SLIM GROUPOIDS

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ABSTRACT. Slim groupoids are groupoids satisfying $x(yz) \approx xz$. We find all simple slim groupoids and all minimal varieties of slim groupoids. Every slim groupoid can be embedded into a subdirectly irreducible slim groupoid. The variety of slim groupoids has the finite embeddability property, so that the word problem is solvable. We introduce the notion of a strongly nonfinitely based slim groupoid (such groupoids are inherently nonfinitely based) and find all strongly nonfinitely based slim groupoids with at most four elements; up to isomorphism, there are just two such groupoids.

We are going to investigate groupoids (algebras with one binary operation) satisfying the equation $x(yz) \approx xz$. Since every term operation of such a groupoid can be represented by a slim term (a term that is a product of a finite sequence of variables with all parentheses grouped to the left), these groupoids are called slim. Similarly as in the case of semigroups, a free object in the variety of slim groupoids is the set of words over a given set of generators; only the multiplication of words differs from that in a free semigroup.

One can expect that the variety of slim groupoids will have similar properties as the variety of semigroups. In some cases it is true. We will see, however, that the variety of slim groupoids has solvable word problem and has the strong amalgamation property.

The purpose of this paper is to introduce and investigate basic properties of the variety of slim groupoids. We are particularly interested in the existence of finite, nonfinitely based slim groupoids. It has been shown by McKenzie [2] that the finite basis problem for equations of finite algebras is unsolvable: there is no algorithm deciding for an arbitrary finite algebra, or a finite groupoid, whether it has a finite basis for its equations. For many varieties, like those of groups or lattices, the problem is solvable in a trivial way: every finite algebra in such a variety has a finite basis. So, it is desirable to look for (natural) examples of varieties with the finite basis problem solvable but in a nontrivial way. Such a variety should be in some sense reasonably small and in another sense reasonably large. Perhaps the variety of slim (or idempotent slim) groupoids could be a good candidate. We introduce the notion of a strongly nonfinitely based slim groupoid (such

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groupoids are inherently nonfinitely based) and find all strongly nonfinitely based slim groupoids with at most four elements; up to isomorphism, there are just two such groupoids.

For the notation and basic notions of universal algebra the reader is referred to [3]. We will work with groupoids, algebras with one binary operation. In most cases, without mention, the operation is denoted multiplicatively: the product of two elements a, b of a groupoid is denoted by $a \cdot b$ or just ab. For elements $a_1, a_2, \dots, a_n$ of a groupoid write $a_1 \dots a_n = (((a_1a_2)a_3)\dots)a_n$. (The parentheses are grouped to the left.) For $n \geq 1$ put $a^n = a_1 \dots a_n$ where $a_i = a$ for all i.

1. Slim groupoids: first concepts

By a slim groupoid we mean a groupoid satisfying the equation $x(yz) \approx xz$.

Let X be a nonempty set. By a term over X we mean an element of the absolutely free groupoid over X. For a term t denote by $\kappa(t)$ the element of X occurring in t at the rightmost position. (The inductive definition: $\kappa(x) = x$ for $x \in X$; $\kappa(uv) = \kappa(v)$.) By a slim term we mean any term $x_1 \dots x_k$ ($k \geq 1$) where $x_1, \dots, x_k \in X$. Every term can be uniquely expressed as $xu_1 \dots u_n$ for an element x of X and some terms $u_1, \dots, u_n$ ($n \geq 0$). For a term $t = xu_1 \dots u_n$ expressed in this way put $t^* = x\kappa(u_1)\dots\kappa(u_n)$. So, t^* is a slim term for any term t.

Theorem 1.1. *The equational theory of slim groupoids can be described by its normal form function $t \mapsto t^*$:*

(1) *for any term t, the equation $t^* \approx t$ is satisfied in all slim groupoids*
(2) *an equation $t \approx u$ is satisfied in all slim groupoids if and only if $t^* = u^*$*
(3) *$t^{**} = t^*$ for any term t*

Proof. This follows easily from the fact that the set of slim terms, considered as a groupoid with respect to the operation $\circ$ defined by $(x_1 \dots x_n) \circ (y_1 \dots y_m) = x_1 \dots x_n y_m$, satisfies $x \circ (y \circ z) = x \circ z$. □

Let X be a nonempty set. By a word over X we mean a nonempty finite sequence of elements of X. A word $\langle x_1, \dots, x_n \rangle$ ($x_i \in X$) can be written as $x_1x_2 \dots x_n$ (and thus identified with a slim term, or also with an element of a free semigroup). We denote by $\mathcal{F}(X)$ the groupoid defined in this way: its underlying set is the set of words over X; the multiplication is given by $(x_1 \dots x_n)(y_1 \dots y_m) = x_1 \dots x_n y_m$.

Theorem 1.2. *For a nonempty set X, the groupoid $\mathcal{F}(X)$ is the free slim groupoid over X.*

Proof. It follows from 1.1. □

For a slim groupoid A we define a binary relation β_A on A as follows: $\langle a, b \rangle \in \beta_A$ if and only if there exists an element $c \in A$ with $ca = cb$.

Theorem 1.3. *Let A be a slim groupoid. Then*

(1) $\langle a,b\rangle \in \beta_A$ *implies* $ca = cb$ *for all* $c \in A$
(2) β_A *is a congruence of* A
(3) $\langle ab,b\rangle \in \beta_A$ *for all* $a,b \in A$*, so that the factor* A/β_A *satisfies* $xy \approx y$
(4) *every block of* β_A *is a subgroupoid of* A *satisfying* $xy \approx xz$

Proof. (1) If $\langle a,b\rangle \in \beta_A$ then $da = db$ for some $d \in A$, so that $ca = c(da) = c(db) = cb$ for all $c \in A$.

(2) It follows from (1) that β_A is an equivalence. If $\langle a,b\rangle \in \beta_A$ then for any $c \in A$ we have $\langle ca,cb\rangle \in \beta_A$, since $c(ca) = ca = cb = c(cb)$; and for any $c \in A$ we have $\langle ac,bc\rangle \in \beta_A$, since $c(ac) = cc = c(bc)$. So, β_A is a congruence.

(3) For $a,b \in A$ we have $a(ab) = ab$, so that $\langle ab,b\rangle \in \beta_A$.

(4) In particular, $\langle aa,a\rangle \in \beta_A$. Thus the block of β_A containing an arbitrary element $a \in A$ is a subgroupoid. Since any two elements of this subgroupoid are β_A-related, the subgroupoid satisfies $xy \approx xz$. □

Next we are going to describe a general construction of arbitrary slim groupoids. Denote by Φ the class of ordered triples $\langle A,\beta,\phi\rangle$ such that A is a nonempty set, β is an equivalence on A and ϕ is a mapping of $A \times A/\beta$ into A with $\phi(a,B) \in B$ for any $\langle a,B\rangle \in A \times A/\beta$. For every such triple we define a groupoid $\mathcal{G}_{A,\beta,\phi}$ with the underlying set A by $ab = \phi(a,b/\beta)$ for all $a,b \in A$.

Theorem 1.4. *A groupoid is slim if and only if it is the groupoid* $\mathcal{G}_{A,\beta,\phi}$ *for a triple* $\langle A,\beta,\phi\rangle \in \Phi$.

Proof. Clearly, $\mathcal{G}_{A,\beta,\phi}$ is a slim groupoid. Now let C be an arbitrary slim groupoid. It is easy to check that $C = \mathcal{G}_{A,\beta,\phi}$ where $A = C$, $\beta = \beta_A$ and ϕ is defined by $\phi(a,a/\beta) = ab$. □

2. Simple slim groupoids and minimal varieties

Lemma 2.1. *The following are equivalent for a groupoid A:*

(1) A *is slim and* $\beta_A = id_A$
(2) A *satisfies* $xy \approx y$

Proof. (1) implies (2) by 1.3. The converse is clear. □

Lemma 2.2. *The following are equivalent for a groupoid A:*

(1) A *is slim and* $\beta_A = A \times A$
(2) A *satisfies* $xy \approx xz$

Proof. (1) implies (2) by 1.3. The converse is clear. □

Theorem 2.3. *The following are up to isomorphism the only simple slim groupoids:*

(1) *the two-element groupoid satisfying* $xy \approx x$
(2) *the two-element groupoid satisfying* $xy \approx y$

(3) *the two-element groupoid satisfying* $xy \approx zu$
(4) *for every prime number* p*, the groupoid with elements* $0, 1, \dots, p-1$ *and multiplication* $\circ$ *given by* $x \circ y = x + 1 \mod p$.

Proof. It is easy to check that all these groupoids are slim and simple. Let A be a simple slim groupoid. Then β_A is either id_A or $A \times A$. If $\beta_A = \mathrm{id}_A$ then A satisfies $xy \approx x$ by 1.3, and then A has just two elements because it is simple. If $\beta_A = A \times A$ then A satisfies $xy \approx xz$ by 1.3, so that A is essentially an algebra with one unary operation; the description of simple algebras with one unary operation belongs to the folklore. □

Theorem 2.4. *The variety of slim groupoids has just three minimal subvarieties:*

(1) *the variety determined by* $xy \approx x$
(2) *the variety determined by* $xy \approx y$
(3) *the variety determined by* $xy \approx zu$

Proof. It follows from 2.3, since every minimal variety contains (and thus is generated by) a simple groupoid. The groupoids 2.3(4) do not generate minimal varieties. They generate varieties determined by $xy \approx xz$ and $x^{p+1} \approx x$, and these contain the variety determined by $xy \approx x$. □

3. Subdirectly irreducible slim groupoids

Theorem 3.1. *Every slim groupoid* A *can be embedded into a subdirectly irreducible slim groupoid* B *such that the monolith of* B *has only singleton blocks and one two-element block, and such that* B *is finite if* A *is finite.*

Proof. Let A be a slim groupoid. Let o be a fixed element of A. For $i \in A$ put $a_i = \langle a, 1\rangle$ and $b_i = \langle a, 2\rangle$. Put $B = A \cup \{a_i : i \in A\} \cup \{b_i : i \in A\}$ and define multiplication on B in this way:

(i) for $i, j \in A$, ij in B is the same as ij in A
(ii) for $i \in A$ put $ia_i = ib_i = a_i$
(iii) for $i, j \in A$ with $i \neq j$ put $ja_i = jb_i = b_i$
(iv) for $i, j \in A$ put $a_i j = b_i j = ij$
(v) for $i, j \in A$ put $b_j a_i = b_j b_i = b_i$
(vi) for $i \in A$ put $a_i a_i = a_i b_i = a_i$
(vii) for $i \in A$ put $a_i a_o = a_i b_o = a_o$
(viii) for $i, j \in A$ with $i \neq j$ and $j \neq o$ put $a_i a_j = a_i b_j = b_j$

It is easy to check that B is a slim groupoid and that the relation $\mu = \{\langle a_o, b_o\rangle, \langle b_o, a_o\rangle\} \cup \mathrm{id}_B$ is a congruence of B. Let $\sim$ be a congruence of B. In order to prove that μ is the monolith of B, we have to show that whenever two distinct elements of B are $\sim$-related then $a_o \sim b_o$. This follows from the following claims. Let i, j, k, m run over elements of A,

Claim 1. If $i \sim j$ where $i \neq j$ then $a_i \sim b_i$. Indeed, $a_i = ia_i \sim ja_i = b_i$.

Claim 2. If $i \sim a_j$ then $k \sim b_j$ for some k. Indeed, take an element $m \in A$ different from j and put $k = mi$; we have $k = mi \sim ma_j = b_j$.

Claim 3. If $i \sim b_j$ then $a_i \sim b_i$. Indeed, $a_i = ib_i \sim b_j b_i = b_i$.

Claim 4. If $a_i \sim b_j$ then $a_o \sim b_o$. Indeed, $a_o = a_i a_o \sim b_j a_o = b_o$.

Claim 5. If $a_i \sim a_j$ where $i \neq j$ and $j \neq o$ then $a_j \sim b_j$. Indeed, $b_j = a_i a_j \sim a_j a_j = a_j$.

Claim 6. If $b_i \sim b_j$ where $i \neq j$ and $j \neq o$ then $a_i \sim b_j$. Indeed, $a_i = a_i b_i \sim a_i b_j = b_j$. □

4. Partial groupoids

By a homomorphism of a partial groupoid A into a partial groupoid B we mean a mapping $f : A \to B$ such that whenever a, b are elements of A such that ab is defined then $f(a)f(b)$ is also defined and $f(ab) = f(a)f(b)$. We say that A is embeddable into B if there exists an injective homomorphism of A into B. For a groupoid B and a nonempty subset S of B we define a partial groupoid $B \restriction S$ with the underlying set S as follows: for $a, b \in S$ the product ab is defined in $B \restriction S$ if and only if this product in B belongs to S, and in this case the product in $B \restriction S$ is equal to the product in B. We say that a partial groupoid A is strongly embeddable into a groupoid B if it is isomorphic to $B \restriction S$ for a nonempty subset S of B.

Clearly, if a partial groupoid A is strongly embeddable into a slim groupoid, then it satisfies the following two conditions:

(P1) whenever $a, b, c \in A$ are such that bc and $a(bc)$ are defined then ac is also defined and $ac = a(bc)$

(P2) whenever $a, b, c \in A$ are such that ac and bc are defined then $a(bc)$ is also defined and $a(bc) = ac$

For a partial groupoid A satisfying (P1) and (P2) we define a groupoid $\mathcal{F}(A)$ as follows. The underlying set of $\mathcal{F}(A)$ is the set of finite nonempty sequences $\langle a_1, a_2, \ldots, a_n \rangle$ of elements of A such that if $n \geq 2$ then $a_1 a_2$ is not defined in A; the multiplication is given by

$$\langle a_1, \ldots, a_n \rangle \langle b_1, \ldots, b_m \rangle = \begin{cases} a_1 b_m \text{ if } n = 1 \text{ and } a_1 b_m \text{ is defined in } A \\ \langle a_1, \ldots, a_n, b_m \rangle \text{ otherwise} \end{cases}$$

For this definition to make sense, we must suppose that no element of A is a finite sequence of length larger than 1. If this is not satisfied then A should be replaced with an isomorphic partial groupoid. Also, we identify an element a of A with $\langle a \rangle$.

Theorem 4.1. *Let A be a partial groupoid satisfying* (P1) *and* (P2)*. Then $\mathcal{F}(A)$ is a slim groupoid; it is the free slim groupoid over A, i.e., it is generated by A and every homomorphism of A into a slim groupoid B can be extended to a homomorphism of $\mathcal{F}(A)$ into B.*

Proof. The most essential is to prove that $\mathcal{F}(A)$ is slim. Let $u = \langle a_1, \ldots, a_n \rangle$, $v = \langle b_1, \ldots, b_m \rangle$ and $w = \langle c_1, \ldots, c_k \rangle$ be three elements of $\mathcal{F}(A)$. We are going to check that $u(wv) = uv$.

Consider first the case when $n = 1$ and $a_1 b_m$ is defined in A, so that $uv = a_1 b_m$. If $k = 1$ and $c_1 b_m$ is defined in A then $u(vw) = a_1(c_1 b_m) = a_1 b_m = uv$ by (P2). Otherwise, $u(wv) = u\langle c_1, \dots, c_k, b_m\rangle = a_1 b_m = uv$.

Consider the remaining case. Now $uv = \langle a_1, \dots, a_n, b_m\rangle$. If $wv = \langle c_1, \dots, c_k, b_m\rangle$ then $u(wv) = \langle a_1, \dots, a_n, b_m\rangle = uv$. Otherwise, $k = 1$, $c_1 b_m$ is defined in A and $wv = c_1 b_m$. If $n = 1$ and $a_1(c_1 b_m)$ is defined in A then $a_1 b_m$ is defined by (P1), which is not possible. So, $wv = \langle c_1, \dots, c_k, b_m\rangle$ and $u(wv) = \langle a_1, \dots, a_n, b_m\rangle = uv$.

Clearly, $\mathcal{F}(A)$ is generated by the set A. A homomorphism f of A into a slim groupoid B can be extended to a homomorphism g of $\mathcal{F}(A)$ into B by setting $g(\langle a_1, \dots, a_n\rangle) = f(a_1)f(a_2)\dots f(a_n)$. □

Corollary 4.2. *A partial groupoid is strongly embeddable into a slim groupoid if and only if it satisfies* (P1) *and* (P2)*.*

For a partial groupoid A denote by γ_A the set of the ordered pairs $\langle a, b\rangle \in A \times A$ such that one of the following three cases takes place:

(1) there exists an element $c \in A$ such that ca, cb are both defiend and $ca = cb$
(2) there exists an element $c \in A$ such that ca is defined and $ca = b$
(3) there exists an element $c \in A$ such that cb is defined and $cb = a$

Denote by β_A the reflexive and transitive closure of γ_A. (If A is a slim groupoid then both γ_A and β_A coincide with the earlier defined congruence β_A.) Of course, β_A is an equivalence on A.

Consider the following condition for a partial groupoid A:

(P0) whenever $\langle a, b\rangle \in \beta_A$, $c \in A$ and ca and cb are both defined then $ca = cb$

Theorem 4.3. *The following three conditions are equivalent for a partial groupoid A:*

(1) *A is embeddable into a slim groupoid*
(2) *A can be completed to a slim groupoid*
(3) *A satisfies* (P0)

Proof. The implications (2) ⇒ (1) ⇒ (3) are trivial. We are going to prove (3) ⇒ (2). Let A satisfy (P0). For every block B of β_A choose one fixed element $\nu(B) \in B$. Define a binary operation $\circ$ on A as follows:

$$a \circ b = \begin{cases} ac \text{ if there is a } c \in A \text{ with } \langle b, c\rangle \in \beta_A \text{ such that } ac \text{ is defined} \\ \nu(b/\beta_A) \text{ otherwise} \end{cases}$$

Correctness of this definition follows from (P0). Clearly, if a, b are two elements of A such that ab is defined in A then $a \circ b = ab$. Thus the groupoid $\langle A, \circ\rangle$ is a completion of the partial groupoid $A = \langle A, \cdot\rangle$. It remains to prove that this groupoid is slim.

Claim 1. $\langle a \circ b, b\rangle \in \beta_A$ for all $a, b \in a$. This is easy to check.

Claim 2. For $a, b, c \in A$ with $\langle a, b\rangle \in \beta_A$ we have $c \circ a = c \circ b$. If $c \circ a = cd$ where $\langle a, d\rangle \in \beta_A$ then $\langle b, d\rangle \in \beta_A$, so that $c \circ b = cd = c \circ a$. The case $c \circ b = cd$ for some d is symmetric. In the remaining case $c \circ a = \nu(a/\beta_A) = \nu(b/\beta_A) = c \circ b$.

Claim 3. For $a, b, c \in A$ we have $a \circ (b \circ c) = a \circ c$. By Claim 1 we have $\langle b \circ c, c\rangle \in \beta_A$ and so $a \circ (b \circ c) = a \circ c$ by Claim 2. □

A variety V is said to have the finite embeddability property if every finite partial algebra that is embeddable into some algebra from V is embeddable into some finite algebra from V.

Corollary 4.4. *The variety of slim groupoids has the finite embeddability property.*

Corollary 4.5. *The variety of slim groupoids has globally decidable word problem.*

This follows from Evans [1]: every finitely based variety with finite embeddability property has globally decidable word problem.

A variety V is said to have the strong amalgamation property if for any two algebras $A, B \in V$ such that the intersection $A \cap B$ is a subalgebra of both A and B, there exists an algebra $C \in V$ such that both A and B are subalgebras of C.

Theorem 4.6. *The variety of slim groupoids has the strong amalgamation property.*

Proof. Let A, B be two slim groupoids such that $A \cap B$ is a subgroupoid of each of them. Define a partial groupoid P with the underlying set $A \cup B$ as follows: for $a, b \in A \cup B$, the product ab is defined in P if and only if either $\{a, b\} \subseteq A$ or $\{a, b\} \subseteq B$; in each case let the product in P coincide with that in either A or B. By Theorem 4.3, it is sufficient to check that P satisfies (P0). Take a fixed element $c \in A \cap B$. Let us first prove that if $\langle a, b\rangle \in \gamma_P$ then $ca = cb$. There exists an element d such that either $da = db$ or $da = b$ or $db = a$. If either $\{a, b\} \subseteq A$ or $\{a, b\} \subseteq B$ then it is easy to see that either $\langle a, b\rangle \in \beta_A$ or $\langle a, b\rangle \in \beta_B$ and hence $ca = cb$. Let, e.g., $a \in A - B$ and $b \in B - A$. Since the products da and db are both defined, we have $d \in A \cap B$ and $da = db \in A \cap B$. Then $ca = c(da) = c(db) = cb$.

Now let $\langle a, b\rangle \in \beta_P$. There exists a finite sequence $a = a_0, a_1, \dots, a_k = b$ such that $\langle a_{i-1}, a_i\rangle \in \gamma_P$ for $i = 1, \dots, n$. We have seen that $ca_{i-1} = ca_i$ for all i. Thus $ca = cb$. From this it follows that $da = db$ whenever both da and db are defined. □

5. Equational theories

Let X be a countably infinite set of variables. The underlying set of $\mathcal{F}(X)$ is a subset of the groupoid $\langle \mathcal{T}(X), \circ\rangle$ of terms over X. The free semigroup $\langle \mathcal{S}(X), *\rangle$ over X has the same underlying set as $\mathcal{F}(X)$. For two elements $x_1 \dots x_n$ and $y_1 \dots y_m$ we have $(x_1 \dots x_n)(y_1 \dots y_m) = x_1 \dots x_n y_m$ and $(x_1 \dots x_n) * (y_1 \dots y_m) = x_1 \dots x_n y_1 \dots y_m$.

An equational theory is a fully invariant congruence of the groupoid $\langle \mathcal{T}(X), \circ \rangle$. By a slim theory we mean a restriction to $\mathcal{F}(X)$ of an equational theory extending the equational theory of slim groupoids. Of course, the lattice of varieties of slim groupoids is antiisomorphic to the lattice of slim theories.

Theorem 5.1. *A binary relation R on $\mathcal{F}(X)$ is a slim theory if and only if it is a congruence of the free semigroup $\mathcal{S}(X)$ satisfying the following three conditions:*

(1) *if $\langle x_1 \dots x_n, y_1 \dots y_m \rangle \in R$ then $\langle f(x_1) \dots f(x_n), f(y_1), \dots, f(y_m) \rangle \in R$ for any mapping f of X into X*
(2) *if $\langle x_1 \dots x_n, y_1 \dots y_m \rangle \in R$ where $x_1 \neq y_1$ then $\langle z x_1 \dots x_n, y_1 \dots y_m \rangle \in R$ for any variable z*
(3) *if there is an equation $\langle x_1 \dots x_n, y_1 \dots y_m \rangle \in R$ such that $x_n \neq y_m$ then $\langle xy, xz \rangle \in R$ for three distinct variables x, y, z*

Proof. Let R be a slim theory, so that $R = R' \cap (\mathcal{F}(X) \times \mathcal{F}(X))$ for an equational theory extending the equational theory of slim groupoids. Let $\langle x_1 \dots x_n, y_1 \dots y_m \rangle \in R$. Condition (1) is satisfied, since R' is a fully invariant congruence of $\mathcal{T}(X)$. If $x_1 \neq y_1$ then substituting $z x_1$ for x_1 yields $\langle z x_1 \dots x_n, y_1 \dots y_m \rangle \in R$. If $x_n \neq y_m$, take a variable z different from both x_n and y_m; we have $\langle z \circ (x_1 \dots x_n), z \circ (y_1 \dots y_m) \rangle \in R'$, so that $\langle z x_n, z y_m \rangle \in R$.

It remains to prove the converse. Denote by R' the set of the equations $\langle u, v \rangle \in \mathcal{F}(X) \times \mathcal{F}(X)$ such that $\langle u^*, v^* \rangle \in R$. (As above, u^* is the only element of $\mathcal{F}(X)$ such that $\langle u, u^* \rangle$ is in the equational theory of slim groupoids.) Clearly, R' is an equivalence. Let $\langle u, v \rangle \in R'$ and w be a term. Since R is a congruence, $\langle u^* \kappa(w), v^* \kappa(w) \rangle \in R$ and hence $\langle u \circ w, v \circ w \rangle \in R'$. We have $(w \circ u)^* = w^* x_n$ and $(w \circ v)^* = w^* y_m$. If $x_n = y_m$, we get $(w \circ u)^* = (w \circ v)^*$ and thus $\langle w \circ u, w \circ v \rangle \in R'$. If $x_n \neq y_m$, the same follows from (3). So, R' is a congruence of $\mathcal{T}(X)$.

Let $\langle u, v \rangle \in R'$ and let f be an endomorphism of $\mathcal{F}(X)$. We have $u = x_0 u_1 \dots u_n$ and $v = y_0 v_1 \dots v_m$ for some variables x_0, y_0 and terms u_i, v_j. Then $\langle u^*, v^* \rangle = \langle x_0 x_1 \dots x_n, y_0 y_1 \dots y_m \rangle \in R$ where $x_i = \kappa(u_i)$ and $y_j = \kappa(v_j)$ for $i, j \geq 1$. Put $f(x_0)^* = z_1 \dots z_r$ and $f(y_0)^* = w_1 \dots w_s$. For $i, j \geq 1$ put $p_i = \kappa(f(x_i))$ and $q_j = \kappa(f(y_j))$. We have $\langle z_r p_1 \dots p_n, w_s q_1 \dots q_m \rangle \in R$ by (1). If $x_0 = y_0$ then $\langle f(u)^*, f(v)^* \rangle = \langle z_1 \dots z_r p_1 \dots p_n, z_1 \dots z_r q_1 \dots q_m \rangle \in R$, since R is a congruence of $\mathcal{S}(X)$. If $x_0 \neq y_0$ then it follows easily from (2) that $\langle z_1 \dots z_r p_1 \dots p_n, w_1 \dots w_s q_1 \dots q_m \rangle \in R$, i.e., $\langle f(u)^*, f(v)^* \rangle \in R$. This shows that R' is a fully invariant congruence of $\mathcal{T}(X)$. Clearly, R' extends the equational theory of slim groupoids and R is its restriction to $\mathcal{F}(X)$. $\square$

By a slim-regular equation we mean an equation $x_1 \dots x_n \approx y_1 \dots y_m$ (x_i and y_j are variables) such that $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$, $x_1 = y_1$ and $x_n = y_m$.

By a slim derivation of an equation $u \approx v$ based on a set B of slim-regular equations we mean a finite sequence $u_0, \dots, u_k$ of words such that $u_0 = u$, $u_k = v$ and for every $i = 0, \dots, k-1$, $\langle u_i, u_{i+1} \rangle$ is an immediate consequence of an equation $\langle x_1 \dots x_n, y_1 \dots y_m \rangle \in B \cup B^{-1}$ in the sense that the word u_{i+1} is obtained from u_i by replacing a subword $f(x_1) \dots f(x_n)$, for a mapping f of the set of variables into itself, with $f(y_1) \dots f(y_m)$.

Theorem 5.2. *Let B be a set of slim-regular equations and let u, v be two terms. The equation $u \approx v$ is satisfied in the variety of slim groupoids determined by B if and only if there exists a slim derivation of $u \approx v$ based on B.*

Proof. It follows from 5.1. □

6. Strongly nonfinitely based finite slim groupoids

A finite groupoid A is said to be nonfinitely based if its equational theory has no finite base. It is said to be inherently nonfinitely based if there is no finitely based, locally finite variety containing A.

By a strongly nonfinitely based slim groupoid we mean a finite slim groupoid A such that whenever A satisfies an equation $\langle u, v \rangle$ where both u, v are slim and u is linear (i.e., every variable occurs at most once in u), then $u = v$.

Theorem 6.1. *Let A be a finite, strongly nonfinitely based slim groupoid. Then A is inherently nonfinitely based.*

Proof. First observe that if an equation $\langle u, v \rangle$ is satisfied in A then $\kappa(u) = \kappa(v)$. Indeed, if $\kappa(u) \neq \kappa(v)$ then A satisfies $xy = xz$, a contradiction. Also observe that if $\langle x, u \rangle$ is satisfied in A where x is a variable x then $u = x$.

Let V be a locally finite variety containing A and suppose that the equational theory E of V has a finite base B. Denote by E_0 the equational theory of A, so that $E \subseteq E_0$. Let q be a positive integer larger than the length of u, for any $\langle u, v \rangle \in B \cup B^{-1}$. For any $i \geq 1$ denote by t_i the term which is the product of the first i variables in the sequence $x_1, \dots, x_q, x_1, \dots, x_q, x_1, \dots, x_q, \dots$. Since V is locally finite, we have $\langle t_i, t_j \rangle \in E$ for some $i \neq j$. Since B is a base for E, there exists a B-derivation $t_i = w_0, w_1, \dots, w_n = t_j$.

Let us prove by induction on $p = 0, 1, \dots$ that $w_p^* = t_i$. For $p = 0$ it is clear. Let $w_p^* = t_i$ for some $p < n$. There exist an equation $\langle u, v \rangle \in B \cup B^{-1}$ and an endomorphism f of the groupoid of terms such that w_{p+1} is obtained from w_p by replacing a subterm $f(u)$ with $f(v)$. We have $w_p = xr_2 \dots r_i$ for a variable x and some terms $r_2, \dots, r_i$ (the same i as above). If $f(u)$ is a subterm of r_m for some m then $w_{p+1} = xr'_2 \dots r'_i$ for some terms r'_i with $r'_c = r_c$ for all $c \neq m$ and $\kappa(r'_m) = \kappa(r_m)$, so that $w_{p+1}^* = w_p^* = t_i$. Otherwise,

$f(u) = xr_2 \dots r_d$ for some d. We have $u = yu_2 \dots u_k$ for a variable y and some terms $u_2, \dots, u_k$ where $k < q$. Then $f(y) = xr_2 \dots r_e$, $f(u_2) = r_{e+1}$, $\dots$, $f(u_k) = r_d$. Since $k < q$, the variables $\kappa(r_e), \kappa(r_{e+1}), \dots, \kappa(r_d)$ are pairwise distinct. Hence $y, \kappa(u_2), \dots, \kappa(u_k)$ are pairwise distinct. Thus u^* is a slim linear term. Since $\langle u^*, v^* \rangle$ is satisfied in A and v^* is slim, we get $u^* = v^*$. Then also $(f(u))^* = (f(v))^*$. We get $w_{p+1}^* = (f(v))^* \kappa(r_{d+1}) \dots \kappa(r_i) = w_p^* = t_i$.

In particular, $w_n^* = t_i$, i.e., $t_j = t_i$, a contradiction. □

Consider the slim groupoid $\mathcal{G}_{4,1}$ with elements a, b, c, d and multiplication table

	a	b	c	d
a	a	a	c	c
b	a	a	d	d
c	b	b	c	c
d	b	b	d	d

Lemma 6.2. *Let h be a homomorphism of the groupoid T of terms into $\mathcal{G}_{4,1}$. Let $t = x_1 \dots x_n$ where $n \geq 2$ and x_i are variables. Then*

(1) *$h(t) = a$ iff $\{h(x_{n-1}), h(x_n)\} \subseteq \{a, b\}$*
(2) *$h(t) = b$ iff $h(x_n) \in \{a, b\}$ and $h(x_{n-1}) \in \{c, d\}$*
(3) *$h(t) = c$ iff $h(x_n) \in \{c, d\}$ and, where k is the least index with $\{h(x_k), \dots, h(x_n)\} \subseteq \{c, d\}$, one of the following three cases takes place:*
    $k = 1$ and $h(x_1) = c$
    $k = 2$ and $h(x_1) = a$
    $k \geq 3$ and $\{h(x_{k-2}), h(x_{k-1})\} \subseteq \{a, b\}$
(4) *$h(t) = d$ in the remaining cases*

Proof. It can be checked easily. □

Lemma 6.3. *Let $x_1 \dots x_n \approx y_1 \dots y_m$ be satisfied in $\mathcal{G}_{4,1}$, where x_i and y_j are variables. Then $x_1 = y_1$, $x_n = y_m$ and if $n = 1$ then $m = 1$.*

Proof. Since $\mathcal{G}_{4,1}$ contains the subgroupoid $\{c, d\}$ satisfying $xy \approx x$, we have $x_1 = y_1$. Since the factor $\mathcal{G}_{4,1}/\beta_{\mathcal{G}_{4,1}}$ is a two-element groupoid satisfying $xy \approx y$, we have $x_n = y_m$. Since $\mathcal{G}_{4,1}$ contains the subgroupoid $\{a, b\}$ satisfying $xy \approx uv$, $\mathcal{G}_{4,1}$ does not satisfy any equation $x \approx x^k$ with $k > 1$. Consequently, if $n = 1$ then $m = 1$. □

Lemma 6.4. *Let $x_1 \dots x_n \approx y_1 \dots y_m$ be satisfied in $\mathcal{G}_{4,1}$, where x_i and y_j are variables. Then $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$.*

Proof. Suppose, for example, that there exists an i with $x_i \notin \{y_1, \dots, y_m\}$ and take the largest index i with this property. By 6.3 we have $1 < i < n$.

Consider first the case $x_{i-1} \neq x_i$. Take the homomorphism $h : T \to \mathcal{G}_{4,1}$ with $h(x_i) = b$ and $h(z) = c$ for all other variables z. Then $h(x_1 \dots x_n) =$

$d \neq c = h(y_1 \dots y_m)$, a contradiction. (For these computations one can use Lemma 6.2.)

Now consider the remaining case $x_{i-1} = x_i$. Take $h : T \to \mathcal{G}_{4,1}$ with $h(x_i) = a$ and $h(z) = d$ for all other variables z. Then $h(x_1 \dots x_n) = c \neq d = h(y_1 \dots y_m)$, a contradiction again. $\square$

Theorem 6.5. *$\mathcal{G}_{4,1}$ is a strongly nonfinitely based slim groupoid.*

Proof. Suppose, on the contrary, that there are pairwise different variables $x_1, \dots, x_n$ and some variables $y_1, \dots, y_m$ such that $x_1 \dots x_n \approx y_1 \dots y_m$ is satisfied in $\mathcal{G}_{4,1}$ but $x_1 \dots x_n \neq y_1 \dots y_m$. We know already that $1 < n \leq m$, $x_1 = y_1$, $x_n = y_m$ and $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$.

Let us prove by induction on $i = 0, \dots, n-1$ that $y_{m-i} = x_{n-i}$. For $i = 0$ it follows from 6.3. Let $i > 0$ and $y_{m-j} = x_{n-j}$ for all $j < n$; suppose that $y_{m-i} \neq x_{n-i}$. If $y_{m-i} \neq x_{n-i+1}$, take the homomorphism $h : T \to \mathcal{G}_{4,1}$ with $h(x_{n-i}) = h(x_{n-i+1}) = a$ and $h(z) = c$ for all other variables z; we have $h(x_1 \dots x_n) \in \{a, c\}$ (a if $i = 1$ and c if $i > 1$), while $h(y_1 \dots y_m) \in \{b, d\}$ (b if $i = 1$ and d if $i > 1$). If $y_{m-i} = x_{n-i+1}$, take $h : T \to \mathcal{G}_{4,1}$ with $h(x_{n-i+1}) = a$ and $h(z) = c$ for all other variables z; we have $h(x_1 \dots x_n) \in \{b, d\}$ (b if $i = 1$ and d if $i > 1$), while $h(y_1 \dots y_m) \in \{a, c\}$ (a if $i = 1$ and c if $i > 1$). In both cases we get a contradiction.

Thus $y_m = x_n, \dots, y_{m-n+1} = x_1$. It remains to show that $m = n$. Suppose, on the contrary, that $m > n$. If $y_{m-n} = x_1$, take $h : T \to \mathcal{G}_{4,1}$ with $h(x_1) = b$ and $h(z) = c$ for all other variables z; we have $h(x_1 \dots x_n) = d$ while $h(y_1 \dots y_m) = c$. If $y_{m-n} \neq x_1$, take $h : T \to \mathcal{G}_{4,1}$ with $h(x_1) = a$ and $h(z) = c$ for all other variables z; we have $h(x_1 \dots x_n) = c$ while $h(y_1 \dots y_m) = d$. $\square$

Now consider the slim groupoid $\mathcal{G}_{4,2}$ with elements a, b, c, d and multiplication table

	a	b	c	d
a	a	a	c	c
b	a	a	d	d
c	b	b	d	d
d	b	b	c	c

Theorem 6.6. *$\mathcal{G}_{4,2}$ is a strongly nonfinitely based slim groupoid.*

Proof. The idea is essentially the same as for $\mathcal{G}_{4,1}$. The main difference is that the analogue of Lemma 6.2, which is then often used for checking, is slightly more complicated. For a homomorphism h of the groupoid T of terms into $\mathcal{G}_{4,2}$ and for a term $t = x_1 \dots x_n$ ($n \geq 2$) we have

(1) $h(t) = a$ iff $\{h(x_{n-1}), h(x_n)\} \subseteq \{a, b\}$
(2) $h(t) = b$ iff $h(x_n) \in \{a, b\}$ and $h(x_{n-1}) \in \{c, d\}$
(3) $h(t) = c$ iff $h(x_n) \in \{c, d\}$ and, where k is the least index with $\{h(x_k), \dots, h(x_n)\} \subseteq \{c, d\}$, one of the following six cases takes place:

$k = 1$, $h(x_1) = c$ and n is odd
$k = 1$, $h(x_1) = d$ and n is even
$k = 2$, $h(x_1) = a$ and n is even
$k = 2$, $h(x_1) = b$ and n is odd
$k \geq 3$, $h(x_{k-2}) \in \{a, b\}$ and $n - k$ is even
$k \geq 3$, $h(x_{k-2}) \in \{c, d\}$ and $n - k$ is odd

(4) $h(t) = d$ in the remaining cases

Let $x_1 \dots x_n \approx y_1 \dots y_m$ be satisfied in $\mathcal{G}_{4,2}$, where x_i and y_j are variables. One can prove in the same way as in Lemma 6.3 that $x_1 = y_1$, $x_n = y_m$ and if $n = 1$ then $m = 1$. In order to prove that $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$, suppose that there is an i with $x_i \notin \{y_1, \dots, y_m\}$ and let i be the largest index with this property. We have $1 < i < n$. Take two homomorphisms $h, h' : T \to \mathcal{G}_{4,2}$ with $h(x_i) = h'(x_i) = a$ and $h(z) = c$, $h'(z) = d$ for all other variables z. It is easy to check that $h(x_1 \dots x_n) = h'(x_1 \dots x_n)$ while $h(y_1 \dots y_m) \neq h'(y_1 \dots y_m)$ in all cases, so that either $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ or $h'(x_1 \dots x_n) \neq h'(y_1 \dots y_m)$.

Let, moreover, $x_1, \dots, x_n$ be pairwise different. The proof will be finished if we derive a contradiction from the assumption $x_1 \dots x_n \neq y_1 \dots y_m$. We have $1 < n \leq m$.

Let us first prove that $y_{m-i} = x_{n-i}$ for $i = 0, \dots, n-1$. Suppose $y_{m-i} \neq x_{n-i}$ for some i, and let i be the least number with this property; then $i > 0$. If $y_{m-i} \neq x_{n-i+1}$ then $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ where $h(x_{n-i}) = h(x_{n-i+1}) = a$ and $h(z) = c$ for all other variables z. If $y_{m-i} = x_{n-i+1}$ then $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ where $h(x_{n-i+1}) = a$ and $h(z) = c$ for all other variables z.

So, $y_m = x_n$, …, $y_{m-n+1} = x_1$. If $x_1 \dots x_n \neq y_1 \dots y_m$, we get $m > n$. If $y_{m-n} = x_1$ then $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ where $h(x_1) = b$ and $h(z) = c$ for all other variables z. If $y_{m-n} \neq x_1$ then $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ where $h(x_1) = a$ and $h(z) = c$ for all other variables c. □

Theorem 6.7. *The groupoids $\mathcal{G}_{4,1}$ and $\mathcal{G}_{4,2}$ are, up to isomorphism, the only two strongly nonfinitely based slim groupoids with at most four elements.*

Proof. It is possible to use a computer program to generate all slim groupoids with at most four elements that do not satisfy at least one of the equations $xy \approx xyyy$, $xyz \approx xyzxyz$, $xyz \approx xyxyz$ and $xyzu \approx xyzuzuzu$. Only two such groupoids are obtained: the groupoid $\mathcal{G}_{4,1}$ and the groupoid $\mathcal{G}_{4,2}$. □

Let us remark that the varieties generated by $\mathcal{G}_{4,1}$ and $\mathcal{G}_{4,2}$ are incomparable: the equation $xxx = xx$ is satisfied in $\mathcal{G}_{4,1}$ but not in $\mathcal{G}_{4,2}$, and the equation $xxyy \approx xyxyyy$ is satisfied in $\mathcal{G}_{4,2}$ but not in $\mathcal{G}_{4,1}$.

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