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DEFINABILITY FOR EQUATIONAL THEORIES OF COMMUTATIVE GROUPOIDS

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Abstract. We find several large classes of equations with the property that every automorphism of the lattice of equational theories of commutative groupoids fixes any equational theory generated by such equations, and every equational theory generated by finitely many such equations is a definable element of the lattice. We conjecture that the lattice has no non-identical automorphisms.

Keywords: simple algebra, idempotent, group

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Introduction

The study of definability in lattices of equational theories was started in the papers [3], [4], [5] and [6] that all together represent a proof of the conjecture formulated in the paper [11]. In the four papers it is proved that for any signature σ (containing either at least one binary or at least two unary operation symbols), the following are true:

(1) the lattice L of equational theories of signature σ has no automorphisms other that the obvious, syntactically defined one;

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(2) every finitely equational theory of signature σ is definable in L up to these automorphisms;
(3) the equational theory of any finite σ-algebra is definable in L up to these automorphisms;
(4) the set of finitely based equational theories, the set of one-based equational theories and the set of the equational theories corresponding to finitely generated varieties of signature σ are definable subsets of the lattice L.

The result does not imply that the same would be true for the lattice of equational theories corresponding to subvarieties of a given variety, but it suggests that the same technique could be used in the cases when the variety is defined by linear equations (equations containing the same variables on the left as on the right, and containing each variable only twice). The most significant varieties of this kind are those of semigroups, commutative semigroups and commutative groupoids.

An attempt to imitate the results of [3] through [6] to obtain the definability for equational theories of semigroups was done in the paper [8]. At first is seemed that everything would go through smoothly. We succeeded to translate (or modify) the papers [3], [4], [6] and also a half of the paper [5]. But then we got stuck; the paper brings only partial results. We still do not know if the lattice of equational theories of semigroups has only the two obvious automorphisms. (See [12] for some more recent development.)

A similar attempt was done for commutative semigroups in the paper [9]. Again, the author got stuck at a place corresponding to the middle of [5]. Proceeding further, the author succeeded however to prove that the desired aim cannot be achieved: there are non-obvious automorphisms of the lattice (and even uncountably many). The problems of definability in the lattice of equational theories of commutative semigroups have been solved completely in [1]. In particular, the group of automorphisms of the lattice has been described.

These two circumstances naturally turn the attention to the equational theories of commutative groupoids. It seemed at first that in this case everything would be easy, since commutative groupoids do not differ so much from general groupoids as semigroups do. The investigation was already started in the paper [7], which is a commutative modification of [4]. (We do not need to fully describe modular elements of the lattice, as in the paper [3], since in [8] we found a way how to avoid it, and the same can be applied to commutative groupoids.) Also, a half of [5] was translated all right. But then, again, one gets stuck.

After several vain attempts to overcome the difficulties, I gave up and the present paper is the summary of the partial results. We obtain definability for some broad classes of equational theories. After Section 1, where we establish the terminology and recall basic facts, each subsequent section demonstrates definability for a class of theories. First, in Section 2, we deal with the so called ideal theories, defined by certain sets of terms, and then with theories based on various types of equations.

We did not succeed to prove that the lattice of equational theories of commutative groupoids has no non-identical automorphisms. We just conjecture it. (There are no other obvious automorphisms in the commutative case.) However, it is also possible that the situation will turn out to be similar to that of commutative semigroups, that there exist unknown automorphisms. We think that this is an interesting and challenging problem.

1. Preliminaries

This paper is a continuation of [7]. The terminology and notation introduced in that paper remain without change; for more general topics see [10]. Let us recall that X is a fixed infinite countable set, the elements of which are called variables, and F is the free commutative groupoid over X; the elements of F are called terms. The length of a term t is denoted by $\lambda(t)$, or also by $|t|$. The depth of a term t is denoted by $\delta(t)$. If b is a subterm of a term a, i.e., if $a = bc_1 \dots c_n$ for some terms $c_1, \dots, c_n$ ($n \geqslant 0$), we write $b \subseteq a$. The set of variables occurring in a term a is denoted by $\mathbf{S}(a)$. The number of occurrences of a variable x in a term a is denoted by $\nu_x(a)$. A term a is linear if $\nu_x(a) \leqslant 1$ for all variables x. A term a is unary if $\operatorname{Card} \mathbf{S}(a) = 1$. We write $b \sim \mathbf{lh}(a)$ if b is the linear hull of a and $b \sim \mathbf{uh}(a)$ if b is the unary hull of a. By a substitution we mean an endomorphism of F. By a substitution instance of a term a we mean any term $f(a)$ where f is a substitution. Given a variable x and a term a, we denote by σ^x_a the substitution f such that $f(x) = a$ and $f(y) = y$ for every variable $y \neq x$. For two terms a, b we write $a \leqslant b$ if a substitution instance of a is a subterm of b. We write $a < b$ if $a \leqslant b$ and $b \nleqslant a$. We write $a \,||\, b$ if neither $a \leqslant b$ nor $b \leqslant a$. We write $a \sim b$ (and say that the two terms are similar) if $a \leqslant b$ and $b \leqslant a$. The block $a/{\sim}$ is called the pattern of a term a.

A term b is said to be a wonderful extension of a term a if $b = ax_1 \dots x_n$ for some $n \geqslant 0$ and some pairwise distinct variables $x_1, \dots, x_n$ not belonging to $\mathbf{S}(a)$.

For two terms a, b we write $a \sqsubseteq b$ if $\nu_x(a) \leqslant \nu_x(b)$ for all variables x. If $a \sqsubseteq b$ and $b \not\sqsubseteq a$, we say that b is essentially longer than a. Observe that if b is essentially longer than a, then $f(b)$ is longer than $f(a)$ for any substitution f.

By an equation we mean an ordered pair of terms. By an (equational) theory we mean a congruence E of the groupoid F such that $(a, b) \in E$ implies $(f(a), f(b)) \in E$ for any substitution f. The set of all theories is a complete lattice under inclusion. This lattice will be denoted by L. The least element 0_L of L is the set of trivial equations (equations (a, a) for $a \in F$) and the greatest element 1_L of L is the set of all equations.

An equation (c, d) is said to be an immediate consequence of an equation (a, b) if there exists a substitution f such that d can be obtained from c by replacing one occurrence of $f(a)$ with $f(b)$. (I.e., if there are terms $u_1, \dots, u_n$ for some $n \geqslant 0$ such that $c = f(a)u_1 \dots u_n$ and $d = f(b)u_1 \dots u_n$.) An equation is said to be an immediate consequence of a set of equations E if it is an immediate consequence of at least one equation from E.

Let E be a set of equations. By an E-derivation of an equation (a, b) we mean a finite sequence $u_0, \dots, u_n$ (with $n \geqslant 0$) of elements of F such that $u_0 = a$, $u_n = b$ and for any $i \in \{1, \dots, n\}$, either (u_{i-1}, u_i) or (u_i, u_{i-1}) is an immediate consequence of E. An equation is said to be derivable from E if it has at least one E-derivation. It is easy to prove that the set of the equations that are derivable from E is just the least theory containing E. It will be denoted by $\mathbf{Cn}(E)$ and called the theory generated by E, or the theory based on E, and its elements will be called consequences of E. For an equation (u, v) put $\mathbf{Cn}(u, v) = \mathbf{Cn}(\{(u, v)\})$; such theories are called one-based.

By a minimal E-derivation of an equation (a, b) we mean an E-derivation $u_0, \dots, u_n$ of (a, b) such that $n \leqslant m$ for any other E-derivation $v_0, \dots, v_m$ of that equation. Clearly, every equation from $\mathbf{Cn}(E)$ has a minimal E-derivation.

By a full set we mean a set $J \subseteq F$ such that $a \in J$ and $a \leqslant b$ imply $b \in J$. If J is a full set, we define $I_J = 0_L \cup J^2$. Clearly, this is a theory. Theories obtained from full sets in this way will be called ideal theories. The mapping $J \to I_J$ is an isomorphism of the distributive lattice of full sets onto the lattice of ideal theories, which is a complete sublattice of L.

For a term a put $I_a = I_J$ where $J = \{t \colon t \geqslant a\}$. The theories I_a (for $a \in F$) will be called principal ideal theories.

We denote by E_s the theory of semilattices. It consists of the equations (a, b) such that $\mathbf{S}(a) = \mathbf{S}(b)$.

A set E of equations is said to be good if there exists a first-order formula $\varphi(x_1, x_2)$ with two free variables x_1, x_2 in the language of ordered sets such that for any pair T_1, T_2 of theories, $\varphi(T_1, T_2)$ is satisfied in L if and only if $T_1 = I_{H(a,b)}$ and $T_2 = \mathbf{Cn}(a, b)$ for some equation $(a, b) \in E$. (The code-terms $H(a, b)$ were introduced in [7].)

Proposition 1.1. *Let E be a good set of equations. Then:*

(1) *The set of the theories based on an equation from E is definable.*
(2) *The set of the theories based on a finite set of equations from E is definable.*
(3) *For every $(a, b) \in E$, the theory $\mathbf{Cn}(a, b)$ is a definable element of L.*
(4) *Every automorphism of L coincides with the identity on all the elements of L that are theories based on a subset of E.*

P r o o f. This is easy. (The results of [7] can be used.) □

Clearly, the union of a finite collection of good sets of equations is good. Every good set of equations is closed under similarity. (Two equations (a,b) and (c,d) are called similar if $\alpha(a)=c$ and $\alpha(b)=d$ for an automorphism α of F.)

Suppose that K_1 is a good set of equations and K_2 is another set of equations, perhaps larger than K_1, for which we prove that whenever $(a,b)\in K_2$ then $\mathbf{Cn}(a,b)$ is the greatest (or perhaps the smallest, or the only) theory T satisfying, together with some more simple, first-order expressible conditions, the following condition: for any $(c,d)\in K_1$, $(c,d)\in T$ if and only if (c,d) is a consequence of (a,b). Then, if K_2 has been defined syntactically in a reasonable way, it follows from the results of [7] that K_2 is also good. (By saying that K_2 has been defined in a reasonable way we mean that the techniques explained in [7] can be used to show that the set of the code-terms $H(a,b)$ with $(a,b)\in K_2$ is definable in the ordered set of term patterns.)

We will prove in Section 3 that the set of strictly parallel equations is good and then continue to build larger good sets of equations in this way. We would get the complete decidability result if this process can lead in finitely many steps to obtain the set of all equations as a good set, similarly as it has been done in [5] for equational theories of universal algebras. In the present paper we will not get that far.

According to a folklore result (every non-regular equational theory is generated by its regular equations together with any one of its non-regular equations), it is sufficient to restrict ourselves to regular equations—equations (a,b) such that $\mathbf{S}(a)=\mathbf{S}(b)$.

2. Definability of ideal theories

Theorem 2.1. *Let J be a full set. Then I_J and $I_J\cap E_s$ are modular elements of L.*

Proof. Let T be either I_J or $I_J\cap E_s$. Let A, B be two theories such that $A\subseteq B$, $B\subseteq A\vee T$ and $B\cap T\subseteq A$. In order to prove that T is modular, we need to show that $A=B$.

Consider first the case when either $T=I_J$ or $A\subseteq E_s$. Let $(a,b)\in B$. There exists an $(A\cup T)$-derivation $a_0,\dots,a_n$ of (a,b). We will prove $(a,b)\in A$ by induction on n. If $n=0$ then $(a,b)=(a,a)\in A$. Let $n>0$. If $(a,a_1)\in A$ then $a_1,\dots,a_n$ is a shorter $(A\cup T)$-derivation of $(a_1,b)\in B$, so $(a_1,b)\in A$ by induction and we get $(a,b)\in A$. If $(a_{n-1},b)\in A$, we get $(a,b)\in A$ similarly. If $(a,a_1)\in T-A$ and $(a_{n-1},b)\in T-A$ then both a and b belong to J. So, if $T=I_J$, we get $(a,b)\in B\cap T\subseteq A$; if $T=I_J\cap E_s$, then $B\subseteq A\vee T\subseteq E_s$, and again $(a,b)\in B\cap T\subseteq A$.

It remains to consider the case when $T=I_J\cap E_s$ and $A\not\subseteq E_s$.

Claim 1: *If $(a,b)\in B$ where $a,b\in J$ and $\mathbf{S}(a)\subseteq\mathbf{S}(b)$, then $(a,b)\in A$.*

It is easy to see that since $A \not\subseteq E_s$, there exists a term $s = s(x, y)$ with $\mathbf{S}(s) = \{x, y\}$ (for two distinct variables x, y) such that $(s(x, y), s(x, x)) \in A$. Choose a variable $x_0 \in \mathbf{S}(a)$. Define two substitutions f, g by $f(z) = g(z) = z$ for $z \in \mathbf{S}(a)$ and $f(z) = s(x_0, z)$ and $g(z) = s(x_0, x_0)$ for the variables z not belonging to $\mathbf{S}(a)$. Since $(f(z), g(z)) \in A$ for all variables z, we have $(f(b), g(b)) \in A$. Now $(a, g(b)) \in B \cap T \subseteq A$, $(g(b), f(b)) \in A$, $(f(b), b)) \in B \cap T \subseteq A$, so that $(a, b) \in A$.

Claim 2: *If* $(a, b) \in B$ *and there is no term* $c \in J$ *with* $(a, c) \in A$*, then* $(a, b) \in A$.

Let $a_0, \dots, a_n$ be an $(A \cup T)$-derivation of (a, b). By induction on $i = 0, \dots, n$ one can easily prove that $(a, a_i) \in A$.

Let $(a, b) \in B$. We need to prove that $(a, b) \in A$. By Claim 2 (and its symmetric version) we can assume that there exist terms $c, d \in J$ with $(a, c) \in A$ and $(b, d) \in A$. If $\mathbf{S}(c) = \mathbf{S}(d)$, then $(c, d) \in B \cap T \subseteq A$ and hence $(a, b) \in A$. So, without loss of generality we can suppose that $\mathbf{S}(c) \not\subseteq \mathbf{S}(d)$. Define a substitution f by $f(x) = cd$ for $x \in \mathbf{S}(c) - \mathbf{S}(d)$ and $f(x) = x$ for all the other variables x. We have $(c, d) \in B$, $(f(c), f(d)) \in B$ where $f(d) = d$, so $(c, f(c)) \in B$. But $c, f(c) \in J$ and $\mathbf{S}(c) \subseteq \mathbf{S}(f(c))$, so $(c, f(c)) \in A$ by Claim 1. Also, $(f(c), d) \in B$ together with $f(c), d \in J$ and $\mathbf{S}(d) \subseteq \mathbf{S}(f(c))$ imply $(f(c), d) \in A$ by Claim 1. Hence $(c, d) \in A$ and we get $(a, b) \in A$. $\square$

Theorem 2.2. *Let* T *be a modular element of* L*. Denote by* U *the set of the terms* a *for which there exists a term* b *such that* $(a, b) \in T$ *and* $b \neq p(a)$ *for any permutation* p *of* $\mathbf{S}(a)$*. Then* U *is a full set and* $(U \times U) \cap E_s \subseteq T$*. If* $T \neq 0_L$*, then* U *is nonempty.*

Proof. *Claim 1*: *For every* $a \in U$ *there exists a term* b *such that* $(a, b) \in T$*,* $b \not\leqslant a$ *and* $\mathbf{S}(a) = \mathbf{S}(b)$.

We have $(a, c) \in T$ for some c such that $c \neq p(a)$ for any permutation p of $\mathbf{S}(a)$. If there exists a variable $x \in \mathbf{S}(a) - \mathbf{S}(c)$, we can take $b = f(a)$ where f is the substitution with $f(x) = aa$ and $f(y) = y$ for all variables $y \neq x$. If $\mathbf{S}(a) \subseteq \mathbf{S}(c)$ and there exists a variable $x \in \mathbf{S}(c) - \mathbf{S}(a)$, take $b = f(c)$ where f is the substitution mapping the variables from $\mathbf{S}(a)$ onto themselves and mapping all other variables onto a. Now let $\mathbf{S}(a) = \mathbf{S}(c)$. If $c \not\leqslant a$, take $b = c$. If $c \leqslant a$, then $c < a$, $a = f(c)a_1 \dots a_k$ for a substitution f and some terms $a_1, \dots, a_k$, and we can take $b = f(a)a_1 \dots a_k$.

Claim 2: *For every* $a \in U$ *there exists a term* b *such that* $(a, b) \in T$*,* $a \subset b$ *and* $\mathbf{S}(a) = \mathbf{S}(b)$.

By Claim 1 there exists a term c such that $(a, c) \in T$, $c \not\leqslant a$ and $\mathbf{S}(a) = \mathbf{S}(c)$. Denote by A the theory generated by (c, cc) and by B the theory generated by (a, aa) and (c, cc). We have $A \subseteq B$ and $(a, aa) \in (A \vee T) \cap B = A \vee (T \cap B)$. So, there exists an $(A \cup (T \cap B))$-derivation of (a, aa). In particular, there exists a term $b \neq a$ such that either $(a, b) \in A$ or $(a, b) \in T \cap B$. Since $c \not\leq a$, we cannot have $(a, b) \in A$. Hence $(a, b) \in T \cap B$ and there exists a B-derivation $u_0, \dots, u_k$ of (a, b). Easily by induction on $i = 0, \dots, k$, $a \subseteq u_i$. Hence $a \subset b$. Since $(a, b) \in B$, we have $\mathbf{S}(a) = \mathbf{S}(b)$.

Claim 3: *If p, q, r, s are terms such that $p \not\leq r$, $q \not\leq r$, $p \not\leq s$, $q \not\leq s$, $r \,||\, s$, $\mathbf{S}(r) = \mathbf{S}(s)$ and $T \cup \{(p, q)\} \models (r, s)$, then $(r, s) \in T$.*

Denote by A the theory generated by (p, q) and by B the theory generated by (p, q) and (r, s). We have $A \subseteq B$ and $(r, s) \in (A \vee T) \cap B = A \vee (T \cap B)$. Let $u_0, \dots, u_k$ be a minimal $A \cup (T \cap B)$-derivation of (r, s). Let us prove by induction on i that u_i can be obtained by a permutation of variables from either r or s, and $(r, u_i) \in T \cap B$. This is clear for $i = 0$. Let $i > 0$ and let u_{i-1} be either $\alpha(r)$ or $\alpha(s)$ for a permutation α of $\mathbf{S}(r)$. Then $p \not\leq u_{i-1}$, $q \not\leq u_{i-1}$ and so (since $u_{i-1} \neq u_i$) $(u_{i-1}, u_i) \notin A$. Hence $(u_{i-1}, u_i) \in T \cap B$. Since $(r, u_{i-1}) \in T \cap B$ by induction, we get $(r, u_i) \in T \cap B$. There is a $\{(p, q), (r, s)\}$-derivation $v_0, \dots, v_m$ of (u_{i-1}, u_i). Now v_0 can be obtained by a permutation of variables from either r or s. Since $r \,||\, s$, it is easy to prove by induction on j that also v_j can be obtained by a permutation of variables from either r or s. In particular, this is true for u_i and we are done with the induction. We get $(r, s) \in T \cap B \subseteq T$.

We say that a term a is well-behaved if $(a, d) \in T$ for every term d such that $a \subseteq d$ and $\mathbf{S}(a) = \mathbf{S}(d)$.

Claim 4: *If $a \in U$ and if there exist a term b and an infinite sequence $x_1, x_2, \dots$ of variables from $\mathbf{S}(a)$ such that $(a, b) \in T$, $a \subset b$, $\mathbf{S}(a) = \mathbf{S}(b)$ and $b \not\leq ax_1 \dots x_k$ for all k, then a is well-behaved.*

We have $b = ab_1 \dots b_m$ for some terms $b_1, \dots, b_m$. Let d be a term such that $a \subset d$ and $\mathbf{S}(a) = \mathbf{S}(d)$. We have $d = ad_1 \dots d_n$ for some terms $d_1, \dots, d_n$. Take k so large that $ax_1 \dots x_k$ is longer than $bb_1 \dots b_m d_1 \dots d_n$. One can easily check that the assumptions of Claim 3 are all satisfied if we put

$$p = bx_1 \dots x_k, \quad q = bb_1 \dots b_m, \quad r = ax_1 \dots x_k, \quad s = b$$

and that they are also satisfied if we put

$$p = bx_1 \dots x_k, \quad q = bb_1 \dots b_m d_1 \dots d_n, \quad r = ax_1 \dots x_k, \quad s = bd_1 \dots d_n.$$

It follows from the first observation that $(ax_1 \dots x_k, b) \in T$, from which we get $(ax_1 \dots x_k, a) \in T$; and from the second observation that $(ax_1 \dots x_k, bd_1 \dots d_n) \in T$, whence $(ax_1 \dots x_k, d) \in T$. But then $(a, d) \in T$.

Claim 5: *If $a \in U$ is not well-behaved, then every term b such that $(a, b) \in T$, $a \subset b$ and $\mathbf{S}(a) = \mathbf{S}(b)$ can be written as $b = ay_1 \dots y_r$ for a sequence $y_1, \dots, y_r$*

of variables such that $r \equiv 0 \bmod n$, where n is the cardinality of $\mathbf{S}(a)$, and $y_i = y_j$ implies $i \equiv j \bmod n$.

We have $b = ay_1 \dots y_r$ for some terms $y_1, \dots, y_r$. Consider the infinite sequence $x_1, x_2, \dots$, where $\{x_1, \dots, x_n\} = \mathbf{S}(a)$ and $x_i = x_{i-n}$ for $i > n$. According to Claim 4, $b \leqslant ax_1 \dots x_k$ for some k. Clearly, this implies that $y_1, \dots, y_r$ are variables and $y_i = y_j$ implies $i \equiv j \bmod n$. We also have $(a, az_1 \dots z_{2r}) \in T$ where $z_i = z_{i+r} = y_i$ for $i = 1, \dots, r$, so we can similarly conclude that $z_i = z_j$ implies $i \equiv j \bmod n$. But this is possible only if $r \equiv 0 \bmod n$.

Claim 6: *Every term $a \in U$ is well-behaved.*

Suppose that a is not well-behaved. By Claim 2 there exists a term b such that $(a, b) \in T$, $a \subset b$ and $\mathbf{S}(a) = \mathbf{S}(b)$. By Claim 5, b can be written as $b = ay_1 \dots y_r$ where $\{y_1, \dots, y_r\} = \mathbf{S}(a)$. Take a variable $x \in \mathbf{S}(a)$. By Claim 4 we have $b \leqslant ax_1 \dots x_k$ for some k, where $x_1 = \dots = x_k = x$. Clearly, this is possible only if $\mathbf{S}(a) = \{x\}$. In particular, $y_1 = \dots y_r = x$. Take a variable $y \neq x$. We have $(ay, by) \in T$, so that $ay \in U$. Moreover, ay contains two variables and we have already proved that every such term, belonging to U, is well-behaved. Hence $(ay, ay \cdot xx) \in T$ and then $(ax, ax \cdot xx) \in T$. From this we get $(a, (ax \cdot xx)y_2 \dots y_r) \in T$, a contradiction by Claim 5.

Claim 7: *U is a full set.*

Let $a \in U$ and $a \leqslant b$. We need to prove that $b \in U$. We have $f(a) \subseteq b$ for a substitution f. By Claim 2 there exists a term c with $(a, c) \in T$, $a \subset c$ and $\mathbf{S}(a) = \mathbf{S}(c)$. Denote by b' the term obtained from b by replacing one occurrence of $f(a)$ with $f(c)$. Since $(b, b') \in T$ and b' is longer than b, we get $b \in U$.

Claim 8: *We have $(a, b) \in T$ for any two terms $a, b \in U$ with $\mathbf{S}(a) = \mathbf{S}(b)$.*

Indeed, by Claim 6 we have $(a, ab) \in T$ and $(b, ab) \in T$.

Claim 9: *If $T \neq 0_L$, then U is nonempty.*

We have $(a, b) \in T$ for some $a \neq b$. We can suppose that $b = p(a)$ for a permutation p of $\mathbf{S}(a)$, since otherwise both a and b belong to U. Denote by $x_1, \dots, x_n$ the variables from $\mathbf{S}(a)$, so that $n > 1$. We have $(ax_1 \dots x_n, bx_1 \dots x_n) \in T$, and clearly $bx_1 \dots x_n \neq p(ax_1 \dots x_n)$ for any permutation p of $\mathbf{S}(a)$. □

Theorem 2.3. *E_s is the only modular coatom T of L with the property that whenever $T = A \vee B$ for two modular elements A, B of L then either $T = A$ or $T = B$. Consequently, E_s is a definable element of L.*

Proof. E_s is modular by Theorem 2.1; of course, it is a coatom of L. Let $E_s = A \vee B$ where A and B are both modular. Let x be a variable. Since $(x, xx) \in E_s$, there exists an $A \cup B$-derivation of (x, xx). Consequently, there exists a term $a \neq x$ such that (x, a) belongs to either A or B. Without loss of generality, $(x, a) \in A$. But then it follows from Theorem 2.2 that $A = E_s$.

Suppose that there exists a modular coatom $T \neq E_s$ of L with the same property. If $(x, a) \in T$ for some variable x and some $a \neq x$, then $T = E_s$ by Theorem 2.2, a contradiction. It follows that $T \subseteq I_J$ where J is the full set of the terms that are not variables. Since T is a coatom, we get $T = I_J$. But I_J is a nontrivial join of two modular elements, e.g., $I_J = (I_J \cap E_s) \vee I_K$ where K is the set of all terms of length at least 3. □

Theorem 2.4. *A theory T is an intersection of a principal ideal theory with E_s if and only if it satisfies the following three conditions:*

(1) *T is modular and $0_L \subset T \subseteq E_s$;*

(2) *for every modular theory S such that $0_L \subset S \subset T$ there exists a theory $U \subseteq T$ for which there is no smallest theory $V \subseteq T$ with the property $U \subseteq (U \cap S) \vee V$;*

(3) *whenever $T = M_1 \vee M_2$ where M_1 and M_2 are both modular theories then either $T = M_1$ or $T = M_2$.*

Consequently, the set of the theories $I_a \cap E_s$, where a is a term, is definable.

Proof. Let $T = I_a \cap E_s$. By Theorem 2.1, T is modular; the rest of (1) is clear.

Let $0_L \subset S \subset T$ where S is modular. Denote by J the set of the terms t for which there exists a term t' such that $(t, t') \in S$ and $t' \neq p(t)$ for any permutation p of $\mathbf{S}(t)$. By Theorem 2.2, J is a nonempty full set and $I_J \cap E_s \subseteq S$. Since $S \subset T$, we have $J \subset I_a$ and $a \notin J$. Put $U = \mathbf{Cn}(a, aa)$, so that $U \subseteq T$, and suppose that there is a smallest theory $V \subseteq T$ with $U \subseteq (U \cap S) \vee V$; we need to obtain a contradiction from this assumption. Denote by W the set of the terms $w \in J$ such that $\mathbf{S}(w) = \{x\}$, where x is a fixed variable. Clearly, W is nonempty. For $w \in W$ we have $wx \in J$, $(w(a), w(a)a) \in U \cap S$, $(a, w(a)) \in T$ and hence $U \subseteq (U \cap S) \vee \mathbf{Cn}(a, w(a))$; consequently, $V \subseteq \mathbf{Cn}(a, w(a))$. For every $w \in W$, $(a, w(a))$ is contained in the theory consisting of the equations (u, v) such that for every variable y, $\nu_y(u) - \nu_y(v)$ is divisible by $\lambda(w) - 1$. Consequently, whenever $(u, v) \in V$ then for every variable y, $\nu_y(u) - \nu_y(v)$ is divisible by $\lambda(w) - 1$. But obviously, for every $w \in W$ there exists a term $w' \in W$ with $\lambda(w') = \lambda(w) + 1$. It follows that $(u, v) \in V$ is possible only if $\nu_y(u) = \nu_y(v)$ for all variables y. Since $(a, aa) \in (U \cap S) \vee V$, there is an $(U \cap S) \cup V$-derivation $u_0, \ldots, u_n$ of (a, aa). Let us prove by induction on i that $\lambda(u_i) = \lambda(a)$ and $\mathbf{S}(u_i) = \mathbf{S}(a)$. This is clear for $u_i = u_0 = a$; let it be true for some u_i with $i < n$. If $(u_i, u_{i+1}) \in V$, then the conclusion for u_{i+1} follows from the above observation. If $(u_i, u_{i+1}) \in U \cap S$, then it follows from $a \notin J$ that $u_{i+1} = p(u_i)$ for a permutation p of $\mathbf{S}(u_i)$, so that $\lambda(u_{i+1}) = \lambda(u_i)$ and $\mathbf{S}(u_{i+1}) = \mathbf{S}(u_i)$. The induction has been finished. In particular, $\lambda(aa) = \lambda(a)$, a contradiction.

Let $T = M_1 \vee M_2$ where M_1 and M_2 are modular. Since $(a, aa) \in T$, there exists a term b such that $(a, b) \in M_i$ for an $i \in \{1, 2\}$ and $b \neq p(a)$ for any permutation p of $\mathbf{S}(a)$. Then it follows from Theorem 2.2 that $T = M_i$.

Now we are going to prove the converse implication. Let T be a theory satisfying the three conditions. Denote by J the set of the terms t for which there exists a term t' such that $(t, t') \in T$ and $t' \neq p(t)$ for any permutation p of $\mathbf{S}(t)$. By Theorem 2.2, J is a nonempty full set and $I_J \cap E_s \subseteq T$. Suppose that $T \neq I_J \cap E_s$. Put $S = I_J \cap E_s$, so that S is modular by Theorem 2.1 and $0_L \subset S \subset T$. Let U be a theory contained in T. For every term $a \in J$ we have $a/S = a/T = \{b \in F\colon\ \mathbf{S}(a) = \mathbf{S}(b)\}$. For every term $a \notin J$ we have $a/S = \{a\}$, and a/T may contain only the terms $p(a)$ where p is a permutation of $\mathbf{S}(a)$ (so that a/T is finite). From this it follows easily that for any theory V contained in T, $U \subseteq (U \cap S) \vee V$ if and only if $U \cap ((F - J) \times (F - J)) \subseteq V$. So, there is a smallest theory among such theories V. This contradiction with (2) proves that $T = I_J \cap E_s$.

Since J is nonempty, there exists a minimal term a in J. Denote by Q the set of the minimal terms of J that are not similar to a and denote by K the full set generated by Q. Clearly, $T = (I_a \cap E_s) \vee (I_K \cap E_s)$. By (3), either $T = I_a \cap E_s$ or $T = I_K \cap E_s$. But then, $T = I_a \cap E_s$. □

Theorem 2.5. *A theory T is an ideal theory if and only if either $T = 0_L$ or else T is modular, $T \not\subseteq E_s$, and there does not exist a modular theory $S \subset T$ such that $S \not\subseteq E_s$ and $U \subseteq S$ for any theory $U \subseteq T$ that is an intersection of a principal ideal theory with E_s. Consequently, the set of ideal theories is definable. Also, the set of principal ideal theories is definable.*

P r o o f. This follows easily from the previous theorems. □

Theorem 2.6. *Every principal ideal theory is definable.*

P r o o f. For two terms a, b we have $I_a \subseteq I_b$ if and only if $a \geqslant b$, so that the ordered set P of principal ideal theories is antiisomorphic to the ordered set of term patterns. By Theorem 2.5, P is a definable subset of the lattice L. According to Theorem 8.1 of [7], every term pattern is a definable element of the ordered set of term patterns. Consequently, every principal ideal theory is a definable element of L. □

For every term a denote by $M(a)$ the set of all equations (u, v) such that either $u = v$ or $u > a$ and $v > a$ or $u \sim v \sim a$ and $\mathbf{S}(u) = \mathbf{S}(v)$. It is easy to check that $M(a)$ is a theory. We have $M(a) = M(b)$ if and only if $I(a) = I(b)$ if and only if $a \sim b$.

Proposition 2.7. *For a term a, $M(a)$ is the largest modular element T of L such that $T \subset I_a$ and $T \not\subseteq E_s$.*

Consequently, the binary relation R, where $(T_1, T_2) \in R$ if and only if $T_1 = I_a$ and $T_2 = M(a)$ for a term a, is definable.

Proof. First we are going to show that $M(a)$ is modular. Let A, B be two theories such that $A \subseteq B$, $B \subseteq A \vee M(a)$ and $B \cap M(a) \subseteq A$. We need to show that $A = B$. Suppose, on the contrary, that there is an equation $(b, c) \in B - A$ and take one for which the length n of a minimal $(A \cup M(a))$-derivation $b_0, \ldots, b_n$ of (b, c) is the smallest possible. We have $(b, c) \notin M(a)$, since otherwise we would have $(b, c) \in B \cap M(a) \subseteq A$. In particular, $a \neq b$ and $n > 0$. If $(b, b_1) \in A$ then $b_1, \ldots, b_n$ is a shorter $(A \cup M(a))$-derivation of the equation $(b_1, c) \in B$, so that $(b_1, c) \in A$ and thus $(b, c) \in A$, a contradiction. We get $(b, b_1) \in M(a) - A$. Similarly, $(b_{n-1}, c) \in M(a) - A$. Since $(b, c) \notin M(a)$, we have $b \sim b_1 \sim a$, $\mathbf{S}(b) = \mathbf{S}(b_1)$ and $b_{n-1}, c > a$ (or vice versa, but the other symmetric case would be handled similarly). Then $n \geqslant 3$. There is a permutation p of $\mathbf{S}(b)$ with $b_1 = p(b)$. Since $(b_1, b_2) \in A$, we have $(p^{-1}(b_1), p^{-1}(b_2)) \in A$, i.e., $(b, p^{-1}(b_2)) \in A$. Now, clearly $b, p^{-1}(b_2), p^{-1}(b_3), \ldots, p^{-1}(b_{n-1}), c$ is a shorter $(A \cup M(a))$-derivation of (b, c), a contradiction.

Clearly, $M(a) \subset I_a$ and $M(a) \not\subseteq E_s$. Conversely, if T is a modular element of L such that $T \subset I_a$ and $T \not\subseteq E_s$, then it follows easily from Theorem 2.2 that $T \subseteq M(a)$. □

For a term a we denote by I_a^* the largest ideal theory properly contained in I_a, i.e., the ideal theory I_J where J is the full set generated by all the covers of a. We have $(u, v) \in I_a^*$ if and only if either $u = v$ or $u, v > a$.

3. Parallel equations

By a parallel equation we mean a regular equation (a, b) such that a, b are two incomparable terms.

For every term a we denote by G_a the set of the permutations p of $\mathbf{S}(a)$ such that $p(a) = a$. Clearly, G_a is a subgroup of the symmetric group on $\mathbf{S}(a)$. (See [2] for an exact description of G_a).

The following two facts can be found in [1] and [2].

Fact 3.1. Let a be a term and p be a permutation of $\mathbf{S}(a)$. Then $G_{p(a)} = pG_ap^{-1}$.

Fact 3.2. Let (a, b) be a parallel equation and p be a permutation of $\mathbf{S}(a)$. Then $(a, p(b)) \in \mathbf{Cn}(a, b)$ if and only if $p \in G_a \vee G_b$ (the join in the lattice of subgroups of the symmetric group on $\mathbf{S}(a)$).

An equation (a,b) is said to be mini-parallel if it is parallel and for any permutation p of $\mathbf{S}(a)$, if $(a,p(b))$ is a consequence of (a,b) then $(a,p(b))$ is equivalent with (a,b).

Lemma 3.3. *A parallel equation (a,b) is mini-parallel if and only if*

$$G_a \vee G_{p(b)} = G_a \vee G_b$$

for every $p \in G_b$.

P r o o f. This follows from Fact 3.2. □

Lemma 3.4. *Every parallel equation (a,b) has a mini-parallel consequence $(a,p(b))$ for some permutation p of $\mathbf{S}(a)$.*

P r o o f. This is evident. □

Example 3.5. The equation $(xyzz,(xx\cdot zz)y)$ is parallel but not mini-parallel; $(xyzz,(xx\cdot yy)z)$ is its mini-parallel consequence.

Lemma 3.6. *Let (a,b) be a parallel equation and T be a theory; put $S=\mathbf{S}(a)=\mathbf{S}(b)$. Then $T=\mathbf{Cn}(a,p(b))$ for some permutation p of S such that $(a,p(b))$ is mini-parallel if and only if the following are satisfied:*

(1) *$T\subseteq E_s$;*
(2) *$T\not\subseteq M(a)\vee M(b)$;*
(3) *$I_a\vee I_b$ is the ideal theory generated by T;*
(4) *whenever U is a theory such that $U\subset T$ then $U\subseteq M(a)\vee M(b)$.*

P r o o f. Clearly, $(u,v)\in M(a)\vee M(b)$ if and only if either $u=v$ or $u\sim v\sim a$ and $\mathbf{S}(u)=\mathbf{S}(v)$ or $u\sim v\sim b$ and $\mathbf{S}(u)=\mathbf{S}(v)$ or each of the terms u, v is (strictly) larger than at least one of the terms a, b. Let $T=\mathbf{Cn}(a,p(b))$ where $(a,p(b))$ is mini-parallel. The first three conditions are obviously satisfied. Let $U\subset T$ and suppose that $U\not\subseteq M(a)\vee M(b)$. Since $U\subseteq I_a\vee I_b$ and $U\not\subseteq M(a)\vee M(b)$, either $(a,a')\in U$ for some $a'\not\sim a$ or $(b,b')\in U$ for some $b'\not\sim b$. But $U\subseteq\mathbf{Cn}(a,p(b))$, so in each case we get $(a,qp(b))\in U$ for some permutation q. Since $(a,p(b))$ is mini-parallel, $(a,qp(b))$ is equivalent with $(a,p(b))$. But then $T=U$, a contradiction.

Conversely, let the four conditions be satisfied. By (2) and (3), either $(a,a')\in T$ for some $a'\not\sim a$ or $(b,b')\in T$ for some $b'\not\sim b$. If $(a,a')\in T$ then $a'\sim b$, since otherwise we would have either $a'>a$ or $a'>b$, $\mathbf{Cn}(a,a')\not\subseteq M(a)\vee M(b)$ and hence $T=\mathbf{Cn}(a,a')$ by (4), a contradiction with (3). So, if $(a,a')\in T$ then $a'\sim b$. Similarly, if $(b,b')\in T$ then $b'\sim a$. In each case we get $(a,p(b))\in T$ for a permutation p

of $\mathbf{S}(a)$. Since $\mathbf{Cn}(a, p(b)) \not\subseteq M(a) \vee M(b)$, by (4) we get $T = \mathbf{Cn}(a, p(b))$. If q is a permutation such that $(a, qp(b))$ is a consequence of $(a, p(b))$, then it follows from (4) that $T = \mathbf{Cn}(a, qp(b))$. Consequently, $(a, p(b))$ is a mini-parallel equation. $\square$

Let a be a term. By an a-permutational theory we mean a theory that has a base consisting of equations $(a, p(a))$, for some permutations p of $\mathbf{S}(a)$.

Proposition 3.7. *Let a be a term. A theory T is a-permutational if and only if either $T = 0_L$ or the following conditions are satisfied:*

(1) *I_a is the ideal theory generated by T;*
(2) *$T \subseteq M(a)$;*
(3) *whenever U is a theory such that $U \subseteq M(a)$ and $U \vee I_a^* = T \vee I_a^*$ then $T \subseteq U$.*

Consequently, the binary relation R where $(T_1, T_2) \in R$ if and only if $T_1 = I_a$ and T_2 is an a-permutational theory for some term a, is definable.

P r o o f. Let T be a-permutational and $T \neq 0_L$. Clearly, the conditions (1) and (2) are satisfied. Let $U \subseteq M(a)$ and $U \vee I_a^* = T \vee I_a^*$. We have $(u, v) \in U \vee I_a^*$ if and only if either $(u, v) \in I_a^*$ or $(u, v) \in U$, $u \sim v \sim a$ and $\mathbf{S}(u) = \mathbf{S}(v)$. We have $(u, v) \in T \vee I_a^*$ if and only if either $(u, v) \in I_a^*$ or $(u, v) \in T$, $u \sim v \sim a$ and $\mathbf{S}(u) = \mathbf{S}(v)$. Since $U \vee I_a^* = T \vee I_a^*$, it follows that for every permutation p of $\mathbf{S}(a)$, $(a, p(a)) \in U$ if and only if $(a, p(a)) \in U$. But T is generated by such equations, so $T \subseteq U$.

Conversely, let (1), (2) and (3) be satisfied. Denote by G the set of the permutations p of $\mathbf{S}(a)$ such that $(a, p(a)) \in T$. Then G is a group and $G_a \subseteq G$; it follows from (1) and (2) that $G_a \subset G$. Denote by U the theory based on the equations $(a, p(a))$ with $p \in G$, so that U is a-permutational and $U \subseteq T$. Clearly, $U \subseteq M(a)$ and $U \vee I_a^* = T \vee I_a^*$. By (3), $T = U$. $\square$

Lemma 3.8. *Let a, b be two terms, f be a substitution and $x_1, \dots, x_n$ $(n \geqslant 0)$ be variables such that*

(1) *$f(a) = bx_1 \dots x_n$;*
(2) *if $1 \leqslant i \leqslant n$ and $i \leqslant \lambda(a)$ then $x_i \notin \mathbf{S}(b)$;*
(3) *if $1 \leqslant i + 1 \leqslant i + k \leqslant n$ and $k \leqslant \lambda(a)$ then $x_{i+1}, \dots, x_{i+k}$ are pairwise distinct.*

Then either a is a slim linear term or $a = a_1 y_1 \dots y_n$ for a term a_1 and pairwise distinct variables $y_1, \dots, y_n$ not belonging to $\mathbf{S}(a_1)$. If $a = b$ and $n \geqslant 1$, then a is a slim linear term.

P r o o f. The first statement will be proved by induction on n. For $n = 0$ it is clear. Let $n \geqslant 1$ and suppose a is not a slim linear term. Then $a = cd$ for two terms c, d with $f(c) = bx_1 \dots x_{n-1}$ and $f(d) = x_n$. Of course, d is a variable. By the induction assumption applied to the terms c, b and the variables $x_1, \dots, x_{n-1}$, there are only two cases to be considered.

Case 1: c is a slim linear term. Then $a = y_1 \dots y_m d$ where $y_1, \dots, y_m$ are pairwise distinct variables and $d = y_i$ for some i. It follows that x_n has at least two occurrences in $bx_1 \dots x_n$, so that $n > \lambda(a) = m + 1$; we have $f(d) = x_n$, $f(y_m) = x_{n-1}, \dots,$ $f(y_3) = x_{n-m+2}$ and $\{f(y_1), f(y_2)\} = \{bx_1 \dots x_{n-m}, x_{n-m+1}\}$. But $bx_1 \dots x_{n-m}, x_{n-m+1}, \dots, x_n$ are pairwise different, so $y_1, \dots, y_m, d$ are pairwise distinct, a contradiction. This case is not possible.

Case 2: $c = c_1 y_1 \dots y_{n-1}$ where $y_1, \dots, y_{n-1}$ are pairwise distinct variables not belonging to $\mathbf{S}(c_1)$. Then $a = c_1 y_1 \dots y_{n-1} d$, $f(c_1) = b$, $f(y_i) = x_i$ and $f(d) = x_n$. Since $\lambda(a) > n$, $x_n \notin \mathbf{S}(bx_1 \dots x_{n-1})$ and so $d \notin \mathbf{S}(c_1 y_1 \dots y_{n-1})$. We can put $a_1 = c_1$ and $y_n = d$.

In order to prove the second statement, let $f(a) = ax_1 \dots x_n$ and suppose that a is not slim and linear. By the first statement, $a = a_1 y_1 \dots y_n$ where $y_1, \dots, y_n$ are pairwise distinct variables not belonging to $\mathbf{S}(a_1)$. Since $f(a_1 y_1 \dots y_n) = a_1 y_1 \dots y_n x_1 \dots x_n$, we have $f(a_1) = a_1 y_1 \dots y_n$ and hence a_1 is a slim linear term (it is obvious in this case, or we could also proceed by induction on the length of a). But then a is a slim linear term. $\square$

An equation (a, b) is said to be strictly parallel if the following conditions are satisfied:

(1) (a, b) is parallel and neither a nor b is a slim linear term;
(2) $G_a = G_b = \mathrm{id}_{\mathbf{S}(a)}$;
(3) whenever a is a wonderful extension of a term a_1 then b is not a substitution instance of a_1;
(4) whenever b is a wonderful extension of a term b_1 then a is not a substitution instance of b_1.

It follows from Lemma 3.3 that every strictly parallel equation is mini-parallel.

Proposition 3.9. *Let (a, b) be a strictly parallel equation and let T be a theory. Then $T = \mathbf{Cn}(a, b)$ if and only if the following two conditions are satisfied:*

(1) *$T = \mathbf{Cn}(a, p(b))$ for a permutation p of $\mathbf{S}(a)$ such that $(a, p(b))$ is mini-parallel;*
(2) *whenever (c, d) is a parallel consequence of (a, b) then $(c, q(d)) \in T$ for a permutation q of $\mathbf{S}(c)$ such that $(c, q(d))$ is mini-parallel.*

Proof. The direct implication is obvious. Let (1) and (2) be satisfied. By (1), $T = \mathbf{Cn}(a, p(b))$ for some permutation p and we only need to prove that p is the identity. Take a number m such that $m \geqslant \lambda(a)$ and $m \geqslant \lambda(b)$. Take a sequence $x_1, \dots, x_n$ of variables such that $\mathbf{S}(a) \subseteq \{x_1, \dots, x_n\}$, whenever $1 \leqslant i + 1 \leqslant i + k \leqslant$

n and $k \leqslant m$ then $x_{i+1}, \ldots, x_{i+k}$ are pairwise distinct and whenever $x_i \in \mathbf{S}(a)$ then $i > m$ and $x_{i-1}, \ldots, x_{i-m} \notin \mathbf{S}(a)$. Clearly, $(ax_1 \ldots x_n, bx_1 \ldots x_n)$ is a parallel consequence of (a, b). So, by (2), there is a permutation q of $\mathbf{S}(ax_1 \ldots x_n)$ such that $(ax_1 \ldots x_n, q(bx_1 \ldots x_n))$ is a consequence of $(a, p(b))$.

Let c be a term such that $(ax_1 \ldots x_n, c)$ is an immediate consequence of either $(a, p(b))$ or $(p(b), a)$. It follows from Lemma 3.8 that $p(b) \not\leq ax_1 \ldots x_n$, so $(ax_1 \ldots x_n, c)$ can be only an immediate consequence of $(a, p(b))$. There exists a substitution f such that $f(a) \subseteq ax_1 \ldots x_n$ and c can be obtained from $ax_1 \ldots x_n$ by replacing an occurrence of $f(a)$ with $fp(b)$. It follows from Lemma 3.8 that $f(a) = a$, hence $c = fp(b)x_1 \ldots x_n$. Since G_a contains only the identity, f is the identity and $c = p(b)x_1 \ldots x_n$.

We can show quite similarly that if c is a term such that $(p(b)x_1 \ldots x_n, c)$ is an immediate consequence of either $(a, p(b))$ or $(p(b), a)$ then $c = ax_1 \ldots x_n$. Since there exists an $(a, p(b))$-derivation of $(ax_1 \ldots x_n, q(bx_1 \ldots x_n))$, it follows that only two terms can be members of this derivation, namely, the terms $ax_1 \ldots x_n$ and $p(b)x_1 \ldots x_n$. In particular, we get $q(bx_1 \ldots x_n) = p(b)x_1 \ldots x_n$. Then $q(b) = p(b)$ and $q(x_i) = x_i$ for all i. Since $\mathbf{S}(b) \subseteq \{x_1, \ldots, x_n\}$, it follows that q is the identity and $p(b) = b$. □

Theorem 3.10. *The set of strictly parallel equations is good.*

P r o o f. The two conditions in Proposition 3.9 can be more formally expressed to obtain the desired first-order formula; the pieces of the form '$T = \mathbf{Cn}(u, g(v))$ for a permutation g such that $(u, g(v))$ is mini-parallel' should be reformulated using Lemma 3.6. □

4. Nice equations

A term a is said to be strongly nice if it is a product of two terms, none of which is a variable; it is said to be weakly nice if it is a product of a variable with a term containing this variable; it is said to be nice if it is either strongly or weakly nice. An equation (a, b) is said to be nice if it is regular and both a and b are nice.

Theorem 4.1. *Let (a, b) be a nice equation. Then $\mathbf{Cn}(a, b)$ is the greatest theory T such that $T \subseteq E_s$ and any strictly parallel equation belongs to T if and only if it is a consequence of (a, b). Consequently, the set of nice equations is good.*

P r o o f. Let T be a such a theory; we need to prove that $T \subseteq \mathbf{Cn}(a, b)$. Let $(c, d) \in T$ and $c \neq d$. Put $m = \max(\lambda(c), \lambda(d))$. Clearly, there exists a sequence $x_1, \ldots, x_n$ of variables such that $n > 2m$, $\mathbf{S}(c) \subseteq \{x_1, \ldots, x_n\}$, $x_1, \ldots, x_m \notin \mathbf{S}(c)$, $x_1, \ldots, x_{n-1}$ are pairwise distinct and $x_n = x_{n-m}$.

Suppose that $cx_1 \dots x_n \leqslant dx_1 \dots x_n$. Since $n > m$, we have $f(cx_1 \dots x_n) = dx_1 \dots x_i$ for some substitution f and some i; clearly, $i > n - m$. Since $n > 2m$, we have $i - m > 1$ and $f(x_n) = x_i$, $f(x_{n-1}) = x_{i-1}, \dots$, $f(x_{n-m}) = x_{i-m}$. If $i \neq n$, we get a contradiction from $x_n = x_{n-m}$ and $x_i \neq x_{i-m}$. So, $i = n$ and $f(cx_1 \dots x_n) = dx_1 \dots x_n$. Consequently, one of the following two cases takes place.

Case 1: $f(c) = d$ and $f(x_i) = x_i$ for all i. Since $\mathbf{S}(c) \subseteq \{x_1, \dots, x_n\}$, we get $f(c) = c$, so that $c = d$, a contradiction.

Case 2: $f(c) = x_1$, $f(x_1) = d$ and $f(x_i) = x_i$ for all $i \geqslant 2$. Then c is a variable, $c = x_j$ for some j and clearly $j \neq 1$, so that $f(c) = c$ and again $c = d$, a contradiction.

We have proved $cx_1 \dots x_n \nleqslant dx_1 \dots x_n$. Quite similarly, $dx_1 \dots x_n \nleqslant cx_1 \dots x_n$. So, $(cx_1 \dots x_n, dx_1 \dots x_n)$ is a parallel equation. Obviously, it is strictly parallel. Since it belongs to T, it is a consequence of (a, b) and there is an (a, b)-derivation $u_0, \dots, u_k$ of this equation.

Let us prove by induction on i that $u_i = v_i x_1 \dots x_n$ for some term v_i such that (c, v_i) is a consequence of (a, b). For $i = 0$ it is clear. Let $i \geqslant 1$. Without loss of generality, (u_{i-1}, u_i) is an immediate consequence of (a, b). There is a substitution f such that $f(a) \subseteq u_{i-1} = v_{i-1} x_1 \dots x_n$ and u_i results from u_{i-1} by replacing $f(a)$ with $f(b)$. If $f(a) \subseteq v_{i-1}$, then $u_i = v_i x_1 \dots x_n$ where v_i results from v_{i-1} by replacing $f(a)$ with $f(b)$, so that (v_{i-1}, v_i) is a consequence of (a, b) and then it follows from the induction assumption that (c, v_i) is a consequence of (a, b). The other case is $f(a) = v_{i-1} x_1 \dots x_r$ for some $r \geqslant 1$. If $r \leqslant m$ then $x_r \notin \mathbf{S}(v_{i-1} x_1 \dots x_{r-1})$, so that a cannot be nice, a contradiction. Hence $r > m$. Then $x_{r-m+1}, \dots, x_r$ are pairwise distinct variables; since $\lambda(a) \leqslant m$ and $f(a) = e x_{r-m+1} \dots x_r$ for some term e, we get that a is a slim linear term; but then a is not nice, a contradiction.

In particular, $dx_1 \dots x_n = v_n x_1 \dots x_n$ where (c, v_n) is a consequence of (a, b). But then (c, d) is a consequence of (a, b). We have proved $T \subseteq \mathbf{Cn}(a, b)$. $\square$

5. Modest equations

An equation (a, b) is said to be modest if it is regular, a, b are of length $\geqslant 3$ and there exists a variable x such that $a = a_1 x$ and $b = b_1 x$ for some terms a_1, b_1 with $x \notin \mathbf{S}(a_1)$ and $x \notin \mathbf{S}(b_1)$.

Denote by E_M the set of the equations (a, b) such that either $a = b$ or (a, b) is either nice or modest.

(The reason why we forbid terms of length less than 3 in the definition of a modest equation is that if we discarded it, then E_M would not be transitive: we would have $(xxy, xy) \in E_M$ and $(xy, yyx) \in E_M$ but $(xxy, yyx) \notin E_M$.)

Proposition 5.1. *E_M is a theory. It is the greatest theory T such that $T \subseteq E_s$, $T \subseteq I_{xyz} \vee I_{xx}$ and whenever (u, v) is either strictly parallel or nice then $(u, v) \in T$ if and only if $(u, v) \in E_M$. Consequently, E_M is a definable element of L.*

P r o o f. One can easily check that E_M is a theory. Let T be a theory with the above mentioned properties; we must prove $T \subseteq E_M$. Suppose, on the contrary, that there exists an equation $(c, d) \in T - E_M$. Without loss of generality, $c = c_1 x$ where $x \in X - \mathbf{S}(c_1)$, while d is not of such a form (with the same x).

We can suppose that c_1 and d are both nice. Indeed, if this was not the case, then instead of (c, d) we could take the equation $(f(c), f(d))$ where f is the substitution with $f(x) = x$ and $f(y) = yy$ for all variables $y \neq x$; we have $(f(c), f(d)) \in T - E_M$, and the terms $f(c_1)$ and $f(d)$ are both nice.

Put $\mathbf{S}(c_1) = \{x_1, \dots, x_n\}$. The equations $(c_1, x_1 x_1 \cdot x_1 x_1 x_2 \dots x_n)$ and $(d, x_1 x_1 x \cdot x_1 x_1 x_1 x_2 \dots x_n)$ are both nice, belong to E_M and hence belong to T. Then also $(c, (x_1 x_1 \cdot x_1 x_1 x_2 \dots x_n) x)$ belongs to T and we get $((x_1 x_1 \cdot x_1 x_1 x_2 \dots x_n) x, x_1 x_1 x \cdot x_1 x_1 x_1 x_2 \dots x_n) \in T$, since $(c, d) \in T$. Clearly, this equation is strictly parallel and so it follows that it belongs to E_M; but it does not belong to E_M and we get a contradiction. $\square$

Theorem 5.2. *Let (a, b) be a modest equation. Then $\mathbf{Cn}(a, b)$ is the greatest theory T such that $T \subseteq E_M$ and any nice equation belongs to T if and only if it is a consequence of (a, b). Consequently, the set of modest equations is good.*

P r o o f. Let $(a, b) = (a_1 x_0, b_1 x_0)$. Let T be such a theory; we need to prove $T \subseteq \mathbf{Cn}(a, b)$. Let $(c, d) \in T$ and $c \neq d$; we are going to prove that $(c, d) \in \mathbf{Cn}(a, b)$. If (c, d) is nice, it is clear. Suppose (c, d) is not nice. Since $(c, d) \in E_M$, it follows that (c, d) is modest. We have $c = c_1 x$ and $d = d_1 x$ for two terms c_1, d_1 and a variable $x \notin \mathbf{S}(c_1) = \mathbf{S}(d_1)$. Take a variable $y \in \mathbf{S}(c_1)$. The equation $(c_1 y, d_1 y)$ is nice and belongs to T, so it is a consequence of (a, b). There is an (a, b)-derivation $w_0, \dots, w_n$ of $(c_1 y, d_1 y)$.

Let us prove by induction on i that $w_i = s_i y$ for a term s_i such that $(c_1 x, s_i x) \in \mathbf{Cn}(a, b)$. For $i = 0$ it is clear. Let $i \geqslant 1$. The equation (w_{i-1}, w_i) is an immediate consequence of either (a, b) or (b, a); without loss of generality, it is sufficient to consider the case when it is an immediate consequence of (a, b). There is a substitution f such that $f(a) \subseteq w_{i-1} = s_{i-1} y$ and w_i results from w_{i-1} by replacing $f(a)$ with $f(b)$. If $f(a) \subseteq s_{i-1}$, then everything is clear. The other case is $f(a) = s_{i-1} y$. Then $f(a_1) = s_{i-1}$, $f(x_0) = y$ and $w_i = f(b_1) y$. Put $s_i = f(b_1)$, so that $w_i = s_i y$. Denote by g the substitution with $g(x_0) = x$ and $g(z) = f(z)$ for all variables $z \neq x_0$. Since g coincides with f on $\mathbf{S}(a_1) = \mathbf{S}(b_1)$, we have $f(a_1) = g(b_1)$. Then $g(a) = g(a_1) x =$

$f(a_1)x = s_{i-1}x$ and $g(b) = g(b_1)x = f(b_1)x = s_i x$. Since $(g(a), g(b)) \in \mathbf{Cn}(a, b)$, we get $(s_{i-1}x, s_i x) \in \mathbf{Cn}(a, b)$ and hence $(c_1 x, s_i x) \in \mathbf{Cn}(a, b)$.

In particular, for $i = n$ we get $(c_1 x, d_1 x) \in \mathbf{Cn}(a, b)$, i.e., $(c, d) \in \mathbf{Cn}(a, b)$. □

6. Unary equations

An equation (a, b) is said to be unary if $\mathbf{S}(a) = \mathbf{S}(b) = \{x\}$ for a variable x.

Theorem 6.1. *Let (a, x) be a unary equation such that x is a variable and $a \neq x$. Then $\mathbf{Cn}(a, x)$ is the greatest theory T such that $T \subseteq E_s$ and any nice equation belongs to T if and only if it is a consequence of (a, x). Consequently, the set of unary equations is good.*

Proof. Let T be such a theory; we need to prove that $T \subseteq C$ where $C = \mathbf{Cn}(a, x)$. Let $(c, d) \in T$ and $c \neq d$. For every variable $y \in \mathbf{S}(c)$ take four distinct variables y_1, y_2, y_3, y_4 in such a way that if $y \neq z$ then the sets $\{y_1, y_2, y_3, y_4\}$ and $\{z_1, z_2, z_3, z_4\}$ are disjoint. Denote by f the substitution with $f(y) = y_1 y_2 \cdot y_3 y_4$ for all $y \in \mathbf{S}(c)$. Since $(f(c), f(d)) \in T$ is a nice equation, we have $(f(c), f(d)) \in C$. Clearly, there exists a substitution g such that $gf(y) = \sigma^x_y \sigma^x_a(a)$ for all $y \in \mathbf{S}(c)$. We have $(\sigma^x_y \sigma^x_a(a), y) \in C$ and thus $(gf(y), y) \in C$ for all $y \in \mathbf{S}(c)$. Hence $(gf(c), c) \in C$ and $(gf(d), d) \in C$; since $(f(c), f(d)) \in C$, we have $(gf(c), gf(d)) \in C$ and we get $(c, d) \in C$.

It follows that the set of the unary equations (a, b) such that either $a \in X$ or $b \in X$ is good. The other nontrivial unary equations are all nice, so the whole set is good. □

7. xy-equations

Throughout this section let x and y be two distinct variables. By an xy-equation we mean a regular equation with the left side equal to xy. The aim of this section is to prove that the set of xy-equations is good.

By a 1-special equation we mean an equation (xy, a) where a is a term such that $\mathbf{S}(a) = \{x, y\}$, $a \neq xy$ and neither xx nor yy is a subterm of a.

Theorem 7.1. *Let (xy, a) be a 1-special equation. Then $\mathbf{Cn}(xy, a)$ is the greatest theory T such that $T \subseteq E_s$ and every equation that is either modest or unary belongs to T if and only if it is a consequence of (xy, a). Consequently, the set of 1-special equations is good.*

P r o o f. Let T be such a theory and $(c,d)\in T$; we need to prove that (c,d) is a consequence of (xy,a). This is clear if (c,d) is either modest or unary. Consider the remaining case only. Since (c,d) is not unary, c, d are of length at least 2. Take a variable z not belonging to $\mathbf{S}(c)=\mathbf{S}(d)$. The equation (cz,dz) is modest and belongs to T, so it is a consequence of (xy,a). There exists an (xy,a)-derivation $u_0,\dots,u_k$ of (cz,dz).

Let us prove by induction on i that whenever u_i can be written as $u_i=vv_1\dots v_m$ where $z\notin\mathbf{S}(v)$ and $z\in\mathbf{S}(v_1)$ (less formally, whenever v is a maximal no z containing occurrence of a subterm in u_i) then (c,v) is a consequence of (xy,a). For $i=0$ it is clear, since $u_0=cz$. Let $i>0$. Then u_i is obtained from u_{i-1} by replacing one occurrence of a subterm pq (for some terms p, q) with the term $r=\sigma^{x,y}_{p,q}(a)$, or vice versa. If a maximal no z containing occurrence of v in u_i is disjoint with pq (with r, respectively), then it is also a maximal no z containing occurrence of v in u_{i-1} and so (c,v) is a consequence of (xy,a) by induction. If it contains pq (or r, respectively) then the same replacement in v transforms v into a maximal no z containing occurrence of a subterm in u_{i-1} and we can again apply induction. The only remaining possibility is that v is a proper subterm of pq (or of r, respectively). But then, in both cases, v is a subterm of either p or q (here we are using the fact that (xy,a) is 1-special) and the induction can be applied again.

Since d is a maximal no z containing occurrence of a subterm in u_k, it follows that (c,d) is a consequence of (xy,a). $\square$

Let K be a set of equations. By a K-related pair we mean a pair of regular theories T_1, T_2 such that $(x,t)\in T_i$ implies $t=x$, there are two terms a_1,a_2 of length $\geqslant 3$ with $(xy,a_i)\in T_i$ for $i=1,2$, and whenever $(u,v)\in K$ then $(u,v)\in T_1$ if and only if $(u,v)\in T_2$.

Lemma 7.2. *Let $T_1\neq T_2$ be a K-related pair where K is the set of the equations that are either strictly parallel or nice or modest or unary or 1-special. For $i=1,2$ denote by H_i the set of the terms t of length $\geqslant 3$ such that $(xy,t)\in T_i$.*

(1) *Let $i\in\{1,2\}$. Then H_i contains a strongly nice term.*
(2) *Let $i\in\{1,2\}$. For every term $t\notin X$ there exists a strongly nice term t' with $(t,t')\in T_i$.*
(3) *$T_1\not\subseteq T_2$ and $T_2\not\subseteq T_1$.*
(4) *$H_1\not\subseteq H_2$ and $H_2\not\subseteq H_1$.*
(5) *Let $i\in\{1,2\}$. There exists a term $a\in H_i$ such that either $xx\subseteq a$ or $yy\subseteq a$.*
(6) *Let $i\in\{1,2\}$. There exists a term $a\in H_i$ such that both $xx\subseteq a$ and $yy\subseteq a$.*
(7) *Let $i\in\{1,2\}$. For every term $t\notin X$ there exists a strongly nice term t' such that $(t,t')\in T_i$ and $xx\subseteq t'$ for all $x\in\mathbf{S}(t)$.*

(8) *Let $i \in \{1,2\}$. Let $t \notin X$ be a term and $x_1, \ldots, x_n$ be all (pairwise distinct) variables occurring in t. Then there exists a strongly nice term t' such that $(t,t') \in T_i$, $x_i x_i \subseteq t'$ for all i and $\nu_{x_1}(t') < \nu_{x_2}(t') < \ldots < \nu_{x_n}(t')$.*

(9) *Let $i \in \{1,2\}$. There exists a positive integer c such that for every term $t \notin X$ there is a positive integer N with the property that for every $k \geqslant 0$ there exists a term t' as in (8) of length $N + kc$.*

(10) *Let u, v be two terms of length $\geqslant 3$. Then $(u,v) \in T_1$ if and only if $(u,v) \in T_2$.*

(11) $H_1 \cap H_2 = \emptyset$.

(12) *Let $i \in \{1,2\}$ and $a \in H_i$. Then either $xx \subseteq a$ or $yy \subseteq a$.*

Proof. (1) There is a term a_i of length $\geqslant 3$ such that $(xy, a_i) \in T_i$. If a_i is not already strongly nice, then (without loss of generality) $a_i = b_i x$ for a term $b_i \notin X$. We have $(xy, \sigma^{x,y}_{b_i,x}(a_i)) \in T_i$ and the right-hand side of this equation is a strongly nice term.

(2) Let $t = uv$. By (1) there is a strongly nice term $b_i \in H_i$. We have $(t, \sigma^{x,y}_{u,v}(b)) \in T_i$ where the right-hand side is a strongly nice term.

(3) Suppose, for example, that $T_1 \subset T_2$. Take an equation $(u,v) \in T_2 - T_1$, so that $u, v \notin X$. By (2) there are nice terms u', v' with $(u,u') \in T_1$ and $(v,v') \in T_1$. Then $(u,u') \in T_2$ and $(v,v') \in T_2$. Since $(u,v) \in T_2$, we get $(u',v') \in T_2$. But (u',v') is nice, so $(u',v') \in T_1$. But then $(u,v) \in T_1$, a contradiction.

(4) Suppose, for example, that $H_1 \subseteq H_2$. Take a strongly nice term $a \in H_1$. If (u_1u_2, v_1v_2) is an arbitrary nontrivial equation from T_1, then $(u_1u_2, \sigma^{x,y}_{u_1,u_2}(a))$ and $(v_1v_2, \sigma^{x,y}_{v_1,v_2}(a))$ both belong to $T_1 \cap T_2$ and so the equation $(\sigma^{x,y}_{u_1,u_2}(a), \sigma^{x,y}_{v_1,v_2}(a))$ belongs to T_1; but it is a nice equation, so it also belongs to T_2 and we get $(u_1u_2, v_1v_2) \in T_2$. Now $T_1 \subseteq T_2$ is a contradiction with (3).

(5) If, for example, no term from H_1 contains either xx or yy as a subterm, then (xy, u) is a 1-special equation for all $u \in H_1$, so that all such equations belong to T_2 and $H_1 \subseteq H_2$, a contradiction with (4).

(6) If $a \in H_i$ where (for example) $xx \subseteq a$ and $yy \not\subseteq a$, then a contains a subterm yv for some term v; the term obtained from a by replacing yv with $\sigma^{x,y}_{y,v}(a)$ belongs to H_i and contains both xx and yy.

(7) Let a be as in (6) and t' be as in (2). Let $x \in \mathbf{S}(t)$. We have $xv \subseteq t'$ for some term v. The term obtained from t' by replacing xv with $\sigma^{x,y}_{x,v}(a)$ is strongly nice, T_i-related with t' and contains xx; it also contains yy for any other variable y whenever t' did, so that we can make this replacement for all variables in $\mathbf{S}(t)$ one by one.

(8) Let t' be as in (7). Take a term $a \in H_i$ and replace an occurrence of x_2x_2 in t', perhaps repeatedly, with $\sigma^{x,y}_{x_2,x_2}(a)$ until t' is transformed into a term with more occurrences of x_2 than of x_1. Then do the same with the variables $x_3, \ldots, x_n$.

(9) Take a term $a \in H_i$ and put $c = \lambda(a) - 2$. For a term t, take a term t' as in (8) and put $N = \lambda(t')$. If we replace a subterm $x_n x_n$ of t' (where x_n is the variable with the largest number of occurrences) with $\sigma^{x,y}_{x_n,x_n}(a)$, we obtain a term of length $N + c$ with the same properties of t' as in (8). We can do this k-times to obtain a term of length $N + kc$.

(10) Let $(u, v) \in T_1$; we are going to prove that $(u, v) \in T_2$.

By (2), there is a nice term w such that $(u, w) \in T_1$. We shall first prove that $(u, w) \in T_1 \cap T_2$. This is clear if u is nice. Otherwise, $u = u_1 z$ for a variable z not occurring in u_1. It follows easily from (9) that there are (perhaps very long) strongly nice terms u_1' and $w' = w_1' w_2'$, both with the properties of t' in (8), such that $\lambda(u_1') + 1 < \lambda(w')$, $\lambda(u_1') > \lambda(w_1')$ and $\lambda(u_1') > \lambda(w_2')$. The equation $(u, u_1' z)$ is modest and belongs to T_1, so $(u, u_1' z) \in T_2$. The equation (w, w') is nice and belongs to T_1, so $(w, w') \in T_2$. The equation $(u_1' z, w')$ is strictly parallel and belongs to T_1, so $(u_1' z, w') \in T_2$. We have obtained $(u, w) \in T_1 \cap T_2$.

Similarly, there exists a nice term $\overline{w}$ such that $(v, \overline{w}) \in T_1 \cap T_2$. Since $(u, v) \in T_1$, we have $(w, \overline{w}) \in T_1$. But $(w, \overline{w})$ is nice, so $(w, \overline{w}) \in T_2$. But then $(u, v) \in T_2$.

(11) If there is a term in $H_1 \cap H_2$, then it follows from (10) that for any equation (u, v) we have $(u, v) \in T_1$ if and only if $(u, v) \in T_2$, so that $T_1 = T_2$, a contradiction.

(12) If $a \in H_i$ and neither $xx \subseteq a$ nor $yy \subseteq a$, then (xy, a) is a 1-special equation, $(xy, a) \in T_1 \cap T_2$ and $a \in H_1 \cap H_2$, a contradiction with (11). $\square$

By a 2-special term we mean a term $t_1 t_2$ where $\mathbf{S}(t_1) = \{x\}$ and $\mathbf{S}(t_2) = \{y\}$. By a 2-special equation we mean an equation (xy, t) where t is a 2-special term of length $\geqslant 3$.

Lemma 7.3. *Let (xy, w) be a consequence of a 2-special equation (xy, t). Then w is 2-special.*

Proof. One can easily see that if (r, s) is an immediate consequence of a 2-special equation then r is 2-special if and only if s is 2-special. $\square$

Theorem 7.4. *Let (xy, a) be a 2-special equation. Then $C = \mathbf{Cn}(xy, a)$ is the only theory T such that $T \subseteq E_s$, the ideal theory generated by T equals I_{xy}, and every equation that is either strictly parallel or nice or modest or unary or 1-special belongs to T if and only if it is a consequence of (xy, a). Consequently, the set of 2-special equations is good.*

Proof. Let T be a theory with these properties. We have $a = u(x)v(y)$ for two unary terms u and v. Since I_{xy} is the ideal theory generated by T, there exists a term b of length $\geqslant 3$ such that $(xy, b) \in T$. We have $\mathbf{S}(b) = \{x, y\}$ and so we can write $b = b(x, y)$. Clearly, $(xy, u'v')$ for two some terms u', v' not belonging to X. Since $(xy, b(x, y)) \in T$, we have $(u'v', b(u', v')) \in T$. This equation is nice, so $(u'v', b(u', v')) \in C$. Then $(xy, b(u', v')) \in C$. By Lemma 7.3, $b(u', v')$ is a 2-special term. From this it follows that b is a 2-special term.

Let (U, V) be an arbitrary immediate consequence of (xy, a), so that $U = pqw_1 \dots w_n$ and $V = u(p)v(q)w_1 \dots w_n$ for some terms $p, q, w_1, \dots, w_n$ $(n \geqslant 0)$. We are going to prove that all 2-special subterms of U are C-equivalent with xy if and only if all 2-special subterms of V are C-equivalent with xy.

Let all 2-special subterms of U be C-equivalent with xy and let t be a 2-special subterm of V. If either $t \subseteq p$ or $t \subseteq q$ or $t \subseteq w_i$ for some i then t is a 2-special subterm of U, so that $(xy, t) \in C$. If $t \subseteq u(p)$ then (since t is a 2-special term) $t \subseteq p$. Similarly, if $t \subseteq v(q)$, then $t \subseteq q$. The only remaining case is $t = u(p)v(q)w_1 \dots w_i$ for some $i \geqslant 0$. Then t is C-equivalent with $pqw_1 \dots w_i$; this is a 2-special subterm of U and so it is C-equivalent with xy.

The converse implication can be proved similarly.

Take a variable $z \notin \{x, y\}$. The equation (xyz, bz) is modest and belongs to T, so it belongs to C and there exists an (xy, a)-derivation of (xyz, bz). The left-hand side of this equation contains a single 2-special subterm, namely, the term xy. It follows from what we have just proved that also every 2-special subterm of bz is C-equivalent with xy. But b is a 2-special subterm of bz, so $(xy, b) \in C$. Hence $(xy, b) \in C \cap T$. Now it follows from Lemma 7.2 (11) that $T = C$. $\square$

By a 3-special equation we mean an equation (xy, a) such that $\mathbf{S}(a) = \{x, y\}$ and $xy \subseteq a$.

Lemma 7.5. *Let (xy, a) be a 3-special equation and $C = \mathbf{Cn}(xy, a)$. Let z be a variable different from both x and y; let $A_0, A_1, \dots, A_n$ be an (xy, a)-derivation where $A_0 = xyz$; let u be a term such that $\mathbf{S}(u) = \{x, y\}$ and $zu \subseteq A_n$. Then there exists a unary term w such that $(u, w(xy)) \in C$.*

Proof. We proceed by induction on n. For $n = 0$ everything is clear. Let $n > 0$, $zu \subseteq A_n$ and $\mathbf{S}(u) = \{x, y\}$. If $zu \subseteq A_{n-1}$, we are done by induction. So, let $zu \nsubseteq A_{n-1}$. There are two cases.

Case 1: $A_{n-1} = a(r, s)p_1 \dots p_k$ and $A_n = rsp_1 \dots p_k$ for some terms r, s, $p_1, \dots, p_k$ $(k \geqslant 0)$. Then $zu \nsubseteq p_i$ for all i, $zu \nsubseteq rs$ (since $rs \subseteq a(r, s) \subseteq A_{n-1}$) and thus $zu = rsp_1 \dots p_j$ for some $j > 0$. Since z is a variable, $z = p_j$ and $u = rsp_1 \dots p_{j-1}$. For $u' = a(r, s)p_1 \dots p_{j-1}$ we have $(u, u') \in C$, $\mathbf{S}(u') = \{x, y\}$, $zu' = a(r, s)p_1 \dots p_j \subseteq A_{n-1}$ and so, by induction, $(u', w(xy)) \in C$ for a unary term w. But then $(u, w(x, y)) \in C$.

Case 2: $A_{n-1} = rsp_1 \dots p_k$ and $A_n = a(r, s)p_1 \dots p_k$ for some terms r, s, $p_1, \dots, p_k$. If $zu = a(r, s)p_1 \dots p_j$ for some $j > 0$, then we can proceed similarly as in

Case 1. Of course, $zu \not\subseteq p_j$ for all j. So, the only remaining case is $zu \subseteq a(r,s)$. We have $zu \not\subseteq r$ and $zu \not\subseteq s$, so that $zu = b(r,s)$ for a non-variable subterm b of a. Since z is a variable not contained in u, this is possible only if either $z = r$ and $u = c(s)$ or else $z = s$ and $u = c(r)$ for a unary term c. By symmetry, it is sufficient to consider the case $z = r$, $u = c(s)$. We have $rs \subseteq A_{n-1}$ where $r = z$ and $\mathbf{S}(s) = \{x, y\}$, so by induction $(s, w(xy)) \in C$ for a unary term w. But then $(c(s), c(w(xy))) \in C$, i.e., $(u, c(w)(xy)) \in C$ where $c(w)$ is a unary term. □

Theorem 7.6. *Let (xy, a) be a 3-special equation. Then $C = \mathbf{Cn}(xy, a)$ is the only theory T such that $T \subseteq E_s$, the ideal theory generated by T equals I_{xy}, and every equation that is either strictly parallel or nice or modest or unary or 1-special or 2-special belongs to T if and only if it is a consequence of (xy, a). Consequently, the set of 3-special equations is good.*

Proof. Let T be a theory with these properties. Since I_{xy} is the ideal theory generated by T, there exists a term t of length $\geqslant 3$ such that $(xy, t) \in T$; we have $\mathbf{S}(t) = \{x, y\}$. Take a variable $z \notin \{x, y\}$. The equation (xyz, tz) is modest and belongs to T, so it belongs to C. By Lemma 7.5, there is a unary term w such that $(t, w(xy)) \in C$.

Suppose $T \neq C$, so that by Lemma 7.2 (11) there is no term b except xy with $(xy, b) \in C \cap T$. In particular, $(xy, t) \notin C$ and thus w is not a variable. Also, t is not 2-special; since $\mathbf{S}(t) = \{x, y\}$, it follows that t is nice. Since w is not a variable, $w(xy)$ is also nice and thus $(t, w(xy))$ is a nice equation; since it belongs to C, we get $(t, w(xy)) \in T$. Hence $(xy, w(xy)) \in T$. But this is a 1-special equation, so $(xy, w(xy)) \in T \cap C$, a contradiction. □

By a 4-special term we mean a term a such that $\mathbf{S}(a) = \{x, y\}$, a is strongly nice and the following two conditions are satisfied:

(1) whenever $u \notin X$ is a proper subterm of a then $f(u) \neq g(a)$ for all substitutions f, g;
(2) whenever $f(a) = g(a)$ for two substitutions f, g then $f(xy) = g(xy)$.

By a 4-special equation we mean an equation (xy, a) such that a is a 4-special term.

Theorem 7.7. *Let (xy, a) be a 4-special equation. Then $C = \mathbf{Cn}(xy, a)$ is the only theory T such that $T \subseteq E_s$, the ideal theory generated by T equals I_{xy}, and every equation that is either strictly parallel or nice or modest or unary or 1-special belongs to T if and only if it is a consequence of (xy, a). Consequently, the set of 4-special equations is good.*

Proof. Let T be a theory with these properties; we need to prove that $T = C$.

Let $a = a_1a_2$ and write a as $a = a(x, y)$. Denote by A the set of the terms t such that $a \not\leq t$. For $u, v \in A$ define a term $u \circ v \in A$ by induction on the length of uv as follows:

$$u \circ v = \begin{cases} uv & \text{if } \ uv \in A, \\ p \circ q & \text{if } \ uv = a(p, q) \text{ for two terms } p \text{ and } q. \end{cases}$$

It follows from (2) that $\circ$ is a correctly defined commutative binary operation on A.

Let h be a homomorphism of the groupoid T of all terms into the groupoid $(A, \circ)$; put $p = h(x)$ and $q = h(y)$. Let us prove by induction on the length of u that if u is a proper subterm of a then $h(u) = u(p, q)$. This is clear if $u \in \{x, y\}$. Now let $u = u_1u_2$. Then $h(u) = h(u_1) \circ h(u_2) = u_1(p, q) \circ u_2(p, q)$ by the induction assumption. It follows from (1) that $u(p, q) \in A$, so that $h(u) = u_1(p, q) \circ u_2(p, q) = u_1(p, q)u_2(p, q) = u(p, q)$ as desired.

In particular, we have $h(a) = h(a_1) \circ h(a_2) = a_1(p, q) \circ a_2(p, q) = p \circ q = h(xy)$. This means that the groupoid $(A, \circ)$ satisfies the equation (xy, a).

Denote by H the extension of the identity on X to a homomorphism of the groupoid T of all terms onto the groupoid $(A, \circ)$. Clearly, $H(u) = u$ for all $u \in A$. Let us prove by induction on the length of a term t that $(t, H(t)) \in C$. This is clear if $t \in X$. Now let $t = t_1t_2$. We have $H(t) = h(t_1) \circ h(t_2)$ where, by the induction assumption, $(t_1, h(t_1)) \in C$ and $(t_2, h(t_2)) \in C$. If $H(t_1) \circ h(t_2) = H(t_1)H(t_2)$, we get $(H(t), t_1t_2) \in C$ as desired. In the opposite case we have $H(t_1)H(t_2) = a(p, q)$ for some $p, q \in A$, and $H(t) = p \circ q$. Since (clearly) pq is shorter than t, by the induction assumption we have $(pq, p \circ q) \in C$. Of course, $(pq, a(p, q)) \in C$; since $a(p, q) = H(t_1)H(t_2)$ and $(H(t_i), t_i) \in C$, we get $(H(t), t) \in C$.

From this it follows that for any terms t and u, $(t, u) \in C$ if and only if $H(t) = H(u)$.

Since I_{xy} is the ideal theory generated by T, there exists a term b of length $\geqslant 3$ such that $(xy, b) \in T$. Take a variable $z \notin \{x, y\}$. The modest equation (xyz, tz) belongs to T, so that it also belongs to C. Consequently, $H(xyz) = H(tz)$. But $H(xyz) = xyz$ and (since a is strictly nice) $H(tz) = H(t)z$. We get $xyz = H(t)z$, so that $xy = H(t)$ and $(xy, t) \in C \cap T$. By Lemma 7.2 we get $T = C$. □

By a 5-special equation we mean an equation (xy, a) such that (xy, a) is not 2-special, $\mathbf{S}(a) = \{x, y\}$ and $xy \not\subseteq a$.

Lemma 7.8. *Let (xy, a) be a 5-special equation. Let t be a term such that $\mathbf{S}(t) = \{x, y\}$ and $xy \not\subseteq t$. Then $(t, uv) \in \mathbf{Cn}(xy, a)$ for two terms u, v such that $\mathbf{S}(u) = \mathbf{S}(v) = \{x, y\}$ and $xy \not\subseteq uv$.*

P r o o f. Since a is not 2-special, without loss of generality $a = a_1a_2$ where $\mathbf{S}(a_2) = \{x, y\}$ and $x \in \mathbf{S}(a_1)$. Let $t = t_1t_2$. We can assume that at least one of the terms t_1, t_2 contains both x and y, because otherwise t could be replaced with $a(t_1, t_2)$. Without loss of generality, $\mathbf{S}(t_2) = \{x, y\}$. Then we can take $uv = a(t_2, t_1)$. $\square$

Lemma 7.9. *Let (xy, a) be a 5-special equation. Let t be a term such that $\mathbf{S}(t) = \{x, y\}$ and $xy \not\subseteq t$. Then $(t, t') \in \mathbf{Cn}(xy, a)$ for a term t' such that $xy \not\subseteq t'$ and t' has a subterm uv with $\mathbf{S}(u) = \{x\}$, $\mathbf{S}(v) = \{y\}$, $u \neq x$ and $v \neq y$.*

Proof. Let w be a minimal subterm of t with $\mathbf{S}(w) = \{x, y\}$. Then $w = w_1 w_2$ where $\mathbf{S}(w_1) = \{x\}$ and $\mathbf{S}(w_2) = \{y\}$. Also, let b be a minimal subterm of a with $\mathbf{S}(b) = \{x, y\}$. Then $b = b_1 b_2$ where $\mathbf{S}(b_1) = \{x\}$ and $\mathbf{S}(b_2) = \{y\}$. Without loss of generality, $b_2 \neq y$. If $w_1 = x$ then $w_2 \neq y$ and we can replace the subterm w of t with the subterm $a(w_2, w_1) \supseteq b_1(w_2) b_2(w_1)$. If $w_2 = y$ then $w_1 \neq x$ and we can replace the subterm w of t with the subterm $a(w_1, w_2) \supseteq b_1(w_1) b_2(w_2)$. $\square$

In the following we are going to prove that every 5-special equation has at least one 4-special consequence. Let (xy, a) be a 5-special equation. It follows from Lemma 7.8 and Lemma 7.9 that we can assume that $a = a_1 a_2 \cdot a_3 a_4$ where

(1) for $j = 1, 2, 3, 4$, a_j contains a subterm $U_j V_j$ with $\mathbf{S}(U_j) = \{x\}$, $\mathbf{S}(V_j) = \{y\}$, $U_j \neq x$, $V_j \neq y$;
(2) a_2 is essentially longer than $a_1 a_3 a_4$.

Denote by $\equiv$ the theory based on (xy, a).

Denote by α the term $a(x, x)$ and write it as $\alpha = xx\alpha_1 \dots \alpha_k$ ($k \geqslant 1$). Of course, $\alpha \equiv xx$. Put $\alpha^0 = xx$ and $\alpha^{i+1} = \alpha^i \alpha_1 \dots \alpha_k$, so that $\alpha^i \equiv xx$ for all $i \geqslant 0$. Denote by $\beta, \beta_1, \dots, \beta_k, \beta^i$ the terms $\alpha, \alpha_1, \alpha_k, \alpha^i$ with x replaced by y. Hence $\beta^i \equiv yy$ for all $i \geqslant 0$.

Put $N = |a| = |\alpha| = |\beta|$.

For $j = 1, \dots, 4$ and any $i \geqslant 0$ denote by U_j^i the term obtained from U_j by replacing one occurrence of xx with α^i, denote by V_j^i the term obtained from V_j by replacing one occurrence of yy with β^i, and denote by a_j^i the term obtained from a_j by replacing one occurrence of $U_j V_j$ with $U_j^i V_j^i$.

Let us take a positive integer m such that a_2^m is essentially longer than a. Put $M = |a_1 a_2^m \cdot a_3 a_4|$.

Lemma 7.10. *For any $i, j \geqslant 0$, every unary subterm of $a_1^i a_2^m \cdot a_3^j a_4$ that is not a variable is a product of two terms, at least one of which is of length $< N$.*

Proof. This is obvious. $\square$

Lemma 7.11. *Let i, j be such that $U_3^j V_3^j$ is essentially longer than $a_1^i a_2^m$ and a_1^i is essentially longer than a_2^m. Then there are no terms p, q with $a_1^i a_2^m(p, q) \subseteq a_3^j a_4(p, q)$.*

P r o o f. Suppose $a_1^i a_2^m(p,q) \subseteq a_3^j a_4(p,q)$. Clearly, $a_1^i a_2^m(p,q)$ is a subterm of either $U_3^j(p)$ or $V_3^j(q)$; without loss of generality, it is sufficient to consider the case $a_1^i a_2^m(p,q) \subseteq U_3^j(p)$. Since $a_1^i(p,q)$ is longer than $a_2^m(p,q)$, it follows from Lemma 7.10 that $a_2^m(p,q) = w(p)$ for a subterm w of U_3^j and $|w| < N$. Now

$$N|p| \leqslant \nu_x(a_2^m)|p| < |a_2^m(p,q)| = |w(p)| < N|p|,$$

a contradiction. □

Lemma 7.12. *Let $i \geqslant M^2$ and u be a unary term of length > 1 such that whenever $w_1 w_2 \subseteq u$ then either $|w_1| < N$ or $|w_2| < N$. Then there are no terms p, q, r with either $a_1^i a_2^m(p,q) = u(r)$ or $a_3^i a_4(p,q) = u(r)$.*

P r o o f. Suppose $a_1^i a_2^m(p,q) = u(r)$. We can write u as $u = u_1 u_2$ where $a_1^i(p,q) = u_1(r)$ and $a_2^m(p,q) = u_2(r)$. Since $a_1^i(p,q)$ is longer than $a_2^m(p,q)$, u_1 is longer than u_2 and hence $|u_2| < N$ by Lemma 7.10.

We have $|U_1^i(p)V_1^i(q)| > i|p|+i|q| \geqslant M^2(|p|+|q|)$. On the other hand, the length of the rest of $a_1^i a_2^m(p,q)$ is less than $M(|p|+|q|)$. Hence the length of $U_1^i(p)V_1^i(q)$ makes more than two-thirds (in particular, more than a half) of the length of $a_1^i a_2^m(p,q)$. From this it follows that $U_1^i(p)V_1^i(p)$ is not a subterm of r, so that $U_1^i(p) = w_1(r)$ and $V_1^i(q) = w_2(r)$ for a subterm $w_1 w_2$ of u.

Suppose $w_1, w_2 \notin X$. We can write w_1 as $w_1 = w_{11}w_{12}$ and U_1^i as $U_1^i = PQ$ where $P(p) = w_{11}(r)$ and $Q(p) = w_{12}(r)$. Without loss of generality, P is longer than Q; but then $|P| > i \geqslant M^2$ and $|Q| < N$. So, $|P| > N|Q|$, $|P(p)| > N|Q(p)|$, $|w_{11}(r)| > N|w_{12}(r)|$, and hence $|w_1| > N$. Similarly, $|w_2| > N$. This is a contradiction, since $w_1 w_2 \subseteq u$.

So, without loss of generality, $w_1 = x$ and $U_1^i(p) = r$. Since the length of $U_1^i(p)V_1^i(q)$ makes more than two-thirds of the length of $a_1^i a_2^m(p,q)$, we cannot have $V_1^i(q) = r$; hence $w_2 \notin X$. We can write $w_2 = w_{21}w_{22}$ and $V_1^i = RS$ where $|R| > i \geqslant M^2$ and $|S| < N$. Without loss of generality, $w_{22}(r) = S(q)$. Then $|w_{22}| < |w_{21}|$, so $|w_{22}| < N$. From this it follows that either $|r| = c|q|$ or $|q| = c|r|$ for some positive integer $c < N$. On the other hand, $|r| = d|p|$ for some $d > M^2$, since $U_1^i(p) = r$ implies that $|r|$ is a multiple of $|p|$ and we have $|U_1^i| > i \geqslant M^2$.

Put $e = \nu_x(a_2^m)$ and $f = \nu_y(a_2^m)$, so that $1 \leqslant e, f \leqslant N$. Then $|u_2||r| = |u_2(r)| = |a_2^m(p,q)| = e|p| + f|q|$.

If $|r| = c|q|$ then $c|u_2||q| = e|p| + f|q|$ means that $e|p|$ is divisible by $|q|$, so that $|q| \leqslant e|p|$, $|r| = c|q| \leqslant ce|p| < M^2|p|$ (since $c, e < N$), a contradiction, since $|r| = d|p|$ where $d > M^2$.

If $|q| = c|r|$ then $|u_2||r| = e|p| + fc|r|$, so that $e|p|$ is divisible by $|r|$ and hence $|r| \leqslant e|p|$ where $e < N$, a contradiction, since $|r| = d|p|$ where $d > M^2$.

This proves that we cannot have $a_1^i a_2^m(p,q) = u(r)$. Quite similarly, we cannot have $a_3^i a_4(p,q) = u(r)$. $\square$

Lemma 7.13. *There exist positive integers i, j with these properties:*

(1) $i > M^2$;
(2) $\nu_x(U_1^i V_1^i) > M^3$ *and* $\nu_y(U_1^i V_1^i) > M^3$;
(3) a_1^i *is essentially longer than* a_2^m;
(4) $U_3^j V_3^j$ *is essentially longer than* $a_1^i a_2^m$;
(5) $M\nu_x(a_1^i a_2) < \nu_x(a_3^j a_4) < M^2 \nu_x(a_1^i a_2^m)$ *and* $M\nu_y(a_1^i a_2) < \nu_y(a_3^j a_4) < M^2 \nu_y(a_1^i a_2^m)$.

P r o o f. One can take i so large that (1), (2) and (3) are satisfied and, moreover, such that (4) and (5) are satisfied if we take $j = Mi + M^2$. (In order to check this, it is useful to realize that if t is any of the terms a_1^k, $a_1^k a_2^m$, U_1^k, U_3^k, $a_3^k a_4$ for some k, then $\nu_x(t) = (N-2)k + d$ and $\nu_y(t) = (N-2)k + d'$ for some $0 \leqslant d, d' < M$.) $\square$

Lemma 7.14. *Let $A = a_1^i a_2^m \cdot a_3^j a_4$ where i, j satisfy the five conditions of Lemma* 7.13, *and let $u \notin X$ be a proper subterm of A. Then there are no terms p, q, r, s with $A(p,q) = u(r,s)$.*

P r o o f. Suppose $A(p,q) = u(r,s)$. We can write u as $u = u_1 u_2$ where $a_1^i a_2^m(p,q) = u_1(r,s)$ and $a_3^j a_4(p,q) = u_2(r,s)$.

Suppose that u_1 is unary. Then, by Lemma 7.12, $u_1 \in X$. If also u_2 is unary then similarly $u_2 \in X$, but clearly $u_1 \neq u_2$, so that xy is a subterm of A, a contradiction. So, $u_1 \in X$ and $\mathbf{S}(u_2) = \{x, y\}$. Then $a_1^i a_2^m(p,q) \subseteq a_3^j a_4(p,q)$, a contradiction with Lemma 7.11.

This proves $\mathbf{S}(u_1) = \{x, y\}$. Similarly, $\mathbf{S}(u_2) = \{x, y\}$ (in this case, instead of an application of Lemma 7.11 we can use the fact that $a_3^j a_4(p,q)$ is longer than $a_1^i a_2^m(p,q)$).

If $|u_2| \leqslant M$ then $|a_3^j a_4(p,q)| = |u_2(r,s)| < M(|r| + |s|) < M|u_1(r,s)| = M|a_1^i a_2^m(p,q)|$, contradicting Lemma 7.13 (5). Hence $|u_2| > M$. Since u_2 contains both x and y, this is possible only if either $U_1^i V_1^i$ or $U_3^j V_3^j$ is a subterm of u_2. Also, since $u_1 u_2 \subset A$, we have $|u_1| < M$ by Lemma 7.10. Since $\nu_x(u_2) \geqslant \nu_x(U_1^i V_1^i) > M^3 > M^2 \nu_x(u_1)$ (and similarly for y), we have $|u_2(r,s)| > M^2 |u_1(r,s)|$, i.e., $|a_3^j a_4(p,q)| > M^2 |a_1^i a_2^m(p,q)|$. On the other hand, it follows from Lemma 7.13 (5) that $|a_3^j a_4(p,q)| < M^2 |a_1^i a_2^m(p,q)|$ and we have obtained the desired contradiction. $\square$

Lemma 7.15. *Let $A = a_1^i a_2^m \cdot a_3^j a_4$ be as in Lemma* 7.14 *and let p, q, r, s be terms such that $A(p,q) = A(r,s)$. Then $pq = rs$.*

P r o o f. Clearly, $U_1^i(p)V_1^i(q) = U_1^i(r)V_1^i(s)$. If $U_1^i(p) = U_1^i(r)$ and $V_1^i(q) = V_1^i(s)$, then $p = r$ and $q = s$. The other case is $U_1^i(p) = V_1^i(s)$ and $V_1^i(q) = U_1^i(r)$.

Suppose $p \neq s$. Then these two terms must be of different lengths, and it is possible to consider, without loss of generality, only the case $|p| > |s|$. Clearly, $|p| \geqslant 2|s|$. We have $|U_1^i| = 2 + i(N-2) + c$ and $|V_1^i| = 2 + i(N-2) + d$ for some $0 \leqslant c, d < N$, so that

$$(2 + i(N-2) + c)|p| = |U_1^i(p)| = |V_1^i(s)| = (2 + i(N-2) + d)|s|$$

from which we get

$$2(2 + i(N-2) + c)|s| \leqslant (2 + i(N-2) + d)|s|$$

and consequently $i < N$. This contradiction proves $p = s$, and $q = r$ can be proved similarly. □

Theorem 7.16. *The set of xy-equations is good.*

P r o o f. It follows from the previous lemmas that the set of 5-special equations is good. The set of xy-equations is the union of the five sets of equations considered and proved to be good in this section. □

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