GEOMETRIES

Jaroslav Ježek

First edition, April 2008

Contents

CHAPTER 1

GEOMETRIC LATTICES

1. Semimodular lattices

By a *semimodular lattice* we mean a lattice satisfying the upper covering condition: for any elements a, b, c of the lattice, $a \prec b$ implies that either $a \vee c \prec b \vee c$ or $a \vee c = b \vee c$.

1.1. Theorem. *A lattice L is semimodular if and only if for any $a, b \in L$, $a \wedge b \prec a$ implies $b \prec a \vee b$.*

Proof. The direct implication is clear. For the converse, let $a \prec b$. If $a \vee c \geq b$ then $a \vee c = b \vee c$. Otherwise, $(a \vee c) \wedge b = a$ and hence $a \vee c \prec (a \vee c) \vee b = b \vee c$. □

1.2. Example. The lattice L in Fig. 1 is an example of a semimodular lattice that is not modular. The subset $\{0, a, c, e, 1\}$ is a sublattice of L isomorphic to the pentagon. Consequently, a sublattice of a semimodular lattice need not be semimodular. Also, the dual of L is not semimodular.

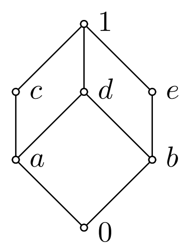

Fig. 1

1.3. Theorem. *Any interval of a semimodular lattice is a semimodular lattice.*

Proof. It is evident. □

By the length of a finite chain $a_0 < a_1 < \cdots < a_n$ we mean the number n (i.e., the cardinality of the chain decreased by 1). A lattice is said to be of finite length n if it contains a subchain of length n and its every subchain is (finite and) of length at most n.

1.4. Theorem. *Let L be a lattice of finite length. Then L is semimodular if and only if for any $a, b \in L$, if $a \wedge b \prec a$ and $a \wedge b \prec b$ then $a \prec a \vee b$ and $b \prec a \vee b$.*

PROOF. The direct implication is clear. In order to prove the converse, let $a \wedge b \prec a$; by 1.1, it is sufficient to prove $b \prec a \vee b$. There exists a chain $a \wedge b = a_0 \prec a_1 \prec \cdots \prec a_k = b$. By induction on i it is easy to prove $a_i \prec a \vee a_i$. In particular, $b = a_k \prec a \vee a_k = a \vee b$. □

1.5. THEOREM. *Let L be a semimodular lattice of finite length. Then any two maximal subchains of L are of the same length.*

PROOF. By induction on n, we are going to prove that if L contains a maximal subchain of length n, then every maximal subchain of L is of length n. For $n \leq 1$, this is obvious. Let $n \geq 2$. Let $a_0 \prec a_1 \prec \cdots \prec a_n$ be a maximal subchain and let $b_0 \prec b_1 \prec \cdots \prec b_m$ be another maximal subchain. If $a_1 = b_1$, then $a_1 \prec \cdots \prec a_n$ and $b_1 \prec \cdots \prec b_m$ are two maximal subchains in the interval from a_1 to a_n, so that they are of the same length by induction and we get $n = m$. Let $a_1 \neq b_1$. Clearly, the two elements a_1 and b_1 are incomparable. Since L is semimodular, we have $a_1 \prec a_1 \vee b_1$ and $b_1 \prec a_1 \vee b_1$. There exists a maximal subchain C in the interval between $a_1 \vee b_1$ and $a_n = b_m$. Since both $a_1 \prec \cdots \prec a_n$ and $\{a_1\} \cup C$ are maximal subchains in the interval from a_1 to a_n, by induction they are of the same length and hence C is of length $n - 2$. Quite similarly, C is of length $m - 2$. Hence $n = m$. □

For any element a of a lattice L with zero, the height of a (in L) is the length of the longest finite subchain of the interval $[0, a]$ (if such a longest finite subchain exists; if not, the height is ∞). The mapping, assigning the height of A to any element a of L, is called the height function on L. If L is a lattice of finite length then every element of L is of finite height.

1.6. THEOREM. *Let L be a semimodular lattice of finite length and let h be the height function on L. Then for any two elements a, b of L we have $h(a) + h(b) \geq h(a \wedge b) + h(a \vee b)$.*

PROOF. Denote by k the length of the interval $[b, a \vee b]$. There exist elements $a \wedge b = a_0 \prec a_1 \prec \cdots \prec a_k = b$. It is easy to see that for every $i < k$, either $a \vee a_i = a \vee a_{i+1}$ or $a \vee a_i \prec a \vee a_{i+1}$. Thus the length of $[a, a \vee b]$ is at most k. We have $h(b) = h(a \wedge b) + k$ and $h(a \vee b) \leq h(a) + k$. From this the inequality follows. □

Let L be a lattice with the least element 0. A subset I of $L - \{0\}$ is said to be independent if for any two finite subsets X and Y of I, $\bigvee X \wedge \bigvee Y = \bigvee (X \cup Y)$.

1.7. THEOREM. *Let L be a semimodular lattice and $a_1, \dots, a_n$ be n different atoms of L. The following three conditions are equivalent:*

(1) *the set $\{a_1, \dots, a_n\}$ is independent*
(2) $(a_1 \vee \cdots \vee a_i) \wedge a_{i+1} = 0$ *for* $i = 1, \dots, n-1$
(3) *the height of* $a_1 \vee \cdots \vee a_n$ *is* n

PROOF. Denote by h the height function on L. Obviously, (1) implies (2). (2) implies (3): Let us prove by induction on i that $h(a_1 \vee \cdots \vee a_i) = i$. For $i = 1$ it is obvious (since a_1 is an atom). If the statement is true for an $i < n$ then $a_1 \vee \cdots \vee a_i \prec a_1 \vee \cdots \vee a_{i+1}$ and thus $h(a_1 \vee \cdots \vee a_{i+1}) = h(a_1 \vee \cdots \vee a_i) + 1 = i + 1$.

(3) implies (1): It is easy to see that $h(\bigvee X) = |X|$ for any subset X of $\{a_1, \dots, a_n\}$. Let X and Y be two subsets of $\{a_1, \dots, a_n\}$. By 1.6 we have $h(\bigvee X) + h(\bigvee Y) \geq h(\bigvee X \wedge \bigvee Y) + h(\bigvee X \vee \bigvee Y)$, i.e., $|X| + |Y| \geq h(\bigvee X \wedge \bigvee Y) + |X \cup Y|$. Thus $h(\bigvee X \wedge \bigvee Y) \leq |X| + |Y| - |X \cup Y| = |X \cap Y|$. On the other hand, $\bigvee X \wedge \bigvee Y \geq \bigvee (X \cap Y)$ and thus $h(\bigvee X \wedge \bigvee Y) \geq h(\bigvee (X \cap Y)) = |X \cap Y|$. Thus $h(\bigvee X \wedge \bigvee Y) = h(\bigvee (X \cap Y))$; since the two elements are comparable, we get $\bigvee X \wedge \bigvee Y = \bigvee (X \cap Y)$. □

Let L be a semimodular lattice and A be a set of atoms of L. A subset U of A is said to span A if for every $a \in A$ there exists a finite subset V of U with $a \leq \bigvee V$.

1.8. THEOREM. *Let L be a semimodular lattice, A be a set of atoms of L and I, U be two subsets of A such that I is independent and U spans A. Then there is an independent subset J such that $I \subseteq J \subseteq U$ and A is spanned by J.*

PROOF. Obviously, the union of a chain of independent subsets of U is an independent subset of U. It follows by Zorn's lemma that there is a maximal independent subset J of U containing I. It is sufficient to show that J spans U. Let $a \in U - J$. Then $J \cup \{a\}$ is not independent and thus there exists a finite subset $\{a_1, \dots, a_n\}$ of J such that $\{a_1, \dots, a_n, a\}$ is not independent. It follows from 1.7 that $(a_1 \vee \cdots \vee a_n) \wedge a \neq 0$; since a is an atom, we get $(a_1 \vee \cdots \vee a_n) \wedge a = a$ and thus $a \leq a_1 \vee \cdots \vee a_n$. □

2. Geometries and geometric lattices

By a closure space we mean an ordered pair $\langle A, G \rangle$ such that A is a set and G is a set of subsets of A, closed under arbitrary intersections (in partricular, $A \in G$). The elements of A are called points of $\langle A, G \rangle$ and the elements of G are called subspaces of $\langle A, G \rangle$. For a subset X of A, the least subspace containing X (the intersection of all subspaces containing X) is called the closure of X in$\langle A, G \rangle$; if G is clear from the context, it is denoted by $\overline{X}$.

A closure space $\langle A, G \rangle$ is called algebraic if it satisfies the following condition: whenever $a \in \overline{X}$ for a subset X of A then $a \in \overline{Y}$ for a finite subset Y of X. It is easy to see that a lattice is algebraic if and only if it is isomorphic to the lattice (with respect to inclusion) of all subspaces of an algebraic closure space. (Recall that an element a of a complete lattice L is said to be *compact* if whenever $a \leq \bigvee S$ for a subset S of L then $a \leq \bigvee S'$ for a finite subset S' of S; a complete lattice L is said to be *algebraic* if every element of L is the join of a set of compact elements.)

By a geometry we mean an algebraic closure space $\langle A, G\rangle$ such that the empty set is a subspace, every one-element subset of A is a subspace, and whenever $a \in \overline{X \cup \{b\}}$ (where $X \subseteq A$ and $a, b \in A$) but $a \notin \overline{X}$ then $b \in \overline{X \cup \{a\}}$.

By a geometric lattice we mean a lattice that is isomorphic to the lattice of all subspaces of some geometry.

2.1. Theorem. *The following are equivalent for a lattice L:*

1. *L is geometric*
2. *L is algebraic, semimodular, and the compact elements of L are exactly the finite joins of atoms of L*
3. *L is complete, semimodular, atomistic (i.e., every element of L is the join of a set of atoms) and all atoms of L are compact.*

Proof. (1) implies (2): Let L be the lattice of subspaces of a geometry $\langle A, G\rangle$. Then L is algebraic and thus compact elements of L are exactly the subspaces $\overline{X}$ with X finite; since atoms of L are just the one-element subspaces, it follows that compact elements are exactly the finite joins of atoms. It remains to prove that L is semimodular.

Let us first prove that if X is a subspace and a is a point not belonging to X then $X \prec \overline{X \cup \{a\}}$. Let Y be a subspace such that $X \subset Y \subseteq \overline{X \cup \{a\}}$. There exists an element $b \in Y - X$. Since $b \in \overline{X \cup \{a\}}$, we have $a \in \overline{X \cup \{b\}} \subseteq Y$ and hence $Y = \overline{X \cup \{a\}}$.

Let X, Y be two subspaces such that $X \prec Y$ and let Z be a subspace. Take a point $a \in Y - X$, so that $Y = \overline{X \cup \{a\}}$. We have $X \vee Z = \overline{X \cup Z}$ and $Y \vee Z = \overline{Y \cup Z} = \overline{X \cup Z \cup \{a\}}$. If $a \in \overline{X \cup Z}$ then $X \vee Z = Y \vee Z$. Otherwise, $X \vee Z \prec Y \vee Z$ according to the above proved claim.

(2) implies (3): This is obvious.

(3) implies (1): Clearly, L is algebraic. Denote by A the set of all atoms of L and by G the set of the subsets X of A such that $X = \{a \in A : a \leq \bigvee X\}$. Clearly, $\langle A, G\rangle$ is an algebraic closure space and every at most one-element subset of A is a subspace. Let a point $a \in \overline{X \cup \{b\}}$ but $a \notin \overline{X}$. So, $a \leq \bigvee(X \cup \{b\})$ and $a \nleq \bigvee X$. Since b is an atom, by semimodularity $\bigvee X \prec \bigvee X \vee b = \bigvee(X \cup \{b\})$. We have $\bigvee X < \bigvee(X \cup \{a\}) \leq \bigvee(X \cup \{b\})$ and thus $\bigvee(X \cup \{a\}) = \bigvee(X \cup \{b\})$, so that $b \leq \bigvee(X \cup \{a\})$ and thus $b \in \overline{X \cup \{a\}}$. We have proved that $\langle A, G\rangle$ is a geometry.

Since every element of L is a join of atoms, for two subspaces X, Y we have $X \subseteq Y$ if and only if $\bigvee X \leq \bigvee Y$. It follows that $X \mapsto \bigvee X$ is an isomorphism of the lattice of subspaces of $\langle A, G\rangle$ onto the lattice L. □

Let L be a geometric lattice. Atoms of L are called points of L. Elements of height 2 are called lines of L, and elements of height 3 are called planes of L. Of course, any line contains (i.e., is above) at least two points and is the join of its any two distinct points.

A geometric lattice is said to be finite dimensional if it is of finite length; in that case, the length of the lattice decreased by 1 is called its dimension.

A small, finite geometry can be often most conveniently described by drawing its picture where points are represented by small circles and lines are represented by straight lines or smooth curves without cusps. For example, the geometry pictured in Fig. 2 has seven points and three lines, each having exactly three points; the geometry pictured in Fig. 3 has six points and seven lines, four of them having three points and three of them having two points.

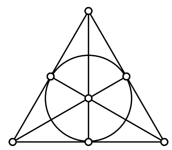

Fig. 2

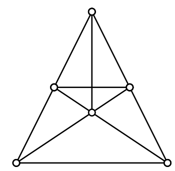

Fig. 3

2.2. THEOREM. *The lattice of all equivalences on any given set X is geometric.*

PROOF. Denote the lattice by L. The atoms of L are exactly the equivalences with precisely one non-singleton block, where this block has precisely two elements. Clearly, the atoms are compact and every equivalence is the join of a set of atoms. For two equivalences r and s we have $r \prec s$ if and only if one block of s is the union of two different blocks of r and all the other blocks of s are blocks of r. From this it easily follows that L is semimodular. □

It is well known that the lattice of all equivalences on any given set is simple and every lattice can be embedded into one such lattice.

2.3. THEOREM. *Any interval of a geometric lattice is a geometric lattice.*

PROOF. Let $[a, b]$ be an interval of a geometric lattice L, so that $[a, b]$ is an algebraic lattice. It is easy to see that an element of $[a, b]$ is an atom of $[a, b]$ if and only if it can be expressed as $a \vee c$ for an atom c of L such that $c \not\leq a$ and $c \leq b$. The rest is easy. □

2.4. LEMMA. *Let a, b, c be three atoms of a geometric lattice L. If $a \leq b \vee c$ and $a \neq b$ then $c \leq a \vee b$.*

PROOF. It is easy. □

2.5. THEOREM. *Let L be a geometric lattice. Then every element of L is the join of an independent set of atoms of L. The set F of elements of L of finite height is an ideal of L; F is a semimodular lattice, every element of F is the join of a finite number of atoms, and L is isomorphic to the ideal lattice of F.*

PROOF. The first statement follows easily from 1.8. Denote by h the height function on L. If $a \in F$, $b \in L$ and $b \leq a$ then $h(b) \leq h(a)$, so that $b \in F$. If $a, b \in F$ then $h(a \vee b) \leq h(a) + h(b) - h(a \wedge b)$ by 1.6, so that $a \vee b \in F$. We have proved that F is an ideal. The rest is easy. □

2.6. Theorem. *Every geometric lattice is relatively complemented.*

Proof. By 2.3, it is sufficient to prove that every geometric lattice L is complemented. Let $a \in L$. By 2.5, a is the join of an independent set I of atoms of L. Denote by K the set of all atoms not less or equal a. There exists a maximal independent set of atoms J with $I \subseteq J \subseteq I \cup K$, and by 1.8 we have $\bigvee J = 1$ (the largest element of L). Put $b = \bigvee(J - I)$, so that $a \vee b = 1$. Suppose $a \wedge b \neq 0$. Then there exists an atom $c \leq a \wedge b$. Since c is a compact element, it follows from $c \leq \bigvee I$ that $c \leq \bigvee I'$ for a finite subset I' of I, and it follows from $c \leq \bigvee(J - I)$ that $c \leq \bigvee J'$ for a finite subset J' of $J - I$. Since J is independent, $c \leq \bigvee I' \wedge \bigvee J' = \bigvee(I' \cap J') = \bigvee \emptyset = 0$, a contradiction. Thus $a \wedge b = 0$ and b is a complement of a in L. □

Two elements of a bounded lattice (i.e., lattice with 0 and 1) are called perspective if they have a common complement.

2.7. Lemma. *Let L be a geometric lattice and a, b be two elements of L. If there exists an element x such that $a \wedge x = b \wedge x = 0$ and $a \vee x = b \vee y$ then a, b are perspective. If a, b are two perspective elements of finite height then there exists an element x' of finite height such that $a \wedge x' = b \wedge x' = 0$ and $a \vee x' = b \vee x'$.*

Proof. Let $a \wedge x = b \wedge x = 0$ and $a \vee x = b \vee y$. By 2.6 there exists a relative complement y of $a \vee x = b \vee x$ in $[x, 1]$. We have $a \wedge y = a \wedge (a \vee x) \wedge y = a \wedge x = 0$ and $a \vee y = a \vee x \vee y = 1$, so that y is a complement of a. Similarly, y is a complement of b.

Let x be a complement of both a and b, where a and b are of finite height. Since $a \leq b \vee x$ and a is compact, there exists an element $x_1 \leq x$ of finite height such that $a \leq b \vee x_1$. Similarly, there exists an element $x_2 \leq x$ such that $b \leq a \vee x_2$. It is sufficient to put $x' = x_1 \vee x_2$. □

2.8. Theorem. *Let L be a geometric lattice. The relation of perspectivity is an equivalence on the set of atoms of L.*

Proof. The relation is reflexive by 2.6. Clearly, it is symmetric. It remains to prove that it is transitive. Denote by h the height function on L. Let a, b, c be three atoms of L such that a, b are perspective and b, c are perspective. By 2.7 there are elements x, y of finite height such that $a \wedge x = b \wedge x = 0$, $a \vee x = b \vee x$, $b \wedge y = c \wedge y = 0$ and $b \vee y = c \vee y$; take x to be of the minimal possible height. Denote by m the height of x and by n the height of y. There is an independent set $\{s_1, \dots, s_m\}$ of atoms in $[0, x]$ and an independent set $\{t_1, \dots, t_n\}$ of atoms in $[0, y]$. For $i = 1, \dots, m$ put $x_i = s_1 \vee \dots \vee s_{i-1} \vee s_{i+1} \vee \dots \vee s_m$.

Suppose $a \leq x_i \vee b$ for some i. Then $x_i \vee a \leq x_i \vee b$, $h(x_i) = m - 1$, $h(x_i \vee a) = h(x_i \vee b) = m$ and thus $x_i \vee a = x_i \vee b$, a contradiction with the minimality of m.

Thus $a \not\leq x_i \vee b$. Hence $h(a \vee b \vee x_i) = 2 + m - 1 = m + 1$. We get $a \vee b \vee x_i = a \vee x = b \vee x$ for all i.

It follows from 1.7 that $\{a, s_1, \ldots, s_m\}$ is an independent set of atoms. By 1.8 there exists a subset $\{t'_1, \ldots, t'_k\}$ of $\{t_1, \ldots, t_n\}$ such that $\{a, s_1, \ldots, s_m, t'_1, \ldots, t'_k\}$ is independent and $a \vee x \vee y = a \vee s_1 \vee \cdots \vee s_m \vee t'_1 \vee \cdots \vee t'_k$. Put $y' = t'_1 \vee \cdots \vee t'_k$. Then $\{a, s_1, \ldots, y'\}$ and $\{b, s_1, \ldots, s_m, y'\}$ are independent sets. Using the definition of independence we get $y' = (x \vee y') \wedge (b \vee x_1 \vee y') \wedge \cdots \wedge (b \vee x_m \vee y')$. Since $c \wedge y = 0$, we have $c \nleq y$. Hence $c \nleq y'$ and we have either $c \nleq x \vee y'$ or $c \nleq b \vee x_i \vee y'$ for some i. In the first case put $z = x \vee y'$, while in the second case put $z = b \vee x_i \vee y'$. Then $c \wedge z = 0$ and $c \leq a \vee z$, since $c \leq b \vee y \leq a \vee x \vee y = a \vee x \vee y' = a \vee b \vee x_i \vee y'$. Since $c \nleq z$, we have $a \nleq z$ and hence $a \wedge z = 0$. Thus $z < z \vee c \leq z \vee a$ where $z \prec z \vee a$, so that $z \vee c = z \vee a$. By 2.7, a, c are perspective. □

2.9. LEMMA. *Let L be a geometric lattice and $A \cup \{a\}$ be a set of atoms of L such that $a \leq \bigvee A$. Then a is perspective to at least one element of A.*

PROOF. Since a is compact, $a \leq \bigvee F$ for a finite subset F of A. Take F to be minimal with this property. Let b be an arbitrary element of F and put $x = \bigvee(F - \{b\})$. Then $b \vee x = \bigvee F$, $a \nleq x$ by the minimality of F, hence $h(a \vee x) = h(x) + 1$ (where h is the height function on L), and so $a \vee x = \bigvee F$. By the minimality, F is independent and thus $b \wedge x = 0$. Since $a \nleq x$, we have $a \wedge x = 0$. By 2.7, a, b are perspective. □

2.10. LEMMA. *Let L be a complete lattice and X be a subset of L. Then $L \simeq \Pi([0, x] : x \in X)$ if and only if every element a of L can be uniquely represented as $a = \bigvee(a_x : x \in X)$ where $a_x \leq x$ for all $x \in X$.*

PROOF. It is easy. □

2.11. THEOREM. *Every geometric lattice is isomorphic to the direct product of directly indecomposable geometric lattices. A geometric lattice is directly indecomposable if and only if its any two atoms are perspective.*

PROOF. Let L be a geometric lattice. Denote by A the set of atoms of L and by Z the set of the blocks of the perspectivity equivalence on A. We are going to show that L is isomorphic to $\Pi([0, \bigvee B] : B \in Z)$. For every $a \in L$ and every $B \in Z$ put $a_B = \bigvee([0, a] \cap B)$. Since a is a join of atoms, we have $a = \bigvee(a_B : B \in Z)$. By 2.10 it is sufficient to prove that this is the only representation of a as the join of elements that are below the elements $\bigvee B$ $(B \in Z)$. Let $a = \bigvee(x_B : B \in Z) = \bigvee(y_B : B \in Z)$ where $x_B \leq \bigvee B$ and $y_B \leq \bigvee B$ for all $B \in Z$, and suppose that $x_B \neq y_B$ for at least one B. Without loss of generality, $y_B \nleq x_B$. We can suppose that $x_B < y_B$, as otherwise we could replace the second representation of a with the join of the two representations.

By 2.9 there exists an atom $t \in B$ such that $t \leq y_B$ and $t \nleq x_B$. Put $a_1 = x_B$ and $a_2 = \bigvee(x_C : C \in Z - \{B\})$. Then $a = a_1 \vee a_2$ and (again using 2.9) $a_1 \wedge a_2 = y_B \wedge a_2 = 0$. We have $t \wedge a_1 = 0$, $a_1 \wedge a_2 = 0$ and $t \leq a_1 \vee a_2$. There exist a relative complement x_1 of $t \vee a_1$ in $[a_1, a]$ and a relative complement x_2 of $a_2 \wedge x_1$ in $[0, a_2]$. (Follow the situation as pictured in Fig. 4.)

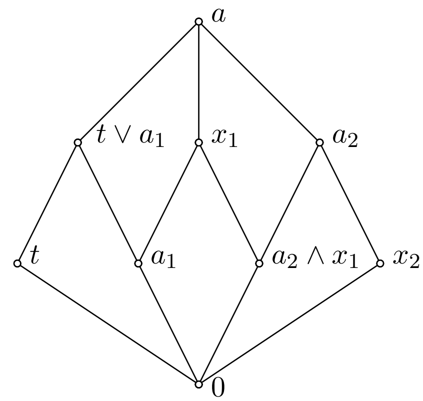

Fig. 4

We have

$$\begin{aligned}&t \wedge x_1 = t \wedge (t \vee a_1) \wedge x_1 = t \wedge a_1 = 0,\\&x_2 \wedge x_1 = a_2 \wedge x_2 \wedge x_1 = x_2 \wedge (a_2 \wedge x_1) = 0,\\&t \vee x_1 = t \vee a_1 \vee x_1 = a,\\&x_2 \vee x_1 = x_2 \vee (a_2 \wedge x_1) \vee x_1 = a_2 \vee x_1 = a.\end{aligned}$$

Take any atom $u \le x_2$ (there exists at least one, since it is easy to see that $x_2 > 0$). Since $x_1 \prec t \vee x_1 = x_1 \vee x_2$, we have $x_1 \vee u = t \vee x_1$ and, of course, $x_1 \wedge u = 0$. Hence t, u are perspective by 2.9. Since $t \in B$, we get $u \in B$. But $u \le a_2 = \bigvee(x_C : C \in Z - \{B\})$, so that u is perspective to an atom not belonging to B, a contradiction.

We have proved that L is isomorphic to $\Pi([0, \bigvee B] : B \in Z)$. Each lattice $[0, \bigvee b]$ with $B \in Z$ has the property its every two atoms are perspective (since they are perspective in L, and the lattice is a direct factor of L). In order to finish the proof, it remains to show that if any two atoms of L are perspective then L is directly indecomposable. Suppose that there exists an isomorphism of L onto the direct product of two nontrivial lattices L_1 and L_2. Denote by p the element of L corresponding to $\langle 0, 1\rangle$ and by q the element of L corresponding to $\langle 1, 0\rangle$. There exist an atom $c \le p$ and an atom $d \le q$. The elements c, d have a common complement e. It is easy to see that e cannot correspond to any element of $L_1 \times L_2$. □

2.12. Theorem. *Let L be a geometric lattice such that any two atoms of L are perspective. Then L is subdirectly irreducible. If, moreover, L is of finite length then L is simple. Consequently, every finite dimensional geometric lattice is isomorphic to the direct product of simple geometric lattices.*

Proof. Let r be a non-identical congruence of L. There exist elements $u, v \in L$ such that $u < v$ and $\langle u, v\rangle \in r$. There exists an atom a of L such that $a \le v$ and $a \not\le u$. Let b be any atom of L. Since a, b are perspective, there exists a common complement x of a and b. It follows from $\langle u, v\rangle \in r$ that $\langle 0, a\rangle \in r$, $\langle x, 1\rangle \in r$ and $\langle 0, b\rangle \in r$. Thus all atoms of L are r-congruent with 0 and L is subdirectly irreducible. If L is of finite length then 1 is the join of finitely many atoms and thus $\langle 0, 1\rangle \in r$. □

3. Projective spaces

By a pre-projective space we mean an ordered pair $\langle A, X\rangle$ (the elements of A are called its points and the elements of X are called its lines) satisfying the following conditions:

(1) Any two distinct points are contained in a unique line
(2) For any five points a, b, c, x, y and any two lines p, q with $a, b, x \in p$ and $b, c, y \in q$ there exist a point z and two lines r, s such that $a, c, z \in r$ and $x, y, z \in s$
(3) Every line has at least two points

By a projective space we mean a pre-projective space every line of which has at least three points.

Condition (2) is called the Pasch axiom and is interpreted in Fig. 5.

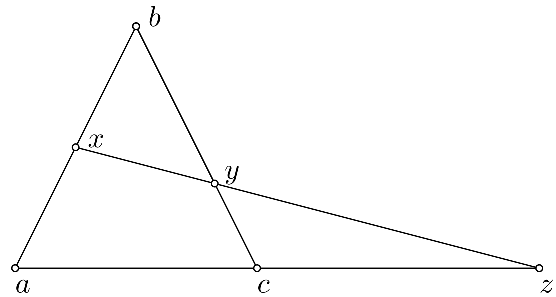

Fig. 5

By a subspace of a pre-projective space $\langle A, X\rangle$ we mean a subset S such that whenever a, b are two different points in S then any point on the line containing a, b belongs to S. For two subsets S, T of A denote by $S + T$ the union of all the lines that contain two distinct points, one from S and the other from T.

3.1. Lemma. *Let S, T be two subspaces of a pre-projective space. Then $S + T$ is also a subspace.*

Proof. For two different points a, b denote by $a + b$ the line containing a, b; for a point a put $a + a = \{a\}$. Let S, T be two subspaces. Let p, q be two different points in $S + T$ and $r \in p + q$. We need to prove that $r \in S + T$.

Let us first prove this under the assumption that $q \in S$. There are points $p_1 \in S$ and $p_2 \in T$ with $p \in p_1 + p_2$. Clearly, we can suppose that $q \neq p_1$ and $r \neq p_2$. By the Pasch axiom there exists a point $t \in (q + p_1) \cap (r + p_2)$. We have $t \in S$. If $t = p_2$ then $p \in S$ and $r \in S \subseteq S|T$. If $t \neq p_2$ then $r \in t + p_2 \in S + T$.

If $q \in T$, the proof is similar. Now let $q \notin S \cup T$. Again, there exists a point $t \in (q + p_1) \cap (r + p_2)$. If $t = p_2$ then $r \in p_1 + p_2 \subseteq S + T$. If $t \neq p_2$ then $r \in t + p_2$ and we get $r \in S + T$ by the previous part of the proof. □

3.2. Theorem. *The set of subspaces of a pre-projective space is a modular geometric lattice (with respect to inclusion). Conversely, if* L *is a modular geometric lattice then* $\langle A, X\rangle$ *is a pre-projective space where* A *is the set of atoms and* X *is the set of lines of* L*. This correspondence between pre-projective spaces and modular geometric lattices is one-to-one* (*up to isomorphism*).

Proof. Let $\langle A, X\rangle$ be a pre-projective space. Clearly, the set of subspaces is a complete lattice; by 3.1, $S \vee T = S + T$. For any set U of points, the subspace generated by U can be obtained as the union $\bigcup_{i\in\omega} U_i$ where $U_0 = U$ and U_{i+1} is the set of the points contained in a line containing two different points from U_i. From this we can see that the lattice is algebraic.

Let us prove that the lattice is modular. Let S, T, U be three subspaces with $S \subseteq U$. We need to prove that $(S + T) \cap U \subseteq S + (T \cap U)$. Let $p \in (S + T) \cap U$. There exist points $p_1 \in S$ and $p_2 \in T$ such that p belongs to the line containing p_1, p_2. If $p = p_1$, everything is clear. If $p \neq p_1$ then p_2 belongs to the line containing p and p_1, so that $p_2 \in U$ and thus $p_2 \in T \cap U$. Hence $p \in S + (T \cap U)$.

Conversely, let L be a modular geometric lattice. Denote by A the set of atoms and by $X$the set of lines (elements of height 2) of L. It is easy to see that $\langle A, X\rangle$ is a pre-projective space. In order to prove that the corresponding lattice is isomorphic to L, we need to prove that every subspace S of $\langle A, X\rangle$ corresponds to an element of L. Being a subspace means that $S \subseteq A$ and whenever $p, q \in S$, $r \in A$ and $r \leq p \vee q$ then $r \in S$. It is sufficient to prove that whenever $p_1, \dots, p_n \in S$, $r \in A$ and $r \leq p_1 \vee \dots \vee p_n$ then $r \in S$. This will be proved by induction on n. For $n \leq 2$ it is clear. Let $n \geq 3$ and $r \leq p_1 \vee \dots \vee p_n$. We can suppose that $p_1, \dots, p_n$ are independent, since otherwise we can use induction. Put $a = p_2 \vee \dots \vee p_n$. We can suppose that $r \not\leq a$, since otherwise we can use induction. Since $r \vee a \leq p_1 \vee a$ and both $r \vee a$ and $p_1 \vee a$ are of height n, we have $r \vee a = p_1 \vee a$. Put $t = (p_1 \vee r) \wedge a$. If $t = 0$, we get a contradiction by modularity. Hence $t \neq 0$, $t \in A$ and, by induction, $t \in S$. But then $r \leq t \vee p_1$ and $r \in S$. □

3.3. Lemma. *Two points* p, q *of a modular geometric lattice* L *are perspective if and only if there exists a point* r *of* L *such that* $r \neq p$, $r \neq q$ *and* $r \leq p \vee q$.

Proof. Let p, q be perspective. There exists a common complement x of p and q. Put $r = (p\vee q)\wedge x$. If $r = 0$, we get a contradiction by modularity. Thus $r \neq 0$ and it follows that r is an atom. Of course, $r \leq p \vee q$. The converse is obvious. □

3.4. THEOREM. *Under the correspondence between pre-projective spaces and modular geometric latticec, projective spaces correspond precisely to simple modular geometric lattices. Every modular geometric lattice is isomorphic to a direct product of modular geometric lattices corresponding to projective spaces.*

PROOF. It follows from 2.12 and 3.3. □

4. Desargues' theorem and arguesian lattices

Let $\langle A, X\rangle$ be a projective space. Three points are said to be collinear if they are contained in one line of $\langle A, X\rangle$. By a triangle (in $\langle A, X\rangle$) we mean an ordered triple of points that are not collinear (in particular, they are pairwise distinct.) Two triangles $\langle a_0, a_1, a_2\rangle$ and $\langle b_0, b_1, b_2\rangle$ are said to be perspective with respect to a point d if $a_i \vee a_j \neq b_i \vee b_j$ for $0 \le i, j \le 2$ and d is on the line $a_i \vee b_i$ for $0 \le i \le 2$. Two triangles $\langle a_0, a_1, a_2\rangle$ and $\langle b_0, b_1, b_2\rangle$ are said to be perspective with respect to a line p if $a_i \vee a_j \neq b_i \vee b_j$ for $0 \le i, j \le 2$ and the points $c_0 = (a_1 \vee a_2) \wedge (b_1 \vee b_2)$, $c_1 = (a_0 \vee a_2) \wedge (b_0 \vee b_2)$, $c_2 = (a_0 \vee a_1) \wedge (b_0 \vee b_1)$ all lie on p. A projective space is said to satisfy the Desargues' theorem if its any two triangles that are perspective with respect to a point, are also perspective with respect to a line. (See Fig. 6.)

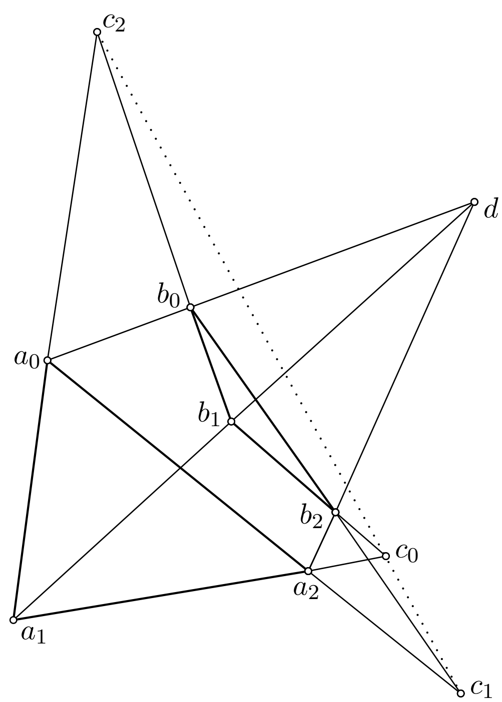

Fig. 6

A lattice L is called arguesian if for any elements $a_0, a_1, a_2, b_0, b_1, b_2 \in L$,

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq ((c \vee a_1) \wedge a_2) \vee ((c \vee b_1) \wedge b_2)$$

where

$$\begin{aligned} c_0 &= (a_1 \vee a_2) \wedge (b_1 \vee b_2), \\ c_1 &= (a_0 \vee a_2) \wedge (b_0 \vee b_2), \\ c_2 &= (a_0 \vee a_1) \wedge (b_0 \vee b_1), \\ c &= c_0 \wedge (c_1 \vee c_2). \end{aligned}$$

Clearly, the class of arguesian lattices is a variety.

4.1. Theorem. *Let L be a simple modular geometric lattice. Then L is arguesian if and only if the corresponding projective space satisfies the Desargues' theorem.*

Proof. The direct implication is clear. Let us prove the converse. Let L be a simple modular geometric lattice such that the corresponding projective space satisfies the Desargues' theorem.

Claim 1. The arguesian identity holds whenever $a_0, a_1, a_2, b_0, b_1, b_2$ are atoms of L. Put $d = (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2)$. We have to prove that $d \leq ((c \vee a_1) \wedge a_2) \vee ((c \vee b_1) \wedge b_2)$. Assume first that $\langle a_0, a_1, a_2 \rangle$ and $\langle b_0, b_1, b_2 \rangle$ are triangles perspective with respect to d. Then c_1, c_2, c_3 are pairwise distinct atoms; since they are collinear, $c = c_0 \wedge (c_1 \vee c_2) = c_0$. Then $((c \vee a_1) \wedge a_2) \vee ((c \vee b_1) \wedge b_2) = a_2 \vee b_2$; of course, $d \leq a_2 \vee b_2$. If the assumption about the triangles is not satisfied then there are several cases to be considered (e.g., $d = 0$, d is a line, a_i are collinear, b_i are collinear, $a_i = b_i$ for some i); in all these cases it is trivial to check that $d \leq ((c \vee a_1) \wedge a_2) \vee ((c \vee b_1) \wedge b_2)$ (without use of the Desargues' theorem).

Claim 2. Let $t(x_1, \dots, x_n)$ be an n-ary term function (in the language of lattices) in which each variable occurs at most once. Then for an atom a and elements $b_1, \dots, b_n \in L$ we have $a \leq t(b_1, \dots, b_n)$ if and only if there exist elements $a_1, \dots, a_n \in L$ such that $a \leq t(a_1, \dots, a_n)$, $a_i \leq b_i$ for all i, and a_i is an atom whenever $b_i \neq 0$. We are going to prove the direct implication by induction on n (the converse is clear). If $n = 1$, $t = x_i$ for some i, we have $a \leq b_i$ and so we can take $a_i = a$. Observe that if either $t = u \wedge v$ or $t = u \vee v$ for two term functions u and v then the sets of variables occurring in u and v are disjoint and in each of the two terms some of the n variables are missing. Let $t = u \wedge v$. We have $a \leq u(b_1, \dots, b_n)$ and $a \leq v(b_1, \dots, b_n)$, so that the existence of the desired elements a_i follows from the induction assumption. Finally, let $t = u \vee v$. We have $a \leq u(b_1, \dots, b_n) \vee v(b_1, \dots, b_n)$. If either $u(b_1, \dots, b_n)$ or $v(b_1, \dots, b_n)$ equals 0, the existence of the elements a_i follows easily from the induction assumption. Let both these elements be non-zero. It follows from 3.1 that there exist two atoms a', a'' such that $a \leq a' \vee a''$, $a' \leq u(b_1, \dots, b_n)$ and $a'' \leq v(b_1, \dots, b_n)$. Now the existence of the elements a_i again follows from the induction assumption.

Claim 3. Let $u(x_1, \dots, x_n)$ and $v(x_1, \dots, x_n)$ be two term functions such that each variable occurs at most once in u. If the inequality $u \leq v$ holds for *all elements of L that are either atoms or zero, then it holds for all elements of L.* Let $b_1, \dots, b_n \in L$; we need to prove $u(b_1, \dots, b_n) \le v(b_1, \dots, b_n)$. It is sufficient to show that whenever a is an atom and $a \le u(b_1, \dots, b_n)$ then $a \le v(b_1, \dots, b_n)$. By Claim 1 there are elements $a_1, \dots, a_n$ such that $a \le u(a_0, \dots, a_n)$, $a_i \le b_i$ for all i, and each a_i is either an atom or zero. Thus $u(a_1, \dots, a_n) \le v(a_1, \dots, a_n)$. We get $a \le u(a_1, \dots, a_n) \le v(a_1, \dots, a_n) \le v(b_1, \dots, b_n)$.

Since Claim 1 can be easily extended to include the case when the elements $a_0, a_1, a_2, b_0, b_1, b_2$ are either atoms or zero, the proof is finished by Claim 3. □

4.2. THEOREM. *Every arguesian lattice is modular.*

PROOF. Suppose that an arguesian lattice is not modular, so that it has a pentagon sublattice described by $O \prec C \prec I$ and $O \prec A \prec B \prec I$. Put $a_0 = a_1 = b_2 = A$, $b_1 = B$ and $a_2 = b_0 = C$. Then $c_0 = B$, $c_1 = I$, $c_2 = A$, $c = B$, $(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) = B$ and $((c \vee a_1) \wedge a_2) \vee ((c \vee b_1) \wedge b_2) = A$, so that $B \le A$, a contradiction. □

4.3. THEOREM. *The congruence lattice of an algebra with permutable congruences is arguesian.*

PROOF. Congruence permutability means that if r, s are two congruences of the algebra and $\langle x, y\rangle \in r \vee s$ then there exists an element z with $\langle x, z\rangle \in r$ and $\langle z, y\rangle \in s$. Let $a_0, a_1, a_2, b_0, b_1, b_2$ be congruences of a congruence permutable algebra; let c_0, c_1, c_2, c have the same meaning as above. Let $\langle x, y\rangle \in (a_0 \vee b_0) \cap (a_1 \vee b_1) \cap (a_2 \vee b_2)$. There exist elements z_0, z_1, z_2 with $\langle x, z_i\rangle \in a_0$ and $\langle z_i, y\rangle \in b_0$ for $i = 0, 1, 2$. We have $\langle z_1, z_2\rangle \in c_0 \cap (c_1 \vee c_2) = c$, $\langle x, z_2\rangle \in (c \vee a_1) \cap a_2$ and $\langle z_2, y\rangle \in (c \vee b_1) \cap b_2$, so that $\langle x, y\rangle \in ((c \vee a_1) \cap a_2) \vee ((c \vee b_1) \cap b_2)$. Thus $(a_0 \vee b_0) \cap (a_1 \vee b_1) \cap (a_2 \vee b_2) \subseteq ((c \vee a_1) \cap a_2) \vee ((c \vee b_1) \cap b_2)$. □

4.4. THEOREM. *The geometric lattice corresponding to a projective space of dimension at least 3 is arguesian.*

PROOF. We must prove that if L is a simple geometric modular lattice of length at least 4 then L is arguesian. Let $\langle a_0, a_1, a_2\rangle$ and $\langle b_0, b_1, b_2\rangle$ be two triangles perspective with respect to the point $d = (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2)$. Put $\alpha = a_0 \vee a_1 \vee a_2$ and $\beta = b_0 \vee b_1 \vee b_2$, so that α and β are planes (elements of height 3).

Consider first the case when $\alpha \ne \beta$. We have $\alpha \vee \beta = \alpha \vee d = \beta \vee d$, from which it follows that the height of $\alpha \vee \beta$ is 4 and then that the height of $\alpha \wedge \beta$ is 2, so that $\alpha \wedge \beta$ is a line. For $i \ne j$, the lines $a_i \vee a_j$ and $b_i \vee b_j$ are in the plane $a_i \vee a_j \vee d$ and thus they intersect in a point. If i, j, k is a permutation of 0,1,2 then the lines $a_i \vee a_j$ and $b_i \vee b_j$ are in the plane $a_i \vee a_j \vee d$, so their intersection c_k is a point. Since $a_i \vee a_j$ is in the plane α and $b_i \vee b_j$ is in the plane β, c_k is in the line $\alpha \wedge \beta$. Hence c_0, c_1, c_2 are collinear.

Now consider the case $\alpha = \beta$. Since the length of L is at least 4, we have $\alpha \ne 1$ and there exists an element p of L such that α is covered by p; the height of p is 4. Denote by s a relative complement of α in the interval $[d,p]$, so that s is a line. Except the point d, this line contains at least two more points d' and d''. Put $e_i = (d' \vee a_i) \wedge (d'' \vee b_i)$. Then $\langle e_0, e_1, e_2\rangle$ is a triangle and the plane $\gamma = e_0 \vee e_1 \vee e_2$ is different from the plane α. The triangles $\langle a_0, a_1, a_2\rangle$ and $\langle e_0, e_1, e_2\rangle$ are perspective with respect to the point d''. By the first case, the points $(a_0 \vee a_1) \wedge (e_0 \vee e_1)$ and $(b_0 \vee b_1) \wedge (e_0 \vee e_1)$ are contained in the line $\delta = \alpha \wedge \gamma$ and so $(a_0 \vee a_1) \wedge (b_0 \vee b_1) \in \delta$. Similarly $(a_0 \vee a_2) \wedge (b_0 \vee b_2) \in \delta$ and $(a_1 \vee a_2) \wedge (b_1 \vee b_2) \in \delta$. □

4.5. Theorem. *If a projective space satisfies the Desargues' theorem then it also satisfies the following* (*dual*) *statement: if two triangles are perspective with respect to a line then they are also perspective with respect to a point.*

Proof. Under the above notation, let c_0, c_1, c_2 be contained in a line s. Put $d = (a_1 \vee b_1) \wedge (a_2 \vee b_2)$. The triangles $\langle a_2, b_2, c_1\rangle$ and $\langle a_1, b_1, c_2\rangle$ are perspective with respect to the point c_0. Hence the points $(a_2 \vee b_2) \wedge (a_1 \vee b_1) = d$, $(a_2 \vee c_1) \wedge (a_1 \vee c_2) = a_0$ and $(b_2 \vee c_1) \wedge (b_1 \vee c_2) = b_0$ are collinear. We get $d = (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2)$. □

5. Coordinatization of projective spaces

The most important example of a projective space can be obtained in the following way. Take a division ring D and a cardinal number $\kappa \geq 2$. As one can easily check, the lattice of all submodules of the κ-dimensional vector space over D is a simple arguesian geometric lattice (it is arguesian by 4.3). Consequently, the lattice gives rise to a projective space.

The coordinatization theorem states that the converse is true: If L is a simple arguesian geometric lattice of length at least 3 then there exists a division ring D and a cardinal number $\kappa \geq 2$ such that L is isomorphic to the lattice of all subspaces of the κ-dimensional vector space over D.

The division ring D can be constructed in the following way. Take an arbitrary line u and its three distinct points denoted by $0, 1, \infty$. Put $D = s - \{\infty\}$. For two elements $a, b \in D$ put

$$a + b = (((a \vee c) \wedge (d \vee \infty)) \vee ((c \vee \infty) \wedge (b \vee d))) \wedge s$$

and

$$ab = (q \vee ((0 \vee ((1 \vee p) \wedge (q \vee b))) \wedge (p \vee a))) \wedge s$$

where c, d are two distinct points not in s such that $0, c, d$ are collinear, and p, q are two distinct points not in s such that p, q, ∞ are collinear. We omit the lengthy proof.

Consequently, according to 4.4, if we assume that all division rings and all their vector spaces are known then all projective spaces of dimension at least 3 are known. Projective spaces of dimension 2, or projective planes, are more difficult to describe.

CHAPTER 2

PROJECTIVE PLANES

1. Incidence structures

By an incidence structure we mean an ordered triple $\langle A, B, I\rangle$ such that A and B are two disjoint sets, and I is a subset of $A \times B$. Incidence structures are also called bipartite graphs. The elements of A are called points and the elements of B are called lines. For a point a and a line b we will write $a < b$ instead of $\langle a, b\rangle \in I$ and say that a is on b, or that b is on a, or that a lies on b, or that b goes through a.

For an incidence structure $\langle A, B, I\rangle$, the incidence structure $\langle B, A, I^{-1}\rangle$ is called dual to $\langle A, B, I\rangle$.

Let $\langle A, B, I\rangle$ be an incidence structure. Assuming that 0 and 1 are not elements of $A \cup B$, the set $A \cup B \cup \{0, 1\}$ is an ordered set with respect to the relation $\leq$ where $x \leq y$ means that either $x = y$ or $x = 0$ or $y = 1$ or $\langle x, y\rangle \in I$. We call this the ordered set corresponding to $\langle A, B, I\rangle$. Observe that in general an incidence structure is not uniquely determined by its corresponding ordered set.

Let us call an incidence structure admissible if it has at least one point, every point is on at least one line and every line is on at least one point. In the following we will consider only admissible incidence structures (in fact, incidence structures satisfying much stronger conditions). Clearly, there is a bijection between admissible incidence structures and the bounded ordered sets of length 3 in which no atom is a coatom. Under this bijection, the dual of an incidence structure corresponds to the dual of the corresponding ordered set. Every admissible incidence structure is uniquely determined by its corresponding ordered set. The difference between an admissible incidence structure and its corresponding ordered set is almost negligible — just adding two new elements 0 and 1. It will be useful to assume that an admissible incidence structure is already an ordered set. Thus points are precisely the atoms and lines are precisely the coatoms. A set of points is said to be collinear if there exists a line containing all its elements. Two admissible incidence structures are (called) isomorphic if they are isomorphic as ordered sets.

It is easy to see that an admissible incidence structure is a lattice if and only if its every two distinct points lie on at most one line. We will investigate only admissible incidence structures with this property.

2. Projective planes

By a projective plane we mean an incidence structure satisfying the following three conditions:

(PP1) Every two distinct points are on precisely one line.
(PP2) Every two distinct lines are on precisely one point.
(PP3) There exist four distinct points, no three of which are collinear.

Clearly, every projective plane is an admissible incidence structure and can be considered as a lattice. An ordered quadruple of points satisfying (PP3) is called a frame of the projective plane.

2.1. THEOREM. *The following are equivalent for a lattice L:*

(1) *L is a projective plane*
(2) *L is a simple modular geometric lattice of length 3*
(3) *L is a modular lattice of length 3 in which every element except $0, 1$ has at least two complements*

PROOF. It follows from the results of the last chapter that the first two conditions are equivalent; the only thing that we need to verify is that in a projective plane, every line has at least three points. Suppose that there is a line p with at most two points. By (PP3) there exist four distinct points a, b, c, d, no three of which are collinear. Since $p \wedge (a \vee b)$, $p \wedge (a \vee c)$ and $p \wedge (a \vee d)$ are three points of p, at least two of them are equal. From this it follows that $a \leq p$. Similarly we get $b \leq p$, $c \leq p$ and $d \leq p$, so that p has at least four points, a contradiction

The equivalence of (2) and (3) is easy. □

2.2. THEOREM. *The dual of a projective plane is a projective plane.*

PROOF. It follows from 2.1(3). □

2.3. THEOREM. *For every projective plane there exists a cardinal number $\kappa \geq 3$ such that every line contains precisely κ points and every point is contained in precisely κ lines.*

PROOF. Let us fix a line p and denote by κ the cardinality of the set of the points contained in p, so that $\kappa \geq 3$. Let q be any other line. It is easy to see that there exists a point a not contained in p and not contained in q. The mapping assigning to any point b of p the line $a \vee b$ is a bijection of the set of points of p onto the set of lines containing a. Similarly, the mapping assigning to any point c of q the line $a \vee c$ is a bijection of the set of points of q onto the same set of lines. Thus any line contains precisely κ points. By 2.2, there exists a cardinal number κ' such that every point is contained in precisely κ' lines. But the point a is contained in precisely κ lines, so $\kappa = \kappa'$. □

2.4. THEOREM. *Let L be a finite projective plane. Then there exists a unique natural number $n \geq 2$ such that every line of L contains precisely*

$n+1$ points and every point of L is contained in precisely $n+1$ lines. The projective plane has precisely n^2+n+1 points and precisely n^2+n+1 lines.

PROOF. The first statement follows from 2.3. Let uf fix a point a. There are $n+1$ points going through a, each of them with precisely n points different from a; all these points are pairwise different and any point is contained in a line going through a, so there are $(n+1)n+1 = n^2+n+1$ points in total. By duality, the total number of lines is the same. □

For a finite projective plane, the number n from 2.4 is called its order. Thus the order of a finite projective plane is a natural number greater or equal 2.

2.5. EXAMPLE. There is, up to isomorphism, precisely one projective plane of order 2. Its geometry is pictured in Fig. 2.

2.6. EXAMPLE. Let O, I, X, Y be four distinct elements. For every natural number n define three sets A_n, B_n, I_n with $I_n \subseteq A_n \times B_n$ as follows. $A_0 = \{O, I, X, Y\}$; $B_0 = \{\{a,b\} : a, b \in A_0$ and $a \neq b\}$; for $a \in A_0$ and $p \in B_0$, $\langle a, p\rangle \in I_0$ if and only if $a \in p$. If A_n, B_n, I_n are defined then A_{n+1} is the union of A_n with the set of the pairs $\langle\{p,q\},0\rangle$ such that $p, q \in L_n$ and there is no $a \in A_n$ with $\langle a,p\rangle \in I_n$ and $\langle a,q\rangle \in I_n$; B_{n+1} is the union of B_n with the set of the pairs $\langle\{a,b\},1\rangle$ such that $a, b \in A_n$ and there is no $p \in B_n$ with $\langle a,p\rangle \in I_n$ and $\langle b,p\rangle \in I_n$; I_{n+1} is the union of I_n with the set of the pairs $\langle\langle\{p,q\},0\rangle,p\rangle$ with $\langle\{p,q\},0\rangle \in A_{n+1} - A_n$ and the pairs $\langle a,\langle\{a,b\},1\rangle\rangle$ with $\langle\{a,b\},1\rangle \in B_{n+1} - B_n$. Put $A = \bigcup_{n=0}^{\infty} A_n$, $B = \bigcup_{n=0}^{\infty} B_n$ and $I = \bigcup_{n=0}^{\infty} I_n$. It is easy to check that $\langle A, B, I\rangle$ is a projective plane. It is called the free projective plane; the elements O, I, X, Y are its generators.

3. Affine planes

By an affine plane we mean an incidence structure satisfying the following three conditions:

(AP1) Every two distinct points are on precisely one line.
(AP2) For every point a and every line b with $a \not\leq b$ there exists a unique line c such that $a \leq c$ and b, c have no point in common.
(AP3) There exist four distinct points, no three of which are collinear.

Clearly, every affine plane is an admissible incidence structure and can be considered as a lattice.

Two lines of an affine plane are said to be parallel if they are either equal or have no point in common. We write $p||q$ if p, q are two parallel lines.

3.1. THEOREM. *Parallelism is an equivalence relation on the set of lines of an affine plane.*

PROOF. Let p, q, r be three lines such that $p||q$ and $q||r$. We need to prove $p||r$. Suppose that p, r are not parallel. Then $a = p \wedge r$ is a point. Clearly, $p \neq q$ and $q \neq r$. There exists a unique line containing a and parallel with q. Since both p and r have this property, we get $p = r$. □

3.2. Corollary. *In an affine plane, any line intersecting another line, intersects all lines parallel to this other line.*

3.3. Theorem. *Let A be an affine plane. There exist two cardinal numbers κ and κ' such that every line of A contains precisely κ points and every point of A is contained in precisely κ' lines. If κ is infinite then $\kappa' = \kappa$. If $\kappa = n$ is finite then $\kappa' = n + 1$, there are precisely n^2 points and there are precisely $n^2 + n$ lines.*

Proof. It is easy. □

For a finite affine plane, the number $n \geq 2$ from 3.3 is called its order.

For a projective plane P and its any one line p we define an affine plane $\mathrm{AP}(P, p)$ as follows. Its points are all points of P except the points contained in p; its lines are all lines of P except the line p; $a \leq q$ in $\mathrm{AP}(P, p)$ if and only if $a \leq q$ in P. One can easily check that we obtain an affine plane.

For an affine plane A we define a projective plane $\mathrm{PP}(A)$ as follows. Its points are the points of A and, moreover, for each block D of the parallelism relation one new point ∞_D; its lines are the lines of A and, moreover, one new line ∞; for a point a and a line p, $a < p$ if and only if one of the following three cases takes place:

(1) $a < p$ in A
(2) $p = \infty$ and a is one of the new points
(3) $a = \infty_D$ for some D and $p \in D$

One can easily check that we obtain a projective plane.

This correpsondence between affine and projective planes is not one-to-one. For a projective plane P and its any line p, the projective plane $\mathrm{PP}(\mathrm{AP}(P, p))$ is isomorphic to P. For an affine plane A, we have $A = \mathrm{AP}(\mathrm{PP}(A), \infty))$ but we may obtain an affine plane not isomorphic with A if we use other line than ∞. Nevertheless, the correspondence is almost one-to-one. It is not necessary to study affine planes for themselves, since we can always embed them into projective planes and proceed from there.

For a field K we define an affine plane $\mathrm{AP}(K)$ as follows. Its points are the elements of $K \times K$; for each triple $\langle a, b, c\rangle \in K^3 - (\{0\} \times \{0\} \times K)$ we have a line $p_{a,b,c} = \{\langle x, y\rangle \in K^2 : ax + by + c = 0\}$ (there are different triples producing the same line); a point is on a line if and only if it is its element. One can easily check that we obtain an affine plane.

For a field K put $\mathrm{PP}(K) = \mathrm{PP}(\mathrm{AP}(K))$. This projective plane is isomorphic to the lattice of all subspaces of the 3-dimensional vector space over K. As we know from before, this projective plane is arguesian.

For the two-element field $\mathbf{Z}_2$, $\mathrm{AP}(\mathbf{Z}_2)$ is an affine plane of order 2. It is pictured in Fig 7.

The projective plane $\mathrm{PP}(\mathbf{Z}_2)$ of order 2 is pictured in Fig. 2.

3.4. Theorem. *For every prime power p^k (p is a prime number, $k \geq 1$) there is an arguesian projective plane of order p^k.*

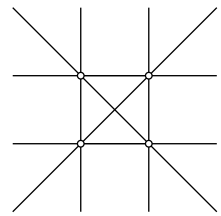

Fig. 6

Proof. It is well known that for any prime power p^k there is a field with p^k elements. □

4. Perspectivities and projectivities

Let p, q be two distinct lines of a projective plane and a be a point with $a \not\leq p$ and $a \not\leq q$. The mapping, assigning to any point x of p the point $(p \vee x) \wedge q$ of q is called the perspectivity from p to q with center a. Clearly, this mapping is a bijection of the set of points on p onto the set of points on q.

Let p, q be two lines of a projective plane and let n be a positive natural number. A mapping of the set of points on p into the set of points on q is said to be a projectivity from p to q of order n if there exist lines $p_0, \dots, p_n$ such that $p_0 = p$, $p_n = q$ and the mapping is a composition of some perspectivities from p_{i-1} to p_i $(i = 1, \dots, n)$. Of course, a projectivity is a bijection.

4.1. Theorem. *Let p, p' be two distinct lines of a projective plane, a, b, c be three distinct points on p, and a', b', c' be three distinct points on p'. Then there exists a projectivity of order 2 from p to p', taking a to a', b to b' and c to c'.*

Proof. We can assume without loss of generality that $a, a', p \wedge p'$ are three distinct points. (If a, a' do not have this property then either b, b' or c, c' can be taken instead.) There exists a point $d \leq a \vee a'$ different from both a and a'. There exists a line p'' containing a' and different from both $a \vee a'$ and p'. The perspectivity with center d takes a, b, c to three distinct points a'', b'', c'' on p''; we have $a'' = a'$. Put $e = (b' \vee b'') \wedge (c' \vee c'')$.

Claim 1. e is a point with $e \not\leq p'$ and $e \not\leq p''$. This follows from the following computations.

$$\begin{aligned}
e \wedge p' &= ((b' \vee b'') \wedge p') \wedge ((c' \vee c'') \wedge p') \\
&= (b' \vee (b'' \wedge p')) \wedge (c' \vee (c'' \vee p')) = (b' \vee 0) \wedge (c' \vee 0) = 0, \\
e \wedge p'' &= 0 \text{ similarly}, \\
e \vee p' &= ((b' \vee b'') \wedge (c' \vee c'') \vee b' \vee c' \\
&= ((b' \vee b'') \wedge (c' \vee c'' \vee b')) \vee c' \\
&= (b' \vee b'' \vee c') \wedge (c' \vee c'' \vee b') = (p' \vee b'') \wedge (p' \vee c'') = 1,
\end{aligned}$$

$e \vee p'' = 1$ similarly.

Claim 2. The perspectivity from p'' to p' with center e takes a'' to a', b'' to b' and c'' to c'. Clearly, it takes a'' to $a' = a''$. We have

$$\begin{aligned} p' \wedge (e \vee b'') &= p' \wedge (b'' \vee ((b' \vee b'') \wedge (c' \vee c''))) \\ &= p' \wedge ((b' \vee b'') \wedge (b'' \vee c' \vee c'')) \\ &= p' \wedge ((b' \vee b'') = (p' \wedge b'') \vee b' = b' \end{aligned}$$

and similarly $p' \wedge (e \vee c'') = c'$. □

5. Collineations

Automorphisms of a projective plane are called its collineations.

Let f be a collineation of a projective plane. A point a is called a center of f if $f(x) = x$ for all $x \geq a$. A line p is called an axis of f if $f(x) = x$ for all $x \leq p$. A collineation is called central if it has a center; it is called axial if it has an axis.

5.1. Example. Let A be an affine plane over a field K and P be the corresponding projective plane; let v be a non-zero vector. The translation by v is a collineation of P with the infinite line serving as the axis and the infinite point corresponding to the lines parallel with v serving as the center.

5.2. Theorem. *Let f be a non-identical collineation of a projective plane. Then f has at most one center and at most one axis.*

Proof. Suppose that f has two different centers a and b. For every point $c \not\leq a \vee b$ we have $f(c) = f((a \vee c) \wedge (b \vee c)) = f(a \vee c) \wedge f(b \vee c) = (a \vee c) \wedge (b \vee c) = c$. Every line other than p contains at least three points, two of which are not on $a \vee b$ and so is fixed by f. Consequently, every line is fixed by f. But then also all points are fixed by f and f is the identity, a contradiction. By duality, f has also at most one axis. □

5.3. Lemma. *Let f be a collineation of a projective plane with center a and let p be a line such that $f(p) \neq p$. Then $f(p) \wedge p$ is fixed by f.*

Proof. Since $p \neq f(p)$, we have $a \not\leq p$, $a \not\leq f(p)$ and $p \wedge f(p) = p \wedge (a \vee (p \wedge f(p))) = f(p) \wedge (a \vee (p \wedge f(p)))$. Therefore $f(p \wedge f(p)) = f(p \wedge (a \vee (p \wedge f(p)))) = f(p) \wedge f(a \vee (p \wedge f(p))) = f(p) \wedge (a \vee (p \wedge f(p))) = p \wedge f(p)$. □

5.4. Lemma. *Let f be a collineation of a projective plane with center a and let p be a line such that either $a \not\leq p$ or else $a \leq p$ and there exist points $b, c \leq p$ such that a, b, c are distinct and all of them are fixed by f. Then p is an axis of f.*

Proof. Consider first the case when $a \not\leq p$. For any point $b \leq p$ we have $b = p \wedge (a \vee b)$; since p and $a \vee b$ are fixed by f, the point b is fixed by f and p is an axis for f.

Now let $a \leq p$ and let $b, c \leq p$ be as above. Suppose that there exists a point $d < p$ with $f(d) \neq d$. Take a line q with $p \wedge q = d$. We have $f(q) \neq q$

and by 5.3, $q \wedge f(q)$ is a point fixed by f. But then $b \vee (q \wedge f(q))$ and $c \vee (q \wedge f(q))$ are fixed lines not containing a, so that by the first case they are two different axes for f, a contradiction with 5.2. Therefore all points on p are fixed by f and p is an axis. □

5.5. THEOREM. *A collineation of a projective plane is central if and only if it is axial.*

PROOF. Let f be a non-identical collineation with center a. Take a point b with $f(b) \neq b$. The proof can be easily finished, using the previous lemmas, if we consider two other distinct lines on b. □

5.6. THEOREM. *Let a be a point and p be a line of a projective plane. Then every collineation with center a and axis p is uniquely determined by its value at any point different from a and not on p.*

PROOF. Let f be a collineation with center a and axis p, and let b be a point with $b \neq a$ and $b \nleq p$. For any point c with $c \nleq a \vee b$ and $c \nleq p$ we have $c = (a \vee c) \wedge (b \vee (p \wedge (b \vee c)))$ and therefore $f(c) = f(a \vee c) \wedge (f(b) \vee f(p \wedge (b \vee c))) = (a \vee c) \wedge (f(b) \vee (p \wedge (b \vee c)))$. Now for any line q with $q \neq p$ and $q \neq a \vee b$ there exists a point c with $c \leq q$, $c \nleq p$ and $c \nleq a \vee b$; we have $q = c \vee (p \wedge q)$ and so $f(q) = (p \wedge q) \vee f(c) = (p \wedge q) \vee ((a \vee c) \wedge (f(b) \vee (p \wedge (b \vee c))))$. Therefore $f(q)$ is determined for all lines q by $f(b)$ and hence f is determined by $f(b)$. □

Let a be a point and p be a line of a projective plane. The projective plane is said to be $a - p$ transitive if for any pair a_0, b_0 such that a, a_0, b_0 are three distinct collinear points with $a_0 \nleq p$ and $b_0 \nleq p$ there exists a collineation f with center a and axis p such that $f(a_0) = b_0$. (This means that all possible collineations with center a and axis p do exist.) A projective plane is said to be p-transitive (or transitive with respect to p) if it is $a - p$ transitive for every point a on p.

5.7. THEOREM. *A projective plane is arguesian if and only if it is $a - p$ transitive for its any point a and its any line p.*

PROOF. Let us start with the converse implication. Let $\langle a_0, a_1, a_2 \rangle$ and $\langle b_0, b_1, b_2 \rangle$ be two triangles perspective with respect to a point d; define c_0, c_1, c_2 as before. There exists a collineation f with center d and axis $c_0 \vee c_1$. We have $a_0 = (d \vee b_0) \wedge (a_2 \vee c_1)$ and $a_1 = (d \vee b_1) \wedge (a_2 \vee c_0)$, so that $f(a_0) = (d \vee b_0) \wedge (f(a_2) \vee c_1) = (d \vee b_0) \wedge (b_2 \vee c_1) = b_0$ and similarly $f(a_1) = b_1$. Hence $(a_0 \vee a_1) \wedge (c_0 \vee c_1) = f((a_0 \vee a_1) \wedge (c_0 \vee c_1)) = f(a_0 \vee a_1) \wedge (c_0 \vee c_1) = (f(a_0) \vee f(a_1)) \wedge (c_0 \vee c_1) = (b_0 \vee b_1) \wedge (c_0 \vee c_1)$. Since $a_0 \vee a_1 \neq b_0 \vee b_1$, we get $c_2 = (a_0 \vee a_1) \wedge (b_0 \vee b_1) \leq c_0 \vee c_1$.

It remains to prove the direct implication. Let a be a point and p be a line. Denote by U the set of the ordered pairs $\langle a_i, b_i \rangle$ of points such that a, a_i, b_i are pairwise distinct and collinear and neither a_i nor b_i is on the line p. For $\langle a_i, b_i \rangle \in U$ and any point b with $b \nleq a \vee a_i$ and $b \nleq p$ put

$f_i(b) = (a \vee b) \wedge (b_i \vee (p \wedge (b \vee a_i)))$. It is easy to check that $a \vee b = a \vee f_i(b)$ and $f_i(b) \nleq p$.

Claim 1. For $\langle a_0, b_0\rangle \in U$ *and* $\langle a_1, b_1\rangle \in U$ *with* $a_0 \wedge (a \vee a_1) = 0 = a_1 \wedge (a \vee a_0)$ *we have* $f_0(a_1) = b_1$ *if and only if* $p \wedge (a_0 \vee a_1) = p \wedge (b_0 \vee b_1)$ *if and only if* $f_1(a_0) = b_0$. It is sufficient to prove the equivalence of the first two conditions. If $f_0(a_1) = b_1$ then $p \wedge (b_0 \vee b_1) = p \wedge (b_0 \vee ((a \vee a_1) \wedge (b_0 \vee (p \wedge (a_0 \vee a_1))))) = p \wedge (b_0 \vee a \vee a_1) \wedge (b_0 \vee (p \wedge (a_0 \vee a_1))) = (a \vee a_0 \vee a_1) \wedge ((p \wedge b_0) \vee (p \wedge (a_0 \vee a_1))) = p \wedge (a_0 \vee a_1)$. If $p \wedge (a_0 \vee a_1) = p \wedge (b_0 \vee b_1)$ then $f_0(a_1) = (a \vee a_1) \wedge (b_0 \vee (p \wedge (a_0 \vee a_1))) = (a \vee b_1) \wedge (b_0 \vee (p \wedge (b_0 \vee b_1))) = (a \vee b_1) \wedge (b_0 \vee b_1) = b_1$.

Claim 2. Let $\langle a_0, b_0\rangle \in U$ *and* $\langle a_1, b_1\rangle \in U$ *be such that* $a_0 \wedge (a \vee a_1) = 0 = a_1 \wedge (a \vee a_0)$ *and* $f_0(a_1) = b_1$. *Then for any point* $a_2 \nleq p$ *with* $a_2 \wedge (a \vee a_0) = 0 = a_2 \wedge (a \vee a_1)$ *we have* $f_0(a_2) = f_1(a_2)$. Consider the triangles $\langle a, a_2, a_1\rangle$ and $\langle b_0, p \wedge (a_0 \vee a_2), p \wedge (a_0 \vee a_1)\rangle$. We have $(a \vee b_0) \wedge (a_2 \vee (p \wedge (a_0 \vee a_2))) = (a \vee b_0) \wedge (a_0 \vee a_2) = a_0$ and $a_1 \vee (p \wedge (a_0 \vee a_1)) = a_0 \vee a_1 \geq a_0$, so that the triangles are perspective with respect to the point a_0. Since the lattice is arguesian, we get $c_2 \leq c_0 \vee c_1$ where

$$
\begin{aligned}
c_2 &= (a \vee a_2) \wedge (b_0 \vee (p \wedge (a_0 \vee a_2))) = f_0(a_2),\\
c_1 &= (a \vee a_1) \wedge (b_0 \vee (p \wedge (a_0 \vee a_1))) = f_0(a_1) = b_1,\\
c_0 &= (a_2 \vee a_1) \wedge ((p \wedge (a_0 \vee a_2)) \vee (p \wedge (a_0 \vee a_1))) \leq p \wedge (a_1 \vee a_2).
\end{aligned}
$$

Hence $f_0(a_2) \leq b_1 \vee (p \wedge (a_1 \vee a_2))$ and thus $f_0(a_2) \leq (a \vee a_2) \wedge (b_1 \vee (p \wedge (a_1 \vee a_2))) = f_1(a_2)$. Since both $f_0(a_2)$ and $f_1(a_2)$ are complements of p, we get $f_0(a_2) = f_1(a_2)$.

Claim 3. Let $\langle a_0, b_0\rangle, \langle a_1, b_1\rangle, \langle a_2, b_2\rangle$ *be elements of* U *such that* $a_1 \wedge (a \vee a_0) = 0 = a_2 \wedge (a \vee a_0)$, $f_0(a_1) = b_1$ *and* $f_0(a_2) = b_2$, *Then* $f_1(a_3) = f_2(a_3)$ *for all points* $a_3 \leq a \vee a_0$. If $(a \vee a_1) \wedge (a \vee a_2) = a$ then, since $a_3 \wedge (a \vee a_1) = 0 = a_3 \wedge (a \vee a_2)$, $f_1(a_3) = f_2(a_3)$ by Claim 2. The only remaining case is $a \vee a_1 = a \vee a_2$. If a_3 is any point on $a \vee a_0$ different from both a and a_0, then $a_4 = (a_0 \vee a_2) \wedge (a_1 \vee a_3)$ is a point not on $a \vee a_0$ and not on $a \vee a_1$ and we have $a_4 \wedge (a \vee a_0) = 0 = a_4 \wedge (a \vee a_1)$. Put $b_4 = f_0(a_4)$. By Claim 2, $f_1(a_4) = f_0(a_4) = b_4$. Applying Claim 2 again to the pairs $\langle a_1, b_1\rangle$ and $\langle a_4, b_4\rangle$ we obtain $f_1(a_3) = f_4(a_3)$ for all $a_3 \leq a \vee a_0$. Since this works for $\langle a_2, b_2\rangle$ as well, we get $f_1(a_3) = f_2(a_3)$.

We are now able to extend the definition of a single f_0, for $\langle a_0, b_0\rangle \in U$, to all points: with $\langle a_1, b_1\rangle \in U$, $p \wedge (a_0 \vee a_1) = p \wedge (b_0 \vee b_1)$ and $a_0 \wedge (a \vee a_1) = 0$ we can define

$$
f(b) = \begin{cases} b \text{ if } b \leq p,\\ (a \vee b) \wedge (b_0 \vee (p \wedge (a_0 \vee b))) \text{ if } b \nleq a \vee a_0,\\ (a \vee b) \wedge (b_1 \vee (p \wedge (a_1 \vee b))) \text{ if } b \nleq a \vee a_1. \end{cases}
$$

Let q be a line. If $a \leq q$ or $q = p$, put $f(q) = q$. For q otherwise we can find a pair $\langle a_3, b_3\rangle \in U$ with $f(a_3) = b_3$ and $q = (q \wedge (a \vee a_3)) \vee (q \wedge p)$. Moreover, if we let $a_4 = q \wedge (a \vee a_3)$, we can determine $b_4 = f(a_4)$. We define

$f(q) = b_4 \vee (q \wedge p)$. Now for a point $b \leq q$, $b \neq a_4$ and $b \neq q \wedge p$, we have $f(b) = (a \vee b) \wedge (b_4 \vee (p \wedge (a \vee a_4)))$ since $b \wedge (a \vee a_4) = b \wedge (a \vee a_1) = 0$. But then $f(b) \leq f(q)$ as desired. Our collineation f is completely defined. □

By an elation of a projective plane we mean a collineation with an axis and a center on the axis.

5.8. Theorem. *Let f, g be two non-identical elations of a projective plane with the same axis p but different centers a and b. Then gf is an elation with axis p and center different from both a and b.*

Proof. Clearly, gf is a collineation with axis p. By 5.5 it has a center c. Suppose that c is not on p. We have $f(a) = a$, $g(b) = b$ and $gf(c) = c$. The points $a, c, f(c)$ and also the points $b, c, g^{-1}(c)$ are collinear; but $f(c) = g^{-1}(c)$, so the two lines are the same and $a = b$, a contradiction. Thus gf is an elation. If $c = a$ then g has center a, a contradiction. Similarly, we cannot have $c = b$. □

Let P be a projective plane. For any point a and any line p, the set of all collineations with center a and axis p is a subgroup of the group of all collineations of P; it is denoted by $G(a, p)$. It follows from 5.8 that the set of all elations with axis p is also a subgroup; this group is denoted by $G(p)$.

5.9. Theorem. *Let p be a line of a projective plane. If there exist two distinct points a, b on p such that the groups $G(a, p)$ and $G(b, p)$ are both nontrivial then the group $G(p)$ of elations with axis p is abelian and all its non-unit elements are either of infinite order or of the same prime order.*

Proof. Let $f, g \in G(p)$ be two elations with different centers a, b respectively. Let c be any point not on p. Since $f(c) < a \vee c$, we have $gf(c) < g(a) \vee g(c) = a \vee g(c)$. But also $gf(c) < b \vee f(c)$, so $gf(c) = (a \vee g(c)) \wedge (b \vee f(c))$. Quite similarly, $fg(c) = (a \vee g(c)) \wedge (b \vee f(c))$ and we get $gf(c) = fg(c)$. If c is a point on p then $gf(c) = c = fg(c)$. Thus $gf = fg$ for any f, g with different centers.

Now let $f, g \in G(p)$ have the same center a. In order to prove that $gf = fg$, it is sufficient to consider the case when f, g are non-identical. There exist a point $b \neq a$ and a non-identical elation $h \in G(b, p)$. By 5.8, gh is an elation with center c different from both a and b. Thus f commutes with both gh and h; consequently, it commutes with g.

Let there exist a non-unit element of $G(p)$ of a finite order. Then there exists a non-unit element f of $G(p)$ of a prime order s; denote its center by a. If g is any other non-unit element of $G(p)$ with center $b \neq a$ then gf has center c different from both a and b and $(gf)^s = g^s \in G(c, p) \cap G(b, p)$ implies $g^s = 1_{G(p)}$, so g is of order p. Using this g, we can see that also any non-unit element in $G(a, p)$ is of order s. □

5.10. Theorem. *Let p be a line of a projective plane. If there are two distinct points a, b on p such that the plane is both $a - p$ and $b - p$ transitive then it is p-transitive.*

Proof. Let c be a point on p different from both a and b. Let c, a_0, b_0 be three distinct collinear points such that a_0, b_0 are not on p. Denote by d the point $(a \vee a_0)\wedge)b \vee b_0)$. There exist collineations $f \in G(a,p)$ and $g \in G(b,p)$ with $f(a_0) = d$ and $g(d) = b_0$. Clearly, the collineation $h = gf$ belongs to $G(c,p)$ and $h(a_0) = b_0$. □

5.11. Theorem. *Let p, q be two distinct lines of a projective plane such that the plane is both p-transitive and q-transitive. Then the plane is r-transitive for any line r containing the point $p \wedge q$.*

Proof. Let r contain the point $a = p \wedge q$ and be different from both p and q. Take any point b on r different from a and let b_1, b_2 be two points not on r such that b, b_1, b_2 are pairwise distinct and collinear. Put $c = (b \vee b_1 \vee b_2) \wedge p$ and $d = (b \vee b_1 \vee b_2) \wedge q$. There exists an elation f with center c and axis p such that $f(d) = b$. We have $f(q) = r$. The points $d = f^{-1}(b)$, $f^{-1}(b_1)$, $f^{-1}(b_2)$ are collinear; there exists an elation g with center d and axis q such that $gf^{-1}(b_1) = f^{-1}(b_2)$. It is easy to check that fgf^{-1} is an elation with center b and axis r, such that $fgf^{-1}(b_1) = b_2$. Since b was an arbitrary point on p different from a, we can use 5.10 to finish the proof. □

5.12. Corollary. *If a projective plane is transitive with respect to three different lines having no common point then it is transitive with respect to any line.*

CHAPTER 3

COORDINATIZATION

1. Projective planes — ternary rings

By a ternary ring we mean an algebra with one ternary operation T and two constants $0, 1$ such that the two constants are distinct elements and the following five conditions are satisfied:

(T1) $T(0, a, b) = T(a, 0, b) = b$ for all a, b
(T2) $T(1, a, 0) = T(a, 1, 0) = a$ for all a, b
(T3) for any triple a, b, c there exists precisely one x with $T(a, b, x) = c$
(T4) for any quadruple a, b, c, d with $a \neq c$ there exists precisely one x with $T(x, a, b) = T(x, c, d)$
(T5) for any quadruple a, b, c, d with $a \neq c$ there exists precisely one pair x, y with $T(a, x, y) = b$ and $T(c, x, y) = d$

1.1. Example. Every division ring can be considered as a ternary ring with respect to $T(x, y, z) = xy + z$.

1.2. Lemma. *Let a, b, c be three elements of a ternary ring such that $a \neq 0$. Then there exist precisely one element x with $T(x, a, b) = c$ and precisely one element y with $T(a, y, b) = c$.*

Proof. It is easy. □

For elements a, b of a ternary ring we define $a + b = T(a, 1, b)$ and $ab = T(a, b, 0)$. In general, these two operations are neither commutative nor associative; in general, it is not true that $T(x, y, z) = xy + z$.

1.3. Theorem. *For any element a of a ternary ring D we have*

$a + 0 = 0 + a = a,$
$a0 = 0a = 0,$
$a1 = 1a = a.$

D is a loop with respect to $+, 0$ and $D - \{0\}$ is a loop with respect to $\cdot, 1$.

Proof. It is obvious. □

For every ternary ring D we define an incidence structure $PP(D)$ in the following way. Its points are the elements (a, b), (m) and (∞) where a, b, m are arbitrary elements of D; its lines are the elements $L_{m,b}$, L_a and L_∞ where a, b, m are arbitrary elements of D; a line $L_{m,b}$ contains the points (x, y) with $y = T(x, m, b)$ and the point (m); a line L_a contains the points

(x, y) with $x = a$ and the point (∞); the line L_∞ contains the points (m) with arbitrary m and the point (∞).

1.4. THEOREM. *Let D be a ternary ring. Then $PP(D)$ is a projective plane and the quadruple $(0,0),(1,1),(0),(\infty)$ is its frame.*

PROOF. First we must prove that any two distinct points are on precisely one line. If the points are (a, b) and (a', b') with $a \neq a'$, it follows from (T5). If the points are (a, b) and (m) with $m \in D$, it follows from (T3). In the remaining cases it is clear.

Next we must prove that any two distinct lines are on precisely one point. If the lines are $L_{m,b}$ and $L_{m',b'}$ with $m \neq m'$, it follows from (T4). If the lines are $L_{m,b}$ and $L_{m,b'}$ with $b \neq b'$, it follows from (T3). In the remaining cases it is clear.

One can easily check that no three of the four points $(0,0),(1,1),(0),(\infty)$ are collinear. □

1.5. THEOREM. *For every projective plane P and every frame O, I, X, Y of P there exists a ternary ring D with zero O and unit I and such that $P \simeq PP(D)$.*

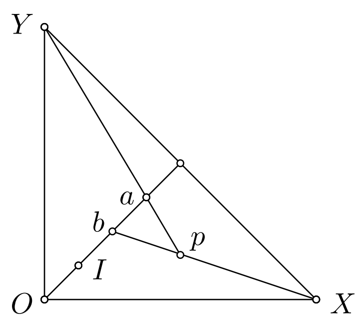

Fig. 7

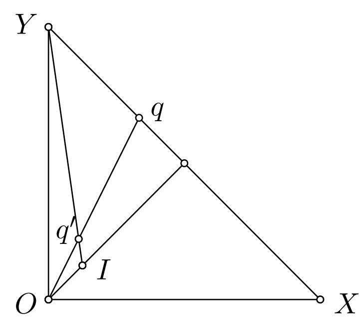

Fig. 8

PROOF. Denote by D the set of the points on the line $O \vee I$ but not on the line $X \vee Y$. For every point p not on $X \vee Y$ put $\phi_0(p) = \langle a, b\rangle$ where $a = (O \vee I) \wedge (p \vee Y)$ and $b = (O \vee I) \wedge (p \vee X)$ (we call a and b the first and the second coordinate of p respectively), so that ϕ_0 is a bijection of the set of points not on $X \vee Y$ onto $D \times D$; for $\langle a, b\rangle \in D \times D$ we have $\phi_0^{-1}(a, b) = (a \vee Y) \wedge (b \vee X)$. Clearly, a point not on $X \vee Y$ has the first coordinate O if and only if it is on $O \vee Y$; it has the first coordinate I if and only if it is on $I \vee Y$. For every point q on $X \vee Y$ other than Y denote by $\phi_1(q)$ the second coordinate of the point $q' = (I \vee Y) \wedge (O \vee q)$ (the first coordinate is I), so that ϕ_1 is a bijection of the set of points on $X \vee Y$ without Y onto the set D; for $m \in D$ we have $\phi^{-1}(m) = (X \vee Y) \wedge (O \vee \phi_0^{-1}(I, m))$.

It is easy to prove that each line different from both $X \vee Y$ and $O \vee Y$ contains for every $x \in D$ precisely one point with coordinates x, y. For

$m, b, x, y \in D$ write $y = T(x, m, b)$ if the point with coordinates x, y is on the line $\phi_0^{-1}(O, b) \vee \phi_1^{-1}(m)$. Then T is a ternary operation on D and it is easy to check that D is a ternary ring with respect to T, O and I.

The mapping ϕ extending both ϕ_0 and ϕ_1 and assigning ∞ to Y is a bijection of the set of points of P onto the set of points of $PP(D)$. It is easy to extend ϕ to the lines to obtain an isomorphism. □

A ternary ring is called reducible if $T(x, y, z) = xy + z$ for all x, y, z. Such ternary rings can be considered as algebras with two binary operations and two constants.

1.6. Example. The ternary ring of the free projective plane with respect to the frame consisting of the four generators is not reducible and the addition is neither commutative nor associative.

2. Line transitive projective planes — VW-systems

By a double loop we mean an algebra D with two binary operations $+, \cdot$ and two constants $0, 1$ such that D is a loop with respect to $+, 0$, $D - \{0\}$ is a loop with respect to $\cdot, 1$ and $x0 = 0x = 0$ for all $x \in D$. (In particular, the product of two non-zero elements is non-zero.) A double loop is called coordinatizable if it has the following two properties:

(1) for any elements a, b, c, d with $a \neq c$ there exists a unique element x with $xa + b = xc + d$
(2) for any elements a, b, c, d with $a \neq c$ there exists a unique pair of elements x, y with $ax + y = b$ and $cx + y = d$

Clearly, reducible ternary rings can be identified with coordinatizable double loops.

2.1. Theorem. *Let D be a ternary ring. The projective plane $PP(D)$ is $(\infty) - L_\infty$ transitive if and only if the following are true:*

(1) *D is reducible*
(2) *D is a group with respect to $+$*

Proof. Let $PP(D)$ be $(\infty) - L_\infty$ transitive. Let $a, m, b \in D$. Put

$O = (0, 0)$, $X = (0)$, $Y = (\infty)$, $M = (m)$, $Q = (1)$,
$R = (0, b)$, $S = (a, T(a, m, b))$,
$U = (Y \vee S) \wedge (O \vee M)$,
$V = (X \vee U) \wedge (O \vee Q)$,
$W = (V \vee Y) \wedge (R \vee Q)$

(see Fig. 9).

Denote by f the collineation with axis $L_\infty = X \vee Y$ and center $\infty = Y$, such that $f(O) = R$. We have

$f(X) = X$,
$f(U) = f((Y \vee U) \wedge (O \vee M)) = (Y \vee U) \wedge (R \vee M) = S$,
$f(V) = f((Y \vee V) \wedge (O \vee Q)) = (Y \vee V) \wedge (R \vee Q) = W$.

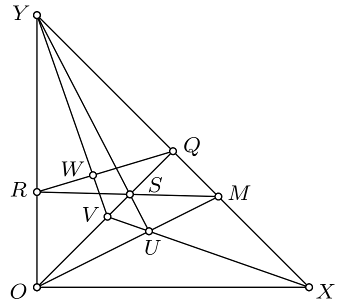

Fig. 9

Since X, U, V are collinear, it follows that X, S, W are collinear. Since $Y \vee S$ has equation $x = a$ and $O \vee M$ has equation $y = xm$, we have $U = (a, am)$. Since $O \vee Q$ has equation $y = x$ and $X \vee U$ has equation $y = am$, we have $V = (am, am)$. Since $V \vee Y$ has equation $x = am$ and $R \vee Q$ has equation $y = x+b$, we have $W = (am, am+b)$. Since X, S, W are collinear, the points S and W have the same second coordinates and thus $T(a, m, b) = am + b$.

In order to prove that D is a group with respect to $+$, it remains to show that the addition is associative. Let $b \in D$ and f be the collineation with axis L_∞ and center ∞, such that $f(0,0) = (0,b)$. The line with equation $y = x$ is mapped to the line with equation $y = x + b$ and the line with equation $x = c$ (where c is an arbitrary element of D) is mapped to itself, so that $f(c,c) = (c + b)$. For any element $a \in D$, the line with equation $y = c$ is mapped to the line with equation $y = c + b$ and the line with equation $x = a$ is mapped to itself, so that $f(a,c) = (a, c + b)$. Let d be an arbitrary element of D and g be the collineation with axis L_∞ and center ∞ such that $g(0,0) = (0,d)$. Then $gf(0,0) = g(0,b) = (0, b+d)$ and thus $gf(a,c) = (a, c + (b + d))$. On the other hand, $gf(a,c) = g(a, c + b) = (a, (c + b) + d)$. We get $c + (b + d) = (c + b) + d$ for arbitrary $b, c, d \in D$.

Conversely, let (1) and (2) be satisfied. For an arbitrary element b of D define a mapping f as follows: a point (a,c) is mapped to $(a, c + b)$, the points on L_∞ to themselves, a line $L_{m,c}$ is mapped to the line $L_{m,c+b}$ and the other lines to themselves. Clearly, it is sufficient to prove that f is a collineation and for this it is sufficient to show that if a point (a,d) lies on $L_{m,c}$ then $(a, d+b)$ lies on $L_{m,c+b}$. If $d = am+c$ then $d+b = (am+c)+b = am + (c + b)$. □

By a VW-system (or Veblen-Wedderburn system) we mean a double loop D satisfying the following conditions:

(VW1) D is an abelian group with respect to addition
(VW2) $(a + b)c = ac + bc$ for all $a, b, c \in D$
(VW3) for any elements $a, b, c \in D$ with $a \neq b$, the equation $xa = xb + c$ has a unique solution

Clearly, every VW-system is a coordinatizable double loop.

2.2. THEOREM. *A ternary ring D is a VW-system if and only if the projective plane $PP(D)$ is L_∞-transitive.*

PROOF. Let D be a VW-system. It is easy to check that for any $r, s \in D$ we get a collineation if we map a point (x, y) to $(x + r, y + s)$, all the other points to themselves, a line with equation $x = c$ to a line with equation $x = r + c$, a line with equation $y = xm + b$ to the line with equation $y = xm - rm + s + b$ and the line L_∞ to itself. For $s = rt$ this collineation is an elation with center (t) and axis L_∞.

Conversely, let the plane be L_∞-transitive. By 2.1, D is a coordinatizable double loop and D is a group with respect to addition. We have seen in the proof of the theorem that for any element $b \in D$ there exists an elation $f_b \in G((\infty), L_\infty)$ such that $f_b(a, c) = (a, c + b)$ for all $a, c \in D$. By 2.5.9, the group $G(L_\infty)$ is abelian. For any $b, d \in D$ we have $f_d f_b = f_b f_d$ and for any $a, c \in D$ $(a, c + b + d) = (a, c + d + b)$, i.e., $b + d = d + b$. Thus D is an abelian group with respect to addition.

Let $a, m, b \in D$. Denote by f the elation with center (0) and axis L_∞, such that $f(0, 0) = (b, 0)$. The line with equation $y = x$ is mapped to the line with equation $y = x - b$ and the line with equation $y = a$ is mapped to itself, so that $f(a, a) = (a + b, a)$. The line with equation $x = a$ is mapped to the line with equation $x = a + b$ and the line with equation $y = am$ is mapped to itself, so that $f(a, am) = (a + b, am)$. Since $f(0, 0) = (b, 0)$, the line with equation $y = xm$ is mapped to the line with equation $y = xm - bm$. Since the point (a, am) is on the line with equation $y = xm$, it follows that $(a + b, am)$ is on the line with equation $y = xm - bm$. Thus $am = (a + b)m - bm$, i.e., $(a + b)m = am + bm$.

We have proved (VW1) and (VW2). Condition (VW3) is also satisfied, since the lines with equations $y = xa$ and $y = xb + c$ contain a single point (x, y). □

2.3. EXAMPLE. Let F be a finite field and $x^2 - rx - s$ be an irreducible quadratic polynomial over F. Put $D = F \times F$, identify the elements of F with the pairs $\langle a, 0\rangle$ $(a \in F)$ and define addition and multiplication on D as follows: $\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$; $\langle a, b\rangle c = \langle ac, bc\rangle$; if w and $\langle a, b\rangle$ are two elements of D with $b \neq 0$ then w can be uniquely expressed as $w = c + \langle a, b\rangle d$ for some $c, d \in F$ and we put $w\langle a, b\rangle = \langle ac + adr + ds, bc + bdr\rangle$. It can be proved that D is a VW-system.

3. Moufang planes — alternative fields

3.1. THEOREM. *Let D be a ternary ring. The projective plane $PP(D)$ is p-transitive for every line p containing (∞) if and only if D is a reducible ternary ring satisfying*

(1) *D is an abelian group with respect to addition*
(2) *$(a + b)c = ac + bc$ for all $a, b, c \in D$*
(3) *$c(a + b) = ca + cb$ for all $a, b, c \in D$*

(4) *for every* $a \in D - \{0\}$ *there exists an element* a^{-1} *such that* $aa^{-1} = a^{-1}a = 1$ *and* $a^{-1}(ab) = b$ *for all* $b \in D$

If these conditions are satisfied then $(y(zy))x = y(z(yx))$ *for all* $x, y, z \in D$.

PROOF. Let the projective plane be p-transitive for every line p containing (∞). By 2.2, D is a reducible ternary ring satisfying (1) and (2). Let $a, b, m \in D$. Let f be the elation with center (∞) and axis $x = 0$, sending the point (0) to (m). Since f sends (0) to (m) and $(0, b)$ to itself, it sends the line $y = b$ to the line $y = xm + b$. Since , moreover, f sends the line $x = a$ to itself, it sends the point (a, b) to $(a, am+b)$. Thus f sends $(1, b)$ to $(1, m+b)$; since it sends $(0, 0)$ to itself, it sends the line $y = xb$ to $y = x(m + b)$. Since the point (a, ab) (which is sent to $(a, am + ab)$) is on the line $y = xb$ (which is sent to $y = x(m + b)$), the point $(a, am + ab)$ is on the line $y = x(m + b)$. This means that $am + ab = a(m + b)$ and we have proved (3).

Let $a \in D - \{0\}$. Let g be the elation with center $(0, 0)$ and axis $x = 0$, sending the point (0) to $(-1 - a, 0)$. Since $(0, 1 + a)$ is sent to itself and (0) is sent to $(-1 - a, 0)$, the line $y = 1 + a$ is sent to the line $y = x + 1 + a$. Since (0) is sent to $(-1 - a, 0)$ and $(0, b + ab)$ (for any $b \in D$) is sent to itself, the line $y = b + ab$ is sent to the line $y = xb + b + ab$. Thus the line $y = 1 + a$ is sent to $y = x + 1 + a$; since, moreover, the line $y = x(1 + a)$ is sent to itself, the point $(1, 1 + a)$ is sent to the point $(d, d + 1 + a)$ for some $d \in D$ satisfying $d(1 + a) = d + 1 + a$. Since (∞) is sent to itself and $(1, 1 + a)$ is sent to $(d, d + 1 + a)$, the line $x = 1$ is sent to the line $x = d$. Since the line $y = x(b + ab)$ is sent to itself and the line $y = b + ab$ is sent to $y = xb + b + ab$, the point $(1, b + ab)$ is sent to the point $(d, d(b + ab))$ and we have $d(b + ab) = db + b + ab$. Put $u = d - 1$. We get $ua = 1$ from $d(1 + a) = d + 1 + a$ and $u(ab) = b$ from $d(b + ab) = db + b + ab$. Similarly as $a \neq 1$ implies $ua = 1$ and $u(ab) = b$ for some u, there exists an element v with $vu = 1$ and $v(ua) = a$. Hence $v = v1 = a$ and $au = 1$. With $a^{-1} = u$ we get (4).

Conversely, let D be a reducible ternary ring satisfying the four conditions. Clearly, D is a VW-system and thus, by 2.2, the projective plane is L_∞-transitive. By 2.5.11, it is sufficient to prove that it is also transitive with respect to the line with equation $x = 1$. For this, it is sufficient to prove that there exists a collineation f mapping L_∞ to this other line. Define f in this way: f sends the point (∞) to itself, the points (m) to $(1, m)$, the points (c, d) with $c \neq 0$ and $c \neq -1$ to $((1+c^{-1})^{-1}, (1+c)^{-1}d)$, the points $(0, d)$ to themselves, the points $(-1, d)$ to $(-d)$, the line L_∞ to the line with equation $x = 1$, the lines $x = c$ with $c \neq 0$ and $c \neq -1$ to the lines $x = (1 + c^{-1})^{-1}$, the line $x = 0$ to itself, the line $x = -1$ to L_∞ and the lines $y = xm + b$ to the lines $y = x(m - b) + b$. In order to verify that this is a collineation, a crucial step is to check that if a point (c, d) with $c \neq 0$ and $c \neq -1$ is on a line $y = xm + b$ then its image is on the image of the line, i.e., to check that $(1 + c)^{-1}(cm + b) = (1 + c^{-1})^{-1}(m - b) + b$. This is a consequence of the following claims.

Claim 1. $(1+c^{-1})^{-1} = 1-(1+c)^{-1}$. Indeed,

$$\begin{aligned} 1 &= (c+1)(1+c)^{-1}, \\ 1 &= c(1+c)^{-1} + (1+c)^{-1}, \\ c^{-1} &= (1+c)^{-1} + c^{-1}(1+c)^{-1} = (1+c^{-1})(1+c)^{-1}, \\ 1 &= (1+c^{-1})(1-(1+c)^{-1}). \end{aligned}$$

Claim 2. $(1+c)^{-1}b = (1+c^{-1})^{-1}(-b)+b$. This follows from Claim 1.

Claim 3. $(1+c)^{-1}(cm) = (1+c^{-1})^{-1}m$. Indeed,

$$\begin{aligned} (1+c)^{-1}((c+1)m) &= m, \\ (1+c)^{-1}(cm) &= m-(1+c)^{-1}m, \\ (1+c)^{-1}(cm) &= (1-(1+c)^{-1})m \end{aligned}$$

and we can use Claim 1.

Let us prove $(y(zy))x = y(z(yx))$ for $x,y,z \in D$. If either $y=0$ or $y=-z^{-1}$, it is evident. Let $y \neq 0$ and $y \neq -z^{-1}$. Put

$$t = (y^{-1}-(y+z^{-1})^{-1})(y(zy)+y).$$

We have

$$\begin{aligned} (y+z^{-1})t &= (y+z^{-1})(zy+1-(y+z^{-1})^{-1}(y(zy))-(y+z^{-1})^{-1}y) \\ &= (y+z^{-1})(zy+1)-y(zy)-y \\ &= y(zy)+y+y+z^{-1}-y(zy)-y = y+z^{-1}, \end{aligned}$$

so that $t=1$. Thus the elements $y^{-1}-(y+z^{-1})^{-1}$ and $y(zy)+y$ are inverse to each other and it follows that

$$(y^{-1}-(y+z^{-1})^{-1})((y(zy))x+yx) = x.$$

Put

$$\begin{aligned} w &= (y^{-1}-(y+z^{-1})^{-1})(y(z(yx))+yx) \\ &= z(yx)+x-(y+z^{-1})^{-1}(y(z(yx))+yx). \end{aligned}$$

We have

$$\begin{aligned} (y+z^{-1})w &= (y+z^{-1})(z(yx)+x)-y(z(yx))-yx \\ &= yx+yx+z^{-1}x-yx = (y+z^{-1})x, \end{aligned}$$

so that $w=x$. Since the expressions of x and w are equal, we get $(y(zy))x = y(z(yx))$. □

By a Moufang plane we mean a projective plane that is p-transitive for any line p.

By an alternative field we mean a double loop D satisfying the following conditions:

(AR1) D is an abelian group with respect to addition
(AR2) $(a+b)c = ac+bc$ for all $a,b,c \in D$

(AR3) $c(a+b) = ca + cb$ for all $a, b, c \in D$
(AR4) for every $a \in D - \{0\}$ there exists an element a^{-1} such that $aa^{-1} = a^{-1}a = 1$ and $a^{-1}(ab) = b$ and $(ba)a^{-1} = b$ for all $b \in D$

3.2. THEOREM. *Let D be a ternary ring. The projective plane $PP(D)$ is a Moufang plane if and only if D is* (*reducible and*) *an alternative field.*

PROOF. Let $PP(D)$ be a Moufang plane. In order to prove that D is an alternative field, by 3.1 it remains to prove $(ba)a^{-1} = b$. Let f be the elation with center $(0,0)$ and axis $y = 0$, sending the point (∞) to $(0,-1)$. Since (∞) is sent to $(0,-1)$ and $(a,0)$ is sent to itself, the line $x = a$ is sent to the line $y = xa^{-1} - 1$. Hence the line $x = 1$ is sent to $y = x - 1$; since the line $y = x(1-ab)$ is sent to itself, the point $(1, 1-ab)$ is sent to $((ab)^{-1}, (ab)^{-1}-1)$. Since, moreover, the point (0) is sent to itself, the line $y = 1 - ab$ is sent to $y = (ab)^{-1} - 1$. Since $x = a$ is sent to $y = xa^{-1} - 1$ and $y = x(a^{-1} - b)$ is sent to itself, the point $(a, 1 - ab)$ is sent to $(b^{-1}, b^{-1}a^{-1} - 1)$. Since, moreover, the point (0) is sent to itself, the line $y = 1 - ab$ is sent to $y = b^{-1}a^{-1} - 1$. But the same line is sent to $y = (ab)^{-1} - 1$ and we get $(ab)^{-1} = b^{-1}a^{-1}$. Now $(ba)a^{-1} = (a^{-1}b^{-1})^{-1}a^{-1} = (a(a^{-1}b^{-1}))^{-1} = (b^{-1})^{-1} = b$.

Conversely, let D be an alternative field. By 3.1, the projective plane is transitive with respect to any line containing (∞). So, by 2.5.12, it remains to prove that it is also transitive with respect to some line not containing (∞). For this, it is sufficient to find a collineation f not sending (∞) to itself. We can construct f in the following way: f sends a point (a,b) to (b,a), a point (m) with $m \neq 0$ to (m^{-1}), the point (0) to (∞), the point (∞) to (0), a line $y = xm + b$ with $m \neq 0$ to $y = xm^{-1} - bm^{-1}$, a line $x = c$ to $y = c$, a line $y = c$ to $x = c$ and the line L_∞ to itself. □

3.3. REMARK. It can be proved that if a projective plane is transitive with respect to two different lines, then it is transitive with respect to any line, so that the conditions of 3.1 are equivalent to those of 3.2; this means that in the definition of an alternative field, the condition $(ba)a^{-1} = b$ is a consequence of the other conditions. Also, it can be proved that for a Moufang plane P, all the ternary rings of P (for all possible frames of P) are (alternative fields and) isomorphic to each other.

4. Arguesian projective planes — division rings

4.1. THEOREM. *Let D be a ternary ring. The projective plane $PP(D)$ is $(0) - L_0$ transitive if and only if D is a reducible ternary ring and the multiplication of D is associative* (*so that $D - \{0\}$ is a group with respect to multiplication*).

PROOF. Let the plane be $(0) - L_0$ transitive. For every $m \in D - \{0\}$ there exists a collineation f_m with center (0) and axis $x = 0$ sending (m) to (1). Since $(0,0)$ is sent to itself and (m) is sent to (1), the line $y = xm$ is sent to $y = x$. Since, moreover, the line $y = am$ (for any $a \in D$) is sent to itself, the point (a, am) is sent to (am, am). Since, moreover, (∞) is sent to itself, the line $x = a$ is sent to $x = am$. Since, moreover, the line $y = c$ (for an arbitrary $c \in D$) is sent to itself, the point (a, c) is sent to (am, c). Since the point $(0, b)$ (for an arbitrary $b \in D$) is sent to itself and (m) is sent to (1), the line $y = T(x, m, b)$ is sent to the line $y = x+b$. Consequently, if a point (a, c) is on the line $y = T(x, m, b)$ then (am, c) is on the line $y = x+b$. We get $T(a, m, b) = am + b$ for all $a, b \in D$.

Let $m, n \in D - \{0\}$ and $a \in D$. The point $(a, 1)$ is sent to $(am, 1)$ by f_m and this point is sent to $((am)n, 1)$ by f_n, so that $(a, 1)$ is sent to $((am)n, 1)$ by $f_n f_m$. In particular, $(1, 1)$ is sent to $(mn, 1)$ by $f_n f_m$. Hence $f_n f_m = f_{mn}$. But $(a, 1)$ is sent to $(a(mn), 1)$ by f_{mn} and we get $(am)n = a(mn)$. The multiplication is associative.

Conversely, let D be reducible and let $G - \{0\}$ be a group. For every $m \neq 0$ we can define a collineation as follows: a point (a, b) is sent to (am, b), a point (n) to $(m^{-1}n)$, the point (∞) to itself, a line $y = xn + n$ is sent to the line $y = xm^{-1}n + b$, a line $x = a$ to $x = am$ and the line L_∞ to itself. If m runs over all non-zero elements of D, we obtain all possible collineations with center (0) and axis L_0. □

4.2. Theorem. *The projective plane corresponding to a division ring is arguesian. Given an arguesian projective plane P, for every frame of P the corresponding ternary ring is reducible and is a division ring; all these division rings (for all frames of P) are isomorphic to each other.*

Proof. It is clear that for a division ring D, the lattice $PP(D)$ is isomorphic to the lattice of submodules, and thus to the congruence lattice of the 2-dimensional vector space over D; it follows from 1.4.3 that the lattice is arguesian.

Let P be an arguesian projective plane. It follows from 2.5.7, 1.5, 3.2 and 4.1 that for every frame of P the corresponding ternary ring is reducible and is a division ring. In order to prove that all these division rings are isomorphic to each other, it is sufficient to prove that for any two frames there exists a collineation sending the points of the first to the points of the second frame (in the given order). We can assume that $P = PP(D)$ for a division ring D and that one of these two frames is the frame $(0, 0), (1, 1), (0), (\infty)$.

By a triangle of P we mean a triple of non-collinear points. By 2.5.7, for any triangle a, b, c and any point c' not on $a \vee b$ there exists a collineation sending a, b, c to a, b, c'. From this it easily follows that for any two triangles a, b, c and a', b', c' there exists a collineation sending a, b, c to a', b', c'.

Thus we may assume that the second frame is $(0, 0), (a, b), (0), (\infty)$ for some $a, b \in D - \{0\}$. A collineation fixing $(0, 0), (0), (\infty)$ and sending (a, b) to $(1, 1)$ can be defined in this way: a point (x, y) is sent to (xa^{-1}, yb^{-1}), a point (m) is sent to (amb^{-1}), the point (∞) is sent to itself, a line $y = xm+c$ is sent to $y = xamb^{-1} + cb^{-1}$, a line $x = c$ is sent to $x = ca^{-1}$ and the line L_∞ is sent to itself. □

CHAPTER 4

FINITE PROJECTIVE PLANES

1. Auxiliary facts from number theory

1.1. Lemma. *For any numbers (more generally, any elements of a commutative ring) $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ we have*

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) =& (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + \\ & (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 + \\ & (x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 + \\ & (x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2\end{aligned}$$

Proof. It can be easily checked. □

1.2. Lemma. (Lagrange's theorem) *Every natural number can be expressed as the sum of four quadrates of natural numbers.*

Proof. By 1.1 it is sufficient to prove that every prime number p can be expressed as the sum of four quadrates. For $p = 2$ we have $p = 1^2 + 1^2 + 0^2 + 0^2$. Let $p \geq 3$.

Claim 1. There exist two natural numbers x, y such that $x, y \leq \frac{1}{2}(p-1)$ and $x^2 + y^2 + 1 \equiv 0 \pmod{p}$. The numbers x^2 with $0 \leq x \leq \frac{1}{2}(p-1)$ are pairwise incongruent modulo p, since if $x^2 \equiv x'^2$ then the number $x^2 - x'^2 = (x - x')(x + x')$ is divisible by p and thus either $x - x'$ or $x + x'$ is divisible by p which is possible only if $x = x'$. Similarly, the numbers $-y^2 - 1$ with $0 \leq y \leq \frac{1}{2}(p-1))$ are pairwise incongruent modulo p. In the union of the two sets there are $p+1$ numbers, so two of them must be congruent. This means that there are x and y with $0 \leq x, y \leq \frac{1}{2}(p-1))$ such that $x^2 \equiv -y^2 - 1 \pmod{p}$.

Claim 2. There exists a number m with $0 < m < p$, such that mp can be expressed as the sum of four quadrates. Let x, y be as in Claim 1. We have $x^2 + y^2 + 1 = mp$ for some $m > 0$. Since $x^2 + y^2 + 1 < \frac{1}{4}p^2 + \frac{1}{4}p^2 + 1 < p^2$, we have $m < p$. We can express p as $p = x^2 + y^2 + 1^2 + 0^2$.

Claim 3. Let m be the least number with the properties from Claim 2. Then $m = 1$. Let $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Suppose first that m is even. Then either all the numbers x_1, x_2, x_3, x_4 are even or they are all odd or two of them are even and two of them are odd. So, without loss of generality we can assume that both $x_1 + x_2$ and $x_3 + x_4$ are even. Then

$$\frac{1}{2}mp = (\frac{x_1 + x_2}{2})^2 + (\frac{x_1 - x_2}{2})^2 + (\frac{x_3 + x_4}{2})^2 + (\frac{x_3 - x_4}{2})^2,$$

a contradiction with the minimality of m. Thus m is odd. Suppose that $m > 1$. Since $0, \pm 1, \dots, \pm\frac{1}{2}(m-1)$ are m numbers pairwise incongruent modulo m, there exist numbers y_1, y_2, y_3, y_4 with $|y_i| < \frac{1}{2}m$ such that $y_i \equiv x_i \pmod{m}$. We have $y_1^2+y_2^2+y_3^2+y_4^2 \equiv x_1^2+x_2^2+x_3^2+x_4^2 \equiv 0 \pmod{m}$, so that $y_1^2+y_2^2+y_3^2+y_4^2 = mn$ for some $n \geq 0$. If $n = 0$ then $y_i = 0$ for all i, $x_i \equiv 0 \pmod{m}$ for all i, $x_1^2+x_2^2+x_3^2+x_4^2 \equiv 0 \pmod{m^2}$, $mp \equiv 0 \pmod{m^2}$ and $p \equiv 0 \pmod{m}$, a contradiction. Thus $n > 0$. Since $mn < 4 \cdot \frac{1}{4}m^2 = m^2$, we have $n < m$. Since $x_1y_1+x_2y_2+x_3y_3+x_4y_4 \equiv x_1^2+x_2^2+x_3^2+x_4^2 \pmod{m}$, we have $x_1y_1+x_2y_2+x_3y_3+x_4y_4 = mz$ for some z. For $i, k = 1, 2, 3, 4$ with $i \neq k$ we have $x_iy_k - x_ky_i \equiv x_ix_k - x_kx_i = 0 \pmod{m}$, so that $x_iy_k - x_ky_i = mz_{i,k}$ for some integers $z_{i,k}$. Now

$$\begin{aligned} mpmn &= (x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2) \\ &= m^2z^2 + m^2(z_{1,2}+z_{3,4})^2 + m^2(z_{1,3}+z_{4,2})^2 + (z_{1,4}+z_{2,3})^2 \end{aligned}$$

by 1.1, so that $pn = z^2 + (z_{1,2}+z_{1,3})^2 + (z_{1,3}+z_{4,2})^2 + (z_{1,4}+z_{2,3})^2$, a contradiction with the minimality of m. □

1.3. Lemma. *A natural number a can be expressed as the sum of two squares of natural numbers if and only if for every prime number p with $p \equiv 3 \pmod{4}$, the largest nonnegative number e such that p^e divides a is even.*

Proof. Let $a = b^2+c^2$ and suppose that there is a prime number $p \equiv 3 \pmod{4}$ such that a is divisible by p^{2k+1} but not by p^{2k} for some k. Denote by m the largest number such that p^m divides b. It is easy to see that m is also the largest number such that p^m divides c, and $m \leq k$. We can suppose that $m = 0$, since otherwise we can replace a with a/p^{2m}, b with b/p^m and c with c/p^m. We have $b^2+c^2 = a \equiv 0 \pmod{p}$, $b \not\equiv 0 \pmod{p}$ and $c \not\equiv 0 \pmod{p}$. Thus there exists an element z of the finite field F with p elements such that $z^2 = -1$ (z is the element corresponding to b divided by the element corresponding to c). As it is well known, the multiplicative group of this field is a cyclic group. Let g be its generator. We have $z = g^c$ for some c, so $g^{2c} = -1$ and $g^{4c} = 1$. Since the group is of order $p-1$, we get $4c = (p-1)d$ for some d. Thus $2c = \frac{p-1}{2}d$ where $\frac{p-1}{2}$ is odd (since $p \equiv 3$ mod 4), d is even and $2c$ is divisible by $p-1$. Hence $g^{2c} = 1$, a contradiction.

The converse implication is a consequence of the following claims 1 and 3. Let us call a natural number expressible if it can be expressed as the sum of two squares.

Claim 1. The product of two expressible numbers is expressible. Indeed, by 1.1 we have $(x_1^2+x_2^2)(y_1^2+y_2^2) = (x_1y_1+x_2y_2)^2 + (x_1y_2-x_2y_1)^2$.

Claim 2. If p is a prime number with $p \equiv 1 \pmod{4}$ then the number $x = (\frac{1}{2}(p-1))!$ satisfies $x^2+1 \equiv 0 \pmod{p}$. By Fermat's theorem we have $x^{p-1} \equiv 1 \pmod{p}$ whenever x is not divisible by p. Thus the polynomial x^{p-1}, considered as a polynomial over the field with p elements, has $p-1$ different roots $1, \dots, p-1$. It follows that $x^{p-1} - 1 \equiv (x-1)(x-2)\dots(x-$

$(p-1))$ (mod p) for all x. In particular, for $x = 0$ we get $(p-1)! \equiv -1$ (mod p) (Wilson's theorem). Thus

$$-1 \equiv (p-1)! \equiv 1 \cdot 2 \cdot \dots \cdot \frac{1}{2}(p-1)(-\frac{1}{2}(p-1)) \cdot \dots \cdot (-2) \cdot (-1)$$
$$\equiv (-1)^{\frac{1}{2}(p-1)}(\frac{1}{2}(p-1))!^2.$$

Since $p \equiv 1 \pmod 4$, the number $\frac{1}{2}(p-1)$ is even, $(-1)^{\frac{1}{2}(p-1)} = 1$ and we get $(\frac{1}{2}(p-1))!^2 \equiv -1 \pmod p$.

Claim 3. Let p be a prime number with $p \equiv 1$ (mod 4). Then p is expressible. By Claim 2 there exists a natural number z with $z^2 + 1 \equiv 0$ (mod p). Denote by e the least natural number with $e^2 > p$. There are more than p (precisely, e^2) ordered pairs $\langle u, v\rangle$ with $0 \le u, v < p$ and thus among the numbers $u+zv$ there must be two that are congruent modulo p. So, there exist natural numbers u', v', u'', v'' less than p such that $u' + zv' \equiv u'' + zv''$ (mod p) and $v' \not\equiv v'' \pmod p$ (if $v' \equiv v''$ then also $u' \equiv u''$, a contradiction). Put $x = u' - u''$ and $y = v' - v''$. We have $x + zy \equiv 0 \pmod p$, so that $zy \equiv -x$. Put $a = |x|$ and $b = |y|$. We have $zb \equiv \pm a$ and $0 < a, b < e$, so that $a^2, b^2 < p$. We have $a^2 + b^2 \equiv z^2b^2 + b^2 = (z^2 + 1)b^2 \equiv 0 \pmod p$, hence $a^2 + b^2 = mp$ for some m; since $a^2, b^2 < p$, the only possibility for m is $m = 1$. Thus $p = a^2 + b^2$. □

1.4. LEMMA. *Let a, b, c, d, x be natural numbers such that $a^2 + b^2 = x(c^2 + d^2)$. Then also x can be expressed as the sum of two quadrates of natural numbers.*

PROOF. It follows from 1.3. □

2. The Bruck–Ryser theorem

2.1. THEOREM. *Let $n \ge 2$ be a natural number congruent with either 1 or 2 modulo 4. If n cannot be expressed as the sum of two quadrates of natural numbers then there is no projective plane of order n.*

PROOF. Suppose there is. Put $N = n^2 + n + 1$, so that there are N points and N lines. Since n is congruent with either 1 or 2, N is congruent with 3 modulo 4. For $i, j \in \{1, \dots, N\}$ put $a_{i,j} = 1$ if the i-th point is on the j-th line, and put $a_{i,j} = 0$ otherwise. Denote by A the matrix with elements $a_{i,j}$. Then $AA^T = A^TA = nI + J$ where A^T is the transpose of A, I is the unit matrix and J is the matrix with all elements equal 1. For every $j = 1, \dots, N$ define a linear form (over the field of rational numbers) in variables $x_1, \dots, x_N$ by $L_i = \sum_{i=1}^{N} a_{i,j}x_i$. Then

$$L_1^2 + \dots + L_N^2 = n(x_1^2 + \dots + x_N^2) + (x_1 + \dots + x_N)^2$$
$$= n(x_2 + \frac{x_1}{n})^2 + \dots + n(x_N + \frac{x_1}{n})^2 + (x_2 + \dots + x_N)^2.$$

Let us introduce new variables $y_1, \dots, y_N$ by $y_1 = x_2 + \dots + x_n$ and $y_i = x_i + \frac{x_1}{n}$ for $i = 2, \dots, N$. Then $y_1, \dots, y_N$ is another base of the n-dimensional vector space over the field of rational numbers. So, if we express each L_j as a linear form in the new variables, we get

$$L_1^2 + \cdots + L_N^2 = y_1^2 + ny_2^2 + \cdots + ny_N^2$$

for all rational numbers $y_1, \dots, y_N$.

By 1.2 there exist natural numbers a_1, a_2, a_3, a_4 such that $n = a_1^2 + a_2^2 + a_3^2 + a_4^2$. Let us introduce new variables $z_1, \dots, z_N$ as follows: $z_1 = y_1$, $z_{N-1} = y_{N-1}$, $z_N = y_N$ and, for $i \equiv 2 \pmod 4$,

$$\begin{aligned} z_i &= a_1 y_i + a_2 y_{i+1} + a_3 y_{i+2} + a_4 y_{i+3}, \\ z_{i+1} &= a_1 y_{i+1} - a_2 y_i + a_3 y_{i+3} - a_4 y_{i+2}, \\ z_{i+2} &= a_1 y_{i+2} - a_3 y_i + a_4 y_{i+1} - a_2 y_{i+3}, \\ z_{i+3} &= a_1 y_{i+3} - a_4 y_i + a_2 y_{i+2} - a_3 y_{i+1}. \end{aligned}$$

It is easy to check that these variables are again independent, so that if we express each L_j as a linear form in the last variables, we get (according to 1.1)

$$L_1^2 + \cdots + L_N^2 = z_1^2 + \cdots + z_{N-2}^2 + n(z_{N-1}^2 + z_N^2)$$

for all rational numbers $z_1, \dots, z_N$. Observe that L_j are now linear forms with different rational coefficients. We have $L_1 = b_1 z_1 + \cdots + b_N z_N$ for some rational numbers $b_1, \dots, b_N$. If $b_1 \neq 1$, substitute L_1 for z_1; if $b_1 = 1$, substitute $-L_1$ for z_1. Then $L_1^2 = z_1^2$ and (after the substitution) $L_2^2 + \cdots + L_N^2 = z_2^2 + \cdots + z_{N-2}^2 + n(z_{N-1}^2 + z_N^2)$. Proceeding in this way we get $L_{N-1}^2 + L_N^2 = n(z_{N-1}^2 + z_N^2)$. Substitute for z_{N-1} and z_N some positive integers that are multiples of all denominators of coefficients of the forms L_{N-1} and L_N. We obtain $a^2 + b^2 = n(c^2 + d^2)$ for some integers a, b, c, d. By 1.4, n is the sum of two quadrates. □

3. Projective planes of small orders

As we already know, for every prime power n there exists a projective plane of order n. It follows from 2.1 that there are no projective planes of orders 6, 14, 21, 22, 30. It has been proved by C. Lam et al. in 1989 (heavy use of computers) that there is no projective plane of order 10. The remaining numbers, up to 30, are 12, 15, 18, 20, 24, 26, 28. For each of these numbers, the existence of a projective plane of that order is an open problem. It is not known if there is a projective plane of an order that is not a prime power.

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