{w\}$, S is a division ring and it follows from 8.1 and 8.9 that S is a proper semifield. Now, assume that $A = S$, i.e., $SS = \{w\}$. The set $B = S + w$ is an ideal of S.

Let $B = S$. For every $a \in S$ there exists an element $b \in S$ with $a = b+w$; we have $a+w = b+w+w = b+w = a$. Thus $w = 0_S$ is an additively neutral element. The set $C = \{c \in S : w \in S + c\}$ is an ideal of S. If $C = S$, then $S(+)$ is a group and S is a ring, a contradiction. Thus $C = \{w\}$, so that $T+T \subseteq T$, where $T = S \setminus \{w\}$. If R is a proper subsemigroup of $T(+)$, then $R \cup \{w\}$ is an ideal of S, a contradiction. Consequently, $T(+)$ has no proper subsemigroups, and hence $|T| = 1$ and $|S| = 2$, again a contradiction.

Next, let $B = \{w\}$. Then w is a bi-absorbing element in S. Let, for a moment, $d \in S$ be such that $S + d = S$. Then $d \neq w$, $d = d + e$ for some $e \in S$ and $e = d + f$ for some $f \in S$. Clearly, $e \neq w \neq f$ and $e + e = d + f + e = d + f = e$. Now, $\{w, e\}$ is an ideal of S, $\{w, e\} = S$ and $|S| = 2$, a contradiction. It means that $S + d \neq S$ for every $d \in S$. But $S + d$ is an ideal of S, $S + d = \{w\}$ and $S + S = \{w\}$.

We have $SS = \{w\} = S + S$. Every subset of S containing the element w is an ideal, and therefore $|S| = 2$, the final contradiction. □

8.15. Theorem. *Semifields are just the semirings of one of the following seven types:*

(1) *The two-element semirings* Z_2, Z_5, Z_6*;*
(2) *Fields;*
(3) *The semifields* $U(G)$*, where* G *is a non-trivial commutative group;*
(4) *The semifields* $V(G)$*, where* G *is a non-trivial commutative group;*
(5) *The semifields* $U(P)$*, where* P *is a parasemifield;*
(6) *The semifields* $Z(P, 0)$*, where* P *is a parasemifield;*
(7) *The semifields* $V(P, T(\cdot))$*, where* P *is a parasemifield and the multiplicative group* $P(\cdot)$ *is a proper subgroup of a commutative group* $T(\cdot)$*.*

Proof. Every semifield is a division semiring and thus the classitication follows from 8.1. □

8.16. Remark. The (ideal-simple) semirings Z_3, Z_4, Z_5, Z_6, $V(G)$ are additively idempotent. The semifields $U(P)$, $Z(P, 0)$ and $V(P, T(\cdot))$ are additively idempotent if and only if the parasemifield P is additively idempotent.

8.17. Remark. The following two statements are equivalent:

(A) A parasemifield is additively idempotent, provided that it is a finitely generated semiring.

(B′) A finitely generated ideal-simple commutative semiring is either additively idempotent or it is finite or it is a copy of the semifield $U(G)$ for an infinite, finitely generated commutative group G.

8.18. Remark. One-generated ideal-simple commutative semirings are just (copies of) the two-element semirings Z_1, Z_2, Z_3, Z_4, finite fields, zero multiplication rings of finite prime order and the semifields $U(G)$ and $V(G)$, where G is a non-trivial finite cyclic group. All these semirings are finite.

8.19. Theorem. *Congruence-simple commutative semirings are just the semirings of one of the following six types:*

(1) *The two-element semirings $Z_1 \dots, Z_6$;*
(2) *Fields;*
(3) *Zero multiplication rings of finite prime order;*
(4) *The semifields $V(G)$, where G is a non-trivial commutative group;*
(5) *The semirings $W(A)$, where A is a subsemigroup of $\mathbf{R}(+)$ with $A \cap \mathbf{R}^+ \neq \emptyset \neq A \cap \mathbf{R}^+$;*
(6) *Subsemirings S of the parasemifield $\mathbf{R}^+$ of positive real numbers such that*
   (6a) *for all $a, b \in S$ there exist $c \in S$ and a positive integer n with $b + c = na$;*
   (6b) *for all $a, b, c, d \in S$ with $a \neq b$ there exist $e, f \in S$ with $ae + bf + c = af + be + d$;*
   (6c) *for all $a, b \in S$ there exist $c, d \in S$ such that $bc + d = a$.*

Proof. This is Theorem 10.1 of [1]. □

8.20. Remark.

(i) Every finitely generated congruence-simple commutative semiring is either finite or additively idempotent.
(ii) One-generated congruence-simple commutative semirings are just (copies of) the two-element semirings Z_1, Z_2, Z_3, Z_4, finite fields, zero multiplication rings of finite prime order and the semifields $V(G)$, where G is a non-trivial finite cyclic group. All these semirings are finite.

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