BINARY EQUALITY LANGUAGES FOR PERIODIC MORPHISMS

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Abstract. Let g and h be binary morphisms defined on $A = \{a, b\}$, and exactly one of them be periodic. It is well known that their equality language is generated by at most one non-empty word. We show that this word is up to the symmetry of the letters a and b equal to $a^i ba^j$, with $i, j \geq 0$. A word of that form is an equality word for all values of i, j.

1. Introduction

Binary equality sets are the simplest non-trivial equality languages. Nevertheless, their precise description is still not known. They were for the first time studied extensively by K. Čulík II and J. Karhumäki in [2]. The binary equality languages of two non-periodic morphisms were recently studied in [3] and [4]. There it is shown that such an equality set is generated by at most two words, and if the rank of $\mathrm{Eq}(g, h)$ is two, it is of the form $\{a^i b, ba^i\}^*$.

The problem is much easier for periodic cases. If both morphisms are periodic and all images have the same primitive root then the equality language contains all words, in which the number of letters a and the number of letters b are in the ratio guaranteeing the length agreement. If just one morphism is periodic, it is easy to show that $\mathrm{Eq}(g, h)$ is generated by at most one non-empty word. The aim of this note is to give a precise description of such a word.

2. Preliminaries

We shall use standard notions of combinatorics on words (see e.g. [1]). From the folklore we need especially the Periodicity Lemma, and the fact that a primitive word u is not an inner factor of uu.

Let g and h be arbitrary morphisms on free monoids. Their *equality language* is the set

$$\mathrm{Eq}(g, h) = \{u \mid g(u) = h(u)\}.$$

Clearly, the empty word ε is always an element of $\mathrm{Eq}(g, h)$, and the language is generated by its minimal elements with respect to the prefix ordering.

Further we shall suppose that two morphisms $g, h : A^* \to A^*$ are given, with $A = \{a, b\}$, and $\mathrm{Eq}(g, h)$ contains a non-empty word. The choice of the target alphabet does not harm generality, since any alphabet can by encoded by two letters.

By symmetry of g and h, we shall suppose that g is periodic and h is not periodic. Let p be a primitive word such that $g(a)$ and $g(b)$ are powers of p. Note that a word w is in $\mathrm{Eq}(g, h)$ if and only if $|g(w)| = |h(w)|$ and $h(w)$ is a power of p.

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It is easy to see that $\mathrm{Eq}(g,h)$ is generated by one non-empty word. Indeed, suppose that α and β are minimal elements of $\mathrm{Eq}(g,h)$. Then

$$h(\alpha^{|h(\beta)|}) = h(\beta^{|h(\alpha)|}),$$

and $\alpha = \beta$, since h is not periodic. This implies a bit more:

Lemma 1. *Let $h(u)$ be a power of p. Then $u \in \mathrm{Eq}(g,h)$.*

3. Characterization

In the sequel we shall omit the cases, which differ from another one just by renaming of the letters.

Case 1.

Let $h(b)$ be a power of p. Since h is not periodic, the word $h(a)$ is not a power of p and thus any word $w \in \mathrm{Eq}(g,h)$ is a power of b. Thus

$$\mathrm{Eq}(g,h) = b^*.$$

Case 2. Suppose now that nor $h(a)$ neither $h(b)$ is a power of p.

Subcase 2.1. Suppose $|h(a)| \geq |p|$ and $|h(b)| \geq |p|$. Let w be a non-empty element of $\mathrm{Eq}(g,h)$. By symmetry, let a be the first letter of w. Then p is a prefix of $h(a)$, and thus p is not a suffix of $h(a)$. This implies that b is the last letter of w, and p is a suffix of $h(b)$.

Clearly, the word w contains a factor ab, and thus $h(ab)$ is a factor of p^∞. The primitivity of p now implies that $h(ab)$ is a power of p. Therefore, by Lemma 1,

$$\mathrm{Eq}(g,h) = (ab)^*.$$

This possibility is fulfilled, for example, by morphisms:

$$g(a) = abab \qquad h(a) = aba$$
$$g(b) = abab \qquad h(b) = babab$$

Subcase 2.2. Suppose now $|h(a)| < |p|$. If $\mathrm{Eq}(g,h)$ contains both letters a and b, the length argument yields $|h(b)| > |p|$. Let s (r resp.) be a suffix (prefix resp.) of p, such that $h(b) = sp^m r$, with $m \geq 0$.

The key idea now is that the position of $h(b)$ within p^∞ is given uniquely up to a shift of length divisible by $|p|$. This follows from the primitivity of p.

Let w be the minimal element of $\mathrm{Eq}(g,h)$ and let $ba^{m'}b$, $m \geq 1$, be a factor of ww. Then $|h(ba^{m'})| = n \cdot |p|$, for some $n > 1$. By the Periodicity Lemma, the length of $h(a^{m'})$ strictly less than $|p| + |h(a)|$. Thus n is given uniquely, and

$$m' = \frac{n \cdot |p| - |h(b)|}{|h(a)|}\ .$$

Suppose that ww contains a factor bb. Then $rs = p$, and both r and s are non-empty (otherwise we have the Case 1). This implies $m' = 0$, a contradiction.

Therefore any element of $\mathrm{Eq}(g,h)$ is of the form $a^i(ba^{m'})^*ba^j$, with $m' = i + j$. Hence $w = a^iba^j$ and

$$\mathrm{Eq}(g,h) = (a^iba^j)^*.$$

For any $i, j \geq 0$ we can find morphisms having this equality language. For example:

$$g(a) = a^i b a^j \qquad h(a) = a$$
$$g(b) = a^i b a^j \qquad h(b) = (ba^{i+j})^{i+j} b$$

We can summarize by

Theorem 1. *Let g and h be binary morphisms, of which exactly one is periodic. Then either* $\mathrm{Eq}(g,h) = \{\varepsilon\}$ *or* $\mathrm{Eq}(g,h) = (a^i b a^j)^*$*, with $i, j \geq 0$ (up to the symmetry of the letters a and b).*

On the other hand, any language $(a^i b a^j)^*$, $i, j \geq 0$, is an equality language for one periodic and one non-periodic binary morphism.

References

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