CONGRUENCE LIFTING OF DIAGRAMS OF FINITE BOOLEAN SEMILATTICES REQUIRES LARGE CONGRUENCE VARIETIES

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ABSTRACT. We construct a diagram $\mathcal{D}_{\bowtie}$, indexed by a finite partially ordered set, of finite Boolean $\langle\vee,0,1\rangle$-semilattices and $\langle\vee,0,1\rangle$-embeddings, with top semilattice $\mathbf{2}^4$, such that for any variety $\mathbf{V}$ of algebras, if $\mathcal{D}_{\bowtie}$ has a lifting, with respect to the congruence lattice functor, by algebras and homomorphisms in $\mathbf{V}$, then there exists an algebra U in $\mathbf{V}$ such that the congruence lattice of U contains, as a 0,1-sublattice, the five-element modular nondistributive lattice M_3. In particular, $\mathbf{V}$ has an algebra whose congruence lattice is neither join- nor meet-semidistributive. Using earlier work of K. A. Kearnes and Á. Szendrei, we also deduce that $\mathbf{V}$ has no nontrivial congruence lattice identity.

In particular, there is no functor Φ from finite Boolean semilattices and $\langle\vee,0,1\rangle$-embeddings to lattices and lattice embeddings such that the composition $\operatorname{Con}\Phi$ is equivalent to the identity (where Con denotes the congruence lattice functor), thus solving negatively a problem raised by P. Pudlák in 1985 about the existence of a functorial solution of the Congruence Lattice Problem.

1. INTRODUCTION

The Congruence Lattice Problem, CLP in short, asks whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice. Most of the recent efforts aimed at solving this problem are focusing on lifting not only individual semilattices, but *diagrams* of semilattices, with respect to the functor Con, that with a lattice associates its congruence lattice. This approach has been initiated by P. Pudlák in [9]. For more information, we refer the reader to the survey papers by G. Grätzer and E. T. Schmidt [6] and by the authors of the present paper [11].

As observed by the authors in [10], the diagram of Figure 1, labeled by Boolean semilattices and $\langle\vee,0,1\rangle$-homomorphisms, where $\pi(x,y)=x\vee y$ and $\varepsilon(x)=\langle x,x\rangle$,

$$\begin{tikzcd} \mathbf{2}^2 \arrow[r, shift left, "\pi"] & \mathbf{2} \arrow[l, shift left, "\varepsilon"] \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

FIGURE 1. A diagram unliftable by any algebras.

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cannot be lifted by lattices, see [10, Theorem 8.1]. In fact, the proof of [10, Theorem 8.1] shows that there is no lifting of this diagram by *algebras*, and so this is a result of universal algebra! However, the map π is not one-to-one, and so the problem remained open whether any diagram of finite Boolean semilattices and $\langle\vee,0\rangle$-*embeddings* could be lifted. This problem was first raised by P. Pudlák in 1985, see the bottom of Page 100 in [9]. It was later attacked by the authors of the present paper, see Problems 1 and 2 in [10]. We shall refer to this problem as *Pudlák's problem.*

In general, it is not known whether it is decidable, for a given finite diagram, to have a lifting by, say, lattices, and tackling even quite simple diagrams may amount to considerable work with *ad hoc* methods greatly varying from one diagram to the other; see, for example, the cube diagrams (indexed by $\mathbf{2}^3$) constructed in [10]. This partly explains why Pudlák's problem had remained open for such a long time.

In the present paper, we solve Pudlák's problem by the negative, by constructing a diagram, indexed by a finite poset, of finite Boolean semilattices and $\langle\vee,0,1\rangle$-embeddings, denoted by $\mathcal{D}_{\bowtie}$, that cannot be lifted, with respect to the Con functor, by lattices, see Theorem 4.1. Again, it turns out that as for the diagram of Figure 1, lattice structure does not really matter, and our negative result holds in any variety satisfying a non-trivial congruence lattice identity. Whether it holds in any variety of algebras is still an open problem.

As the top semilattice of $\mathcal{D}_{\bowtie}$ is $\mathfrak{P}(4)$, this makes it the "shortest" (in terms of the top semilattice) diagram of finite Boolean $\langle\vee,0,1\rangle$-semilattices and $\langle\vee,0,1\rangle$-embeddings that cannot be lifted by lattices.

2. Basic concepts

We denote by $\mathbf{ALatt}$ the category of algebraic lattices and compactness-preserving complete join-homomorphisms, and by $\mathbf{Alg}(\Sigma)$ the category of algebras of a given similarity type Σ with Σ-homomorphisms. For algebras A and B and a homomorphism $f\colon A\to B$, we denote by $\operatorname{Con} f$ the map that with every congruence α of A associates the congruence of B generated by all pairs $\langle f(x),f(y)\rangle$, where $\langle x,y\rangle\in\alpha$. Then $\operatorname{Con} f$ is a compactness-preserving complete join-homomorphism. The correspondence $A\mapsto\operatorname{Con} A$, $f\mapsto\operatorname{Con} f$ is a functor from $\mathbf{Alg}(\Sigma)$ to $\mathbf{ALatt}$, that we shall denote by Con and call the *congruence lattice functor* on $\mathbf{Alg}(\Sigma)$, see [10, Section 5.1]. For an algebra A, we denote by $\mathbf{0}_A$ the identity congruence of A and by $\mathbf{1}_A$ the coarse congruence of A.

A *diagram* of a category $\mathbf{D}$, indexed by a category $\mathbf{C}$, is a functor from $\mathbf{C}$ to $\mathbf{D}$. Most of our diagrams will be indexed by posets, the latter being viewed as categories in which hom sets have at most one element. A *lifting* of a diagram $\Phi\colon\mathbf{C}\to\mathbf{ALatt}$ of algebraic lattices is a diagram $\Psi\colon\mathbf{C}\to\mathbf{Alg}(\Sigma)$ of algebras such that the functor Φ and the composition $\operatorname{Con}\Psi$ are naturally equivalent.

We put $n=\{0,1,\dots,n-1\}$, for every natural number n. For a set X, we denote by $\mathfrak{P}(X)$ the powerset algebra of X.

3. Varieties satisfying a nontrivial congruence lattice identity

We first recall some basic definitions and facts of commutator theory, see R. N. McKenzie, G. F. McNulty, and W. F. Taylor [8, Section 4.13]. For a congruence θ of an algebra A and strings $\vec{a}=\langle a_i\mid i<n\rangle$ and $\vec{b}=\langle b_i\mid i<n\rangle$ of elements of A, let $\vec{a}\equiv_\theta\vec{b}$ hold, if $a_i\equiv_\theta b_i$ holds for all $i<n$. For congruences α, β, and δ of an

algebra A, we say that α *centralizes* β *modulo* δ, in symbol $C(\alpha,\beta;\delta)$, if for all m, $n<\omega$ and every $(m+n)$-ary term t of the similarity type of A,

$$t(\vec{a},\vec{p})\equiv_\delta t(\vec{a},\vec{q}) \implies t(\vec{b},\vec{p})\equiv_\delta t(\vec{b},\vec{q}),$$

for all $\vec{a}$, $\vec{b}\in A^m$ and $\vec{p}$, $\vec{q}\in A^n$ such that $\vec{a}\equiv_\alpha \vec{b}$ and $\vec{p}\equiv_\beta \vec{q}$.

We state in the following lemma a few standard properties of the relation $C(\alpha,\beta;\delta)$, see [8, Lemma 4.149].

Lemma 3.1. *Let A be an algebra.*

(i) *For all α, $\beta\in\operatorname{Con}A$, there exists a least $\delta\in\operatorname{Con}A$ such that $C(\alpha,\beta;\delta)$. We denote this congruence by $[\alpha,\beta]$, the* commutator *of α and β.*

(ii) *For all β, $\delta\in\operatorname{Con}A$, there exists a largest $\alpha\in\operatorname{Con}A$ such that $C(\alpha,\beta;\delta)$. We denote this congruence by $(\delta:\beta)$, the* centralizer *of δ modulo β.*

(iii) *$[\alpha,\beta]\subseteq\alpha\cap\beta$, for all α, $\beta\in\operatorname{Con}A$.*

We say that an algebra A is *Abelian*, if $[\mathbf{1}_A,\mathbf{1}_A]=\mathbf{0}_A$; equivalently, $(\mathbf{0}_A:\mathbf{1}_A)=\mathbf{1}_A$. A *weak difference term* (*resp. weak difference polynomial*) on A (see K. A. Kearnes and Á. Szendrei [5]) is a ternary term (resp., polynomial) d such that

$$d(x,y,y)\equiv_{[\theta,\theta]} x\equiv_{[\theta,\theta]} d(y,y,x),$$

for all $\theta\in\operatorname{Con}A$ and all $\langle x,y\rangle\in\theta$. A weak difference term for a variety $\mathbf{V}$ is a ternary term that is a weak difference term in any algebra of $\mathbf{V}$. The statement of the following lemma has been pointed to the authors by Keith Kearnes.

Lemma 3.2. *Let A be an algebra with a weak difference polynomial $d(x,y,z)$. If there exists a 0,1-homomorphism of M_3 to $\operatorname{Con}A$, then A is Abelian.*

Proof. By assumption, there are α, β, $\gamma\in\operatorname{Con}A$ such that $\alpha\cap\beta=\alpha\cap\gamma=\beta\cap\gamma=\mathbf{0}_A$ and $\alpha\vee\beta=\alpha\vee\gamma=\beta\vee\gamma=\mathbf{1}_A$. It follows from Lemma 3.1(iii) that $[\beta,\alpha]=[\gamma,\alpha]=\mathbf{0}_A$, which can be written $\mathbf{1}_A=\beta\vee\gamma\leq(\mathbf{0}_A:\alpha)$ (see Lemma 3.1(ii)), so that $[\mathbf{1}_A,\alpha]=\mathbf{0}_A$; *a fortiori*, $[\alpha,\alpha]=\mathbf{0}_A$. Similarly, $[\beta,\beta]=[\gamma,\gamma]=\mathbf{0}_A$.

Now let $\langle x,y\rangle\in\alpha\circ\beta$. Pick $z\in A$ such that $x\equiv_\alpha z\equiv_\beta y$. Hence,

$$\begin{aligned}
x&=d(x,z,z) &&\text{(because } x\equiv_\alpha z \text{ and } [\alpha,\alpha]=\mathbf{0}_A)\\
&\equiv_\beta d(x,z,y) &&\text{(because } y\equiv_\beta z \text{ and } d \text{ is a polynomial)}\\
&\equiv_\alpha d(z,z,y) &&\text{(because } z\equiv_\alpha x \text{ and } d \text{ is a polynomial)}\\
&=y &&\text{(because } y\equiv_\beta z \text{ and } [\beta,\beta]=\mathbf{0}_A),
\end{aligned}$$

and hence $\langle x,y\rangle\in\beta\circ\alpha$. Therefore, by symmetry, we have proved that the congruences α, β, and γ are pairwise permutable. The conclusion follows from [8, Lemma 4.153]. ☐

We say that a variety $\mathbf{V}$ has *no nontrivial congruence lattice identity*, if any lattice identity that holds in the congruence lattices of all algebras in $\mathbf{V}$ holds in all lattices.

The following lemma sums up a few deep results of universal algebra, namely Corollaries 4.11 and 4.12 in [5]. Recall that an algebra A is *affine*, if there are a ternary term operation $\mathbf{t}$ of A and an Abelian group operation $\langle x,y\rangle\mapsto x-y$ on A such that $\mathbf{t}(x,y,z)=x-y+z$ for all x, y, $z\in A$ and $\mathbf{t}$ is a homomorphism from A^3 to A. For further information we refer the reader to C. Herrmann [4] or R. Freese and R. N. McKenzie [2, Chapter 5]. We call $\mathbf{t}$ a *difference operation* for A.

Theorem 3.3 (K. A. Kearnes and Á. Szendrei)**.** *For a variety* $\mathbf{V}$ *satisfying a nontrivial congruence lattice identity, the following statements hold.*

(i) $\mathbf{V}$ *has a weak difference term.*
(ii) *Any Abelian algebra of* $\mathbf{V}$ *is affine* (*thus it has permutable congruences*).

In fact, it follows from [5, Theorem 4.8] that (i) implies (ii) in Theorem 3.3. This observation is used in Remark 5.3.

An algebra A is *Hamiltonian*, if every subalgebra of A is a congruence class of A.

Lemma 3.4 (folklore)**.** *Every affine algebra* A *is Hamiltonian.*

Proof. Let $\mathbf{t}$ be a difference operation for A, with an Abelian group operation $-$ on A, and let U be a subalgebra of A. For all $x \in A$ and all u, $v \in U$, it follows from the equation $x + v = \mathbf{t}(x + u, u, v)$ that $x + u \in U$ implies that $x + v \in U$. Hence, picking $u \in U$, we can define a binary relation $\equiv$ on A by setting

$$x \equiv y \iff \mathbf{t}(x, y, u) \in U, \quad \text{for all } x,\ y \in A, \tag{3.1}$$

and the relation $\equiv$ is independent of the choice of u. The relation $\equiv$ is obviously reflexive. Now observe that the equation $\mathbf{t}(y, x, u) = \mathbf{t}(u, x - y + u, u)$ holds for all x, $y \in A$. In particular, if $x \equiv y$, then, as U is closed under $\mathbf{t}$ (because $\mathbf{t}$ is a term operation of A), $\mathbf{t}(y, x, u)$ belongs to U, that is, $y \equiv x$. Hence $\equiv$ is symmetric. Similarly, by using the equation $\mathbf{t}(x, z, u) = \mathbf{t}(x - y + u, u, y - z + u)$, that holds for all x, y, $z \in A$, we obtain that $\equiv$ is transitive. By using the assumption that $\mathbf{t}$ is a homomorphism from A^3 to A and by the independence of (3.1) from u, we obtain that $\equiv$ is compatible with all the operations of A; whence it is a congruence of A. Finally, it is obvious that U is the equivalence class of u modulo $\equiv$. $\square$

We shall use Lemma 3.4 in its following special form: *every subalgebra of a simple affine algebra is either a one-element algebra or the full algebra.*

4. The diagram $\mathcal{D}_{\bowtie}$

We define $\langle \vee, 0, 1\rangle$-homomorphisms $\boldsymbol{e}\colon \mathfrak{P}(1) \to \mathfrak{P}(2)$, $\boldsymbol{f}_i\colon \mathfrak{P}(2) \to \mathfrak{P}(3)$, and $\boldsymbol{u}_i\colon \mathfrak{P}(3) \to \mathfrak{P}(4)$ (for $i < 3$) by their values on atoms:

$$\boldsymbol{e}\colon \{0\} \mapsto \{0,1\};$$

$$\boldsymbol{f}_0\colon \begin{cases} \{0\} & \mapsto \{0,1\} \\ \{1\} & \mapsto \{0,2\} \end{cases}, \qquad \boldsymbol{f}_1\colon \begin{cases} \{0\} & \mapsto \{0,1\} \\ \{1\} & \mapsto \{1,2\} \end{cases}, \qquad \boldsymbol{f}_2\colon \begin{cases} \{0\} & \mapsto \{0,2\} \\ \{1\} & \mapsto \{1,2\} \end{cases},$$

$$\boldsymbol{u}_0\colon \begin{cases} \{0\} & \mapsto \{0\} \\ \{1\} & \mapsto \{1,3\} \\ \{2\} & \mapsto \{2,3\} \end{cases}, \qquad \boldsymbol{u}_1\colon \begin{cases} \{0\} & \mapsto \{0,3\} \\ \{1\} & \mapsto \{1\} \\ \{2\} & \mapsto \{2,3\} \end{cases}, \qquad \boldsymbol{u}_2\colon \begin{cases} \{0\} & \mapsto \{0,3\} \\ \{1\} & \mapsto \{1,3\} \\ \{2\} & \mapsto \{2\} \end{cases}.$$

Our diagram, that we shall denote by $\mathcal{D}_{\bowtie}$, is represented on Figure 2. All its arrows are $\langle \vee, 0, 1\rangle$-embeddings, and none of them except $\boldsymbol{e}$ is a meet-homomorphism. It is easy to verify that the diagram $\mathcal{D}_{\bowtie}$ is commutative.

We denote as usual by M_3 the unique modular nondistributive lattice with five elements. We shall represent it as a semilattice of subsets of 3, by

$$M_3 = \{\varnothing, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}, \tag{4.1}$$

see the left hand side of Figure 3. We are now ready to formulate our main result.

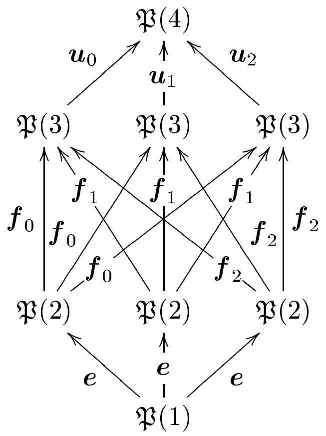

FIGURE 2. The diagram $\mathcal{D}_{\bowtie}$.

Theorem 4.1. *Let* $\mathbf{V}$ *be a variety of algebras. If the diagram* $\mathcal{D}_{\bowtie}$ *admits a lifting, with respect to the congruence lattice functor, by algebras and homomorphisms in* $\mathbf{V}$*, then there exists an algebra* U *in* $\mathbf{V}$ *such that* M_3 *has a* 0,1*-lattice embedding into* $\operatorname{Con} U$*. Furthermore,* $\mathbf{V}$ *satisfies no nontrivial congruence lattice identity.*

The conclusion of Theorem 4.1 can be further strengthened, see Remark 5.3.

In particular, Theorem 4.1 implies immediately that $\mathcal{D}_{\bowtie}$ cannot be lifted by *lattices.*

Lemma 4.2. *The semilattice* $\operatorname{im} \boldsymbol{u}_0 \cap \operatorname{im} \boldsymbol{u}_1 \cap \operatorname{im} \boldsymbol{u}_2$ *is isomorphic to* M_3*.*

Proof. It is straightforward to verify that for all X_0, X_1, $X_2 \subseteq 3$, $\boldsymbol{u}_0(X_0) = \boldsymbol{u}_1(X_1) = \boldsymbol{u}_2(X_2)$ iff $X_0 = X_1 = X_2$ belongs to M_3, see (4.1) and the left hand side of Figure 3. $\square$

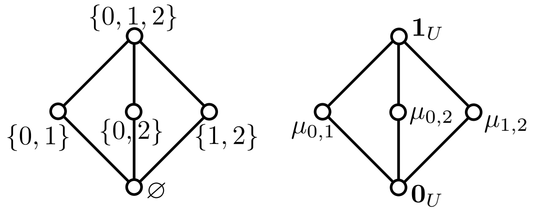

FIGURE 3. Representing M_3 by either subsets or congruences.

5. PROOF OF THEOREM 4.1

We let a diagram $\mathcal{E}$ of algebras in $\mathbf{V}$ lift $\mathcal{D}_{\bowtie}$, and we label $\mathcal{E}$ as on the left hand side of Figure 4. For vertices X and Y of $\mathcal{E}$, denote by $f_{X,Y}$ the unique homomorphism from X to Y arising from $\mathcal{E}$. Furthermore, put $\mathbf{f}_{X,Y} = \operatorname{Con} f_{X,Y}$, a $\langle \vee, 0, 1\rangle$-homomorphism from $\operatorname{Con} X$ to $\operatorname{Con} Y$. For example, $f_{A_0,B}$ is a homomorphism from A_0 to B, with $f_{A_0,B} = f_{B_0,B} \circ f_{A_0,B_0} = f_{B_1,B} \circ f_{A_0,B_1}$, and so on.

We let the equivalence between $\operatorname{Con}\mathcal{E}$ and $\mathcal{D}_{\bowtie}$ be witnessed by isomorphisms ε_X from the corresponding vertex of $\mathcal{D}_{\bowtie}$ onto $\operatorname{Con} X$, for X among A, A_0, A_1, A_2, B_0, B_1, B_2, B. For example, $\varepsilon_{A_i}\colon \mathbf{2}^2 \to \operatorname{Con} A_i$, $\varepsilon_{B_j}\colon \mathbf{2}^3 \to \operatorname{Con} B_j$, and $\varepsilon_{B_j} \circ \boldsymbol{f}_i = \mathbf{f}_{A_i,B_j} \circ \varepsilon_{A_i}$, for all i, $j < 3$.

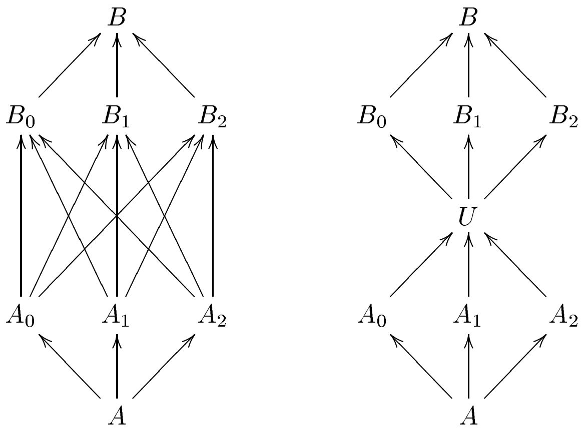

FIGURE 4. The diagram $\mathcal{E}$ and the algebra U.

Since $\mathcal{E}$ lifts $\mathcal{D}_{\bowtie}$ and all arrows of $\mathcal{D}_{\bowtie}$ are $\langle\vee,0\rangle$-embeddings, so that, in particular, they separate 0, all arrows of $\mathcal{E}$ are *embeddings*. In particular, we may replace each vertex X of $\mathcal{E}$ by its image $\operatorname{im} f_{X,B}$ in B, and thus assume that all $f_{X,Y}$-s in $\mathcal{E}$ are (set-theoretical) *inclusion mappings*. We denote by U the subalgebra of B generated by $A_0 \cup A_1 \cup A_2$. Of course, U is contained in $B_0 \cap B_1 \cap B_2$, see the right hand side of Figure 4.

Lemma 5.1. *The equalities* $\mathbf{f}_{A,U}(\mathbf{1}_A) = \mathbf{f}_{A_i,U}(\mathbf{1}_{A_i}) = \mathbf{1}_U$ *hold, for all* $i < 3$.

Proof. It suffices to prove that the congruence $\theta = \mathbf{f}_{A,U}(\mathbf{1}_A)$ is equal to $\mathbf{1}_U$. Fix $a \in A$, and put $V = [a]_\theta$, the θ-block of a in U. As any pair of elements of A is θ-congruent, V is a subalgebra of U. As $\mathcal{E}$ lifts $\mathcal{D}_{\bowtie}$ and $\boldsymbol{e}$ is unit-preserving, all maps $\mathbf{f}_{A,A_i}$, for $i < 3$, are unit-preserving. Hence, for any $x \in A_0$, we obtain the relation $x \equiv a \pmod{\mathbf{f}_{A,A_0}(\mathbf{1}_A)}$, thus, *a fortiori*, $x \equiv a \pmod{\mathbf{f}_{A,U}(\mathbf{1}_A)}$; whence A_0 is contained in V. Similarly, both A_1 and A_2 are contained in V. Since U is the subalgebra of B generated by $A_0 \cup A_1 \cup A_2$, we obtain $V = U$, that is, $\theta = \mathbf{1}_U$. $\square$

Now we put $\xi_{i,j} = \mathbf{f}_{A_i,U} \circ \varepsilon_{A_i}(\{j\})$, for all $i < 3$ and $j < 2$. So $\xi_{i,j}$ is a compact congruence of U. Furthermore, it follows from Lemma 5.1 that

$$\xi_{i,0} \vee \xi_{i,1} = \mathbf{1}_U, \text{ for all } i < 3. \tag{5.1}$$

Now we put

$$\mu_{0,1} = \xi_{0,0} \vee \xi_{1,0}, \quad \mu_{0,2} = \xi_{0,1} \vee \xi_{2,0}, \quad \mu_{1,2} = \xi_{1,1} \vee \xi_{2,1}. \tag{5.2}$$

Lemma 5.2. *There exists a unique* $0,1$*-lattice embedding from* M_3 *into* $\operatorname{Con} U$ *that sends* $\{i,j\}$ *to* $\mu_{i,j}$*, for all* $i < j < 3$.

Proof. It follows from (5.1) that $\mu_{0,1} \vee \mu_{0,2} = \mu_{0,1} \vee \mu_{1,2} = \mu_{0,2} \vee \mu_{1,2} = \mathbf{1}_U$. Now we need to prove that any two $\mu_{i,j}$-s meet at zero. We observe that for all i, $k < 3$ and all $j < 2$,

$$\mathbf{f}_{U,B_k}(\xi_{i,j}) = \mathbf{f}_{U,B_k} \circ \mathbf{f}_{A_i,U} \circ \varepsilon_{A_i}(\{j\}) = \mathbf{f}_{A_i,B_k} \circ \varepsilon_{A_i}(\{j\}) = \varepsilon_{B_k} \circ \boldsymbol{f}_i(\{j\}).$$

This makes it easy to calculate the values $\mathbf{f}_{U,B_k}(\mu_{i,j})$, for $i < j < 3$ and $k < 3$. We obtain, using (5.2),

$$\mathbf{f}_{U,B_k}(\mu_{i,j}) = \varepsilon_{B_k}(\{i,j\}), \text{ for all } i < j < 3 \text{ and } k < 3.$$

It follows immediately that $\mathbf{f}_{U,B_k}(\mu_{0,1} \wedge \mu_{0,2}) = \varepsilon_{B_k}(X_k)$, for some $X_k \subseteq \{0\}$. Hence, applying the observation that $\mathbf{f}_{B_k,B} \circ \mathbf{f}_{U,B_k} = \mathbf{f}_{U,B}$ (independent of k), we obtain that $\boldsymbol{u}_0(X_0) = \boldsymbol{u}_1(X_1) = \boldsymbol{u}_2(X_2)$, hence, by Lemma 4.2, $X_0 = X_1 = X_2$ is not a singleton. Therefore, $X_0 = \varnothing$, so $\mathbf{f}_{U,B_0}(\mu_{0,1} \wedge \mu_{0,2}) = \mathbf{0}_{B_0}$. Since $\mathbf{f}_{U,B_0}$ separates zero, it follows that $\mu_{0,1} \wedge \mu_{0,2} = \mathbf{0}_U$. The proofs that $\mu_{0,1} \wedge \mu_{1,2} = \mathbf{0}_U$ and $\mu_{0,2} \wedge \mu_{1,2} = \mathbf{0}_U$ are similar. Therefore, the right hand side of Figure 3 represents a 0,1-sublattice of $\operatorname{Con} U$ isomorphic to M_3. □

As the subalgebra U of B belongs to $\mathbf{V}$, the result of Lemma 5.2 completes the proof of the first part of Theorem 4.1.

Now suppose that $\mathbf{V}$ (our variety lifting $\mathcal{D}_{\bowtie}$) satisfies a nontrivial congruence lattice identity. By Theorem 3.3(i), $\mathbf{V}$ has a weak difference term. By Lemmas 3.2 and 5.2, the algebra U is Abelian, thus so is A_0. By Theorem 3.3(ii), A_0 is affine. Since $\operatorname{Con} A_0 \cong \mathbf{2}^2$ and A_0 has permutable congruences, we obtain (up to isomorphism) that $A_0 = A' \times A''$, for simple algebras A' and A'', and $\boldsymbol{e}$ is lifted by an embedding of the form $x \mapsto \langle e'(x), e''(x) \rangle$, for embeddings $e' \colon A \hookrightarrow A'$ and $e'' \colon A \hookrightarrow A''$. By Lemma 3.4, both A' and A'' are Hamiltonian, thus both e' and e'' are isomorphisms, whence $A' \cong A''$. Now observe that any term giving a difference operation on the affine algebra A_0 satisfies Mal'cev's equations. Hence, taking $A' = A''$ and using [8, Lemma 4.154], we obtain that the smallest congruence of A_0 collapsing the diagonal of A' is a complement, in $\operatorname{Con} A_0$, of both projection kernels in $A_0 = A' \times A'$, which contradicts $\operatorname{Con} A_0 \cong \mathbf{2}^2$. This concludes the proof of Theorem 4.1.

Remark 5.3. As M_3 does not embed into the congruence lattice of any lattice, a mere solution of Pudlák's problem does not require any use of commutator theory.

On the other hand, it follows from [5, Theorem 4.8] that (i) implies (ii) in the statement of Theorem 3.3. Therefore, the conclusion of Theorem 4.1 can be strengthened into saying that $\mathbf{V}$ has no weak difference term. However, that this is indeed a strengthening is not trivial, and it follows from the deep [5, Corollary 4.12].

6. Discussion

A first immediate corollary of Theorem 4.1 is the following.

Corollary 6.1. *Let* $\mathbf{V}$ *be a variety of algebras with a nontrivial congruence lattice identity. Then there is no functor* Φ *from the category of finite Boolean* $\langle\vee, 0, 1\rangle$*-semilattices and* $\langle\vee, 0, 1\rangle$*-embeddings to* $\mathbf{V}$ *such that the composition* $\operatorname{Con} \Phi$ *is equivalent to the identity.*

The assumption of Corollary 6.1 holds, in particular, for $\mathbf{V}$ being the variety of all lattices, which is congruence-distributive. Therefore, this solves negatively Pudlák's problem. Of course, the level of generality obtained by Theorem 4.1

goes far beyond the failure of congruence-distributivity—for example, it includes congruence-modularity.

These results raise the problem whether there is a 'quasivariety version' of the variety result stated in Corollary 6.1.

Problem 1. Let $\mathbf{V}$ be a variety of algebras. If every finite poset-indexed diagram of finite Boolean semilattices and $\langle\vee,0,1\rangle$-embeddings can be lifted, with respect to the congruence lattice functor, by a diagram in $\mathbf{V}$, then can every finite lattice be embedded into the congruence lattice of some algebra in $\mathbf{V}$?

A possibility would be to introduce more complicated variants of the diagram $\mathcal{D}_{\bowtie}$, which would yield a collection $\langle S_i \mid i<\omega\rangle$ of finite lattices, each of which would play a similar role as M_3 in the proof of Lemma 5.2, and which would generate the quasivariety of all lattices. However, the main difficulty of the crucial Lemma 5.2 is the preservation of meets, for which the lattice M_3 is quite special. Although we know how to extend the method to many finite lattices, we do not know how to get all of them.

Another natural question is whether there is any variety at all satisfying the natural strengthening of the assumption of Theorem 4.1.

Problem 2. Does there exist a variety $\mathbf{V}$ of algebras such that every finite poset-indexed diagram of finite Boolean $\langle\vee,0,1\rangle$-semilattices and $\langle\vee,0,1\rangle$-embeddings can be lifted, with respect to the congruence lattice functor, by algebras in $\mathbf{V}$?

Of course, a similar question can be formulated for finite Boolean $\langle\vee,0\rangle$-semilattices and $\langle\vee,0\rangle$-embeddings.

In view of Bill Lampe's results [7], a natural candidate for $\mathbf{V}$ would be the variety of all *groupoids* (i.e., sets with a binary operation). By the second author's results in [12, 13], any variety satisfying the conclusion of Problem 2 (provided there is any) has the property that every diagram of finite *distributive* $\langle\vee,0,1\rangle$-semilattices and $\langle\vee,0,1\rangle$-embeddings can be lifted, with respect to the congruence lattice functor, by algebras in $\mathbf{V}$. In particular, every distributive algebraic lattice *with compact unit* would be isomorphic to the congruence lattice of some algebra in $\mathbf{V}$. The latter conclusion is known (even without the distributivity restriction!) in case $\mathbf{V}$ is the variety of all groupoids, see [7]. Strangely, whether the corresponding result holds for arbitrary distributive algebraic lattices is still an open problem. Also, there exists an algebraic lattice that is not isomorphic to the congruence lattice of any groupoid—namely, the subspace lattice of any infinite-dimensional vector space over any uncountable field, see [1].

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