VARIETIES OF IDEMPOTENT SLIM GROUPOIDS

J. JEŽEK

Abstract. Idempotent slim groupoids are groupoids satisfying $xx \approx x$ and $x(yz) \approx xz$. We prove that the variety of idempotent slim groupoids has uncountably many subvarieties. We find a four-element, inherently nonfinitely based idempotent slim groupoid; the variety generated by this groupoid has only finitely many subvarieties. We investigate free objects in some varieties of idempotent slim groupoids determined by permutational equations.

This paper is a continuation of the paper [4] which was concerned with general slim groupoids. Here we are going to investigate the idempotent case. An idempotent slim groupoid is a groupoid satisfying $xx \approx x$ and $x(yz) \approx xz$. In [1] idempotent slim groupoids (or their duals) were investigated under the name rectangular groupoids.

We are going to prove in the present paper that the variety of idempotent slim groupoids has uncountably many subvarieties. While all at most three-element idempotent slim groupoids are finitely based, we will find a four-element, inherently nonfinitely based idempotent slim groupoid. It will turn out that the variety $\mathbf{Y}$ generated by this groupoid has the following interesting property: although it is finitely generated and inherently nonfinitely based, it has only finitely many (in fact, precisely six) subvarieties.

We also investigate a descending chain of varieties $\mathbf{W}_n$ of idempotent slim groupoids determined by permutational equations of restricted length. For many pairs k, n of natural numbers we determine whether the free object $\mathcal{F}_{k,n}$ in $\mathbf{W}_n$ with k generators is finite or infinite, and in some cases we compute the cardinality of the free groupoid. The intersection $\mathbf{W}_\infty$ of the varieties $\mathbf{W}_n$ is investigated in a similar way.

The terminology and notation used here are the same as in the paper [4].

1. Uncountably many varieties

By a subword of a word $x_1 \dots x_n$ we mean a word $x_i x_{i+1} \dots x_j$ where $1 \le i \le j \le n$.

A word $x_1 \dots x_n$ (where x_i are variables) is said to be I-reduced if $x_i \neq x_{i+1}$ for $i = 1, \dots, n-1$. The I-reduction of a word $x_1 \dots x_n$ is defined inductively in this way: a variable is its own I-reduction; if $n > 1$ and

1991 *Mathematics Subject Classification.* 20N02.

Key words and phrases. groupoid, variety, nonfinitely based.

The work is a part of the research project MSM0021620839 financed by MSMT and partly supported by the Grant Agency of the Czech Republic, grant #201/05/0002.

$y_1 \dots y_m$ is the I-reduction of $x_1 \dots x_{n-1}$ then the I-reduction of $x_1 \dots x_n$ is $y_1 \dots y_m$ if $x_{n-1} = x_n$ and $y_1 \dots y_m x_n$ if $x_{n-1} \neq x_n$. It is easy to see that an equation $x_1 \dots x_n \approx y_1 \dots y_m$ is satisfied in all idempotent slim groupoids if and only if the I-reductions of $x_1 \dots x_n$ and $y_1 \dots y_m$ are the same.

Theorem 1.1. *The variety of idempotent slim groupoids has $2^{\aleph_0}$ subvarieties.*

Proof. For a word u, variables y, z and a nonnegative integer n we define a word $u[yz]^n$ by induction as follows: $u[yz]^0 = u$; $u[yz]^{n+1} = ((u[yz]^n)y)z$.

Let M be a subset of $\{3, 4, 5, \dots\}$. A word is said to be M-bad if it equals $xyx[zy]^k xyzy$ for some variables x, y, z and some integer $k \in M$. The M-correction of an M-bad word $xyx[zy]^k xyzy$ is the word $xyx[yz]^k xyzy$. An M-significant word is a word that is either M-bad or is the M-correction of an M-bad word. An I-reduced word is said to be M-good if it does not contain any M-bad subword.

Claim 1. Let u be an M-significant I-reduced word, $u = xyx[yz]^k xyzy$ or $u = xyx[zy]^k xyzy$. Then x, y, z are pairwise different variables. Indeed, xy and yz are subwords of u, so $x \neq y$ and $y \neq z$. Either zx (in the first case) or xz (in the second case) is a subword of u, so $x \neq z$.

Claim 2. Let $u = x_1 \dots x_n$ be an I-reduced word and $x_i \dots x_j$ and $x_p \dots x_q$ be its two M-significant subwords. Then either $\langle i, j\rangle = \langle p, q\rangle$ or $q \leq i + 2$ or $j \leq p + 2$. Put $x = x_{j-3}$, $y = x_{j-2}$ and $z = x_{j-1}$. By Claim 1, x, y, z are three different variables. If $j = q$ then it is easy to see that $i = p$ and the two subwords are identical. Let, e.g., $q < j$. For $c \in \{i, i+1, i+3, i+4, \dots, j-6, j-4\}$ we have $q \neq c+3$ since $x_c \in \{x_{c+2}, x_{c+3}\}$ while $x_{q-3} \notin \{x_{q-1}, x_q\}$. Since $x_{q-5} \neq x_{q-3}$ while $x_i = x_{i+2}$, we have $q \neq i+5$. Since $x_{q-1} \in \{x_{q-4}, x_{q-5}\}$ while $x_{j-3} \notin \{x_{j-6}, x_{j-7}\}$, we have $q \neq j-2$.

Claim 3. Any I-reduced word u can be transformed into an M-good word by a finite sequence of replacements of M-bad subwords with their M-corrections. The resulting M-good word is uniquely determined by u and M. By Claim 2, whenever an M-bad subword v is replaced with its M-correction w then any of the later replacements can touch it at most at the first three or the last three positions of its variables; but these positions remain unchanged by the replacements, so w remains unchanged till the end of the process.

The unique M-good word resulting from an I-reduced word u in this way will be called the M-correction of u. Define a groupoid A_M in this way: its underlying set is the set of M-good I-reduced words; its binary operation, denoted by $\circ$, is given by

$$x_1 \dots x_n \circ y_1 \dots y_m = \begin{cases} x_1 \dots x_n \text{ if } y_m = x_n \\ \text{the } M\text{-correction of } x_1 \dots x_n y_m \text{ otherwise} \end{cases}$$

Claim 4. Let $a_1, \dots, a_n$ be elements of A_M. Then $a_1 \circ a_2 \circ \dots \circ a_n$ is the M-correction of the I-reduction of the word $a_1 z_2 \dots z_n$, where z_i is the last variable in the word a_i. It follows from the definition of $\circ$ by induction on n.

Claim 5. A_M is an idempotent slim groupoid. For $k \geq 3$, the equation $xyx[yz]^k xyzy \approx xyx[zy]^k xyzy$ is satisfied in A_M if and only if $k \in M$. It follows from Claim 4.

Since there are $2^{\aleph_0}$ different subsets of $\{3, 4, \dots\}$, it follows from Claim 5 that there are $2^{\aleph_0}$ different varieties of idempotent slim groupoids. $\square$

2. I-strongly nonfinitely based slim groupoids

By an I-strongly nonfinitely based slim groupoid we mean a finite idempotent slim groupoid A such that whenever A satisfies an equation $\langle u, v \rangle$ where u, v are I-reduced words and u is linear then $u = v$.

Theorem 2.1. *Let A be a finite, I-strongly nonfinitely based idempotent slim groupoid. Then A is inherently nonfinitely based.*

Proof. The proof is essentially the same as that of Theorem 6.1 of[4]; the small difference is that for a term t, one should consider (instead of just t^*) the I-reduction of t^*. Observe, however, that our present result is not a consequence of that theorem: an I-strongly nonfinitely based idempotent slim groupoid is not strongly nonfinitely based. $\square$

Consider the idempotent slim groupoid $\mathcal{G}_{4,3}$ with elements a, b, c, d and multiplication table

	a	b	c	d
a	a	a	c	c
b	b	b	c	c
c	a	a	c	c
d	b	b	d	d

Theorem 2.2. *$\mathcal{G}_{4,3}$ is an I-strongly nonfinitely based idempotent slim groupoid.*

Proof. For a homomorphism h of the groupoid T of terms into $\mathcal{G}_{4,3}$ and for a word $t = x_1 \dots x_n$ (where x_i are variables) we have

(1) $h(t) = d$ iff $h(x_1) = d$ and $h(x_i) \in \{c, d\}$ for all i
(2) $h(t) = c$ iff $h(x_n) \in \{c, d\}$ and either $h(x_1) \neq d$ or $h(x_i) \notin \{c, d\}$ for at least one i
(3) $h(t) = b$ iff one of the following two cases takes place:
   - $h(x_1) = b$ and $h(x_i) \in \{a, b\}$ for all i
   - $h(x_1) = d$ and there exists an index $k < n$ such that $h(x_i) \in \{c, d\}$ for all $i \leq k$ and $h(x_i) \in \{a, b\}$ for all $i > k$
(4) $h(t) = a$ in the remaining cases

This will help in the following computations.

Since $\mathcal{G}_{4,3}$ has a two-element subgroupoid satisfying $xy \approx x$ (the subgroupoid $\{a, b\}$ and a two-element factor satisfying $xy \approx y$ (the factor $\mathcal{G}_{4,3}/\beta_{\mathcal{G}_{4,3}}$), any equation satisfied in $\mathcal{G}_{4,3}$ has the same first variables and the same last variables at both sides.

Let $\langle x_1 \dots x_n, y_1 \dots y_m \rangle$ be satisfied in $\mathcal{G}_{4,3}$, where x_i and y_j are variables. Then $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$. In order to prove this, suppose that there exists an i with $x_i \notin \{y_1, \dots, y_m\}$ and let i be the largest index with this property. Take the homomorphism $h : T \to \mathcal{G}_{4,3}$ with $h(x_i) = b$ and $h(z) = d$ for all other variables z. Then $h(x_1 \dots x_n) \in \{a, b, c\}$ while $h(y_1 \dots y_m) = d$, a contradiction.

Let $\langle x_1 \dots x_n, y_1 \dots y_m \rangle$ be satisfied in $\mathcal{G}_{4,3}$, where $x_1 \dots x_n$ and $y_1 \dots y_m$ are both I-reduced and $x_1 \dots x_n$ is linear. Suppose $x_1 \dots x_n \neq y_1 \dots y_m$. We have $1 < n \leq m$.

Let us prove that $y_{m-i} = x_{n-i}$ for $i = 0, \dots, n-1$. Suppose $y_{m-i} \neq x_{n-i}$ for some i, and let i be the least number with this property; then $i > 0$. If $y_{m-i} = x_j$ for some $j < n-i$, then $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ where $h(x_1) = \dots = h(x_{n-i}) = d$ and $h(x_{n-i+1}) = \dots = h(x_n) = b$. If $y_{m-i} = x_j$ for some $j > n-i$, then $h(x_1 \dots x_n) \neq h(y_1 \dots y_m)$ where $h(x_1) = \dots = h(x_{n-i-1}) = d$ and $h(x_{n-i}) = \dots = h(x_n) = b$.

So, $y_m = x_n$, $\dots$, $y_{m-n+1} = x_1$. If $x_1 \dots x_n \neq y_1 \dots y_m$, we get $m > n$. We have $y_{m-n} = x_i$ for some $i \geq 3$. Define h by $h(x_1) = \dots = h(x_{i-1}) = d$ and $h(x_i) = \dots = h(x_n) = b$. Then $h(x_1 \dots x_n) = b$ while $h(y_1 \dots y_m) = c$, a contradiction. □

Theorem 2.3. *The groupoid $\mathcal{G}_{4,3}$ is, up to isomorphism, the only I-strongly nonfinitely based idempotent slim groupoid with at most four elements.*

Proof. It is possible to generate all idempotent slim groupoids with at most four elements that do not satisfy the equation $xyzu \approx xyzuzuzu$. Only one such groupoid is obtained, the groupoid $\mathcal{G}_{4,3}$. □

3. Three-element idempotent slim groupoids

Theorem 3.1. *All idempotent slim groupoids with at most three elements are finitely based.*

Proof. According to Gerhard [2], all varieties of idempotent semigroups are finitely based. According to Jacobs and Schwabauer [3], all varieties of algebras with one unary operation are finitely based. Thus it remains to consider the at most three-element idempotent slim groupoids that are not semigroups and do not satisfy $xy \approx xz$. It is easy to find that there is, up to isomorphism, precisely one such groupoid. It has three elements a, b, c and multiplication

	a	b	c
a	a	a	c
b	b	b	c
c	a	a	c

It has been shown in [1] that the equational theory of this groupoid is based on the three equations $x(zy) \approx xy$, $xx \approx x$ and $xyzu \approx xzyu$. □

Remark 3.2. In idempotent slim groupoids, $xyzx \approx x$ implies $xyz \approx xz$. Indeed, $xyz = xyxzyyz = xyxz = xz$.

Remark 3.3. In idempotent slim groupoids, the equations $xyx \approx x$ and $xyzux \approx xp(y)p(z)p(u)x$ for all permutations p of $\{y, z, u\}$ imply $xyzu \approx xzyu$. Indeed, $xyzu = xyzuxu = xzyuxu = xzyu$.

4. The varieties $\mathbf{W}_n$

For $n \geq 1$ denote by $\mathbf{W}_n$ the variety of idempotent slim groupoids satisfying $xy_1 \dots y_n x \approx xy_{p(1)} \dots y_{p(n)} x$ for all permutations p of $\{1, \dots, n\}$.

Clearly, $\mathbf{W}_1$ is the variety of all idempotent slim groupoids, $\mathbf{W}_2$ is determined (together with the equations of idempotent slim groupoids) by $xyzx \approx xzyx$, $\mathbf{W}_3$ by $xyzux \approx xuzyx \approx xzyux$, etc. We have $\mathbf{W}_1 \supset \mathbf{W}_2 \supset \mathbf{W}_3 \supset \dots$. Denote by $\sim_n$ the equational theory of $\mathbf{W}_n$.

It can be easily checked (with an aid of computer) that every groupoid in $\mathbf{W}_3$ with at most 8 elements belongs to $\mathbf{W}_4$.

We denote by $\mathcal{F}_{k,n}$ the free groupoid in $\mathbf{W}_n$ with k generarors. In the following we are going to describe $\mathcal{F}_{k,n}$ for small numbers k.

Theorem 4.1. *$\mathcal{F}_{2,n}$ is infinite for $n \leq 2$. For $n \geq 3$, $\mathcal{F}_{2,n}$ has 8 elements and its multiplication table is*

$\mathcal{F}_{2,2}$	x	y	xy	yx	xyx	yxy	$xyxy$	$yxyx$
x	x	xy	xy	x	x	xy	xy	x
y	yx	y	y	yx	yx	y	y	yx
xy	xyx	xy	xy	xyx	xyx	xy	xy	xyx
yx	yx	yxy	yxy	yx	yx	yxy	yxy	yx
xyx	xyx	$xyxy$	$xyxy$	xyx	xyx	$xyxy$	$xyxy$	xyx
yxy	$yxyx$	yxy	yxy	$yxyx$	$yxyx$	yxy	yxy	$yxyx$
$xyxy$	xyx	$xyxy$	$xyxy$	xyx	xyx	$xyxy$	$xyxy$	xyx
$yxyx$	$yxyx$	yxy	yxy	$yxyx$	$yxyx$	yxy	yxy	$yxyx$

Proof. Denote the two generators by x and y. Clearly, every word over $\{x, y\}$ is $\sim_n$-equivalent to a word that is a beginning of either $xyxyxy\dots$ or $yxyxyx\dots$. All these words are pairwise $\sim_n$-inequivalent if $n \leq 2$. For $n \geq 3$, we have $xyxyx \sim_n xyx$ and $yxyxy \sim_n yxy$, so every word is $\sim_n$-equivalent to one of the eight words. It is easy to check that the eight-element groupoid belongs to $\mathbf{W}_n$. Consequently, it is the free groupoid. □

We say that a word $x_1 \dots x_n$ precedes a word $y_1 \dots y_m$ if one of the following three cases takes place:

(1) $n < m$
(2) $n = m \geq 3$, $x_1 = x_3$ and $y_1 \neq y_3$
(3) $n = m \geq 4$, $x_1 \neq x_3$, $y_1 \neq y_3$, $x_2 = x_4$ and $y_2 \neq y_4$

A word t is said to be $\sim_n$-minimal if there is no word preceding t and $\sim_n$-equivalent with t. Clearly, every word (in a fixed number of variables) is $\sim_n$-equivalent with at least one $\sim_n$-minimal word.

A word $y_1 \dots y_m$ is said to be an extension of $x_1 \dots x_n$ if $n \le m$ and $x_i = y_i$ for all $i \le n$. Let $t, u, v, \dots$ be words over $\{x_1 \dots, x_k\}$. We write $t \lhd_{k,n} \langle u, v, \dots \rangle$ if $u, v, \dots$ are extensions of u and every $\sim_n$-minimal extension of u (containing only $x_1, \dots, x_k$) is extended by one of the words $u, v, \dots$.

We are now going to describe $\mathcal{F}_{3,3}$. So, in the next lemmas let $\lhd$ stand for $\lhd_{3,3}$. Denote the three generators by x, y, z.

Lemma 4.2. $xyxy \lhd \langle xyxyz \rangle$.

Proof. It follows from $xyxyzx \sim_3 xyxyxzx \sim_3 xyxzx \sim_3 xyzx$. □

Lemma 4.3. $xyxz \lhd \langle xyxzy \rangle$.

Proof. $xyxzyx \sim_3 xyxyzx \sim_3 xyzx$ and $xyxzyz \sim_3 xyzxyz \sim_3 xzyxyz \sim_3 xzyyxz \sim_3 xzyxz \sim_3 xyzxz$. □

Lemma 4.4. $xyzx \lhd \langle xyzx \rangle$.

Proof. $xyzxy \sim_3 xyxzy$ and $xyzxz \sim_3 xzyxz \sim_3 xzxyz$. □

Lemma 4.5. $xyzy \lhd \langle xyzyzx \rangle$.

Proof. $xyzyx \sim_3 xyzx$, $xyzyzy \sim_3 xyzy$, $xyzyzxz \sim_3 xyzxyz$ and $xyzyzxy \sim_3 xyzyxzy \sim_3 xyzxzy$. □

From these lemmas it follows that every word in variables x, y, z is $\sim_3$-equivalent with at least one word that can be extended to a word similar to one of the words $xyxyz, xyxzy, xyzx, xyzyzx$. (Two words are said to be similar if one is obtained by a permutation of variables in the other.) It is not difficult to write all such words; their number is 66. Now we know that $\mathcal{F}_{3,3}$ has at most 66 elements and we suspect that 66 could be the precise number. In order to prove it, we try to write the multiplication table for $\mathcal{F}_{3,3}$; clearly, if the groupoid given by this table satisfies the equations of $\mathbf{W}_3$, it is the free groupoid in $\mathbf{W}_3$. The trouble is that the multiplication table would be too big. However, it is sufficient to consider just a fragment. First of all, instead of the 66 columns it is sufficient to write the three columns corresponding to the three generators: the product of two words is equal to the product of the first word with the last variable in the second. And instead of 66 rows, it is sufficient to write the representative 12 of them; the other ones are obtained by permutations of variables. We obtain the displayed fragment. In this fragment, each of the first 2 rows can be permuted to 3 different rows and each of the next 10 rows to 6.

We can check easily that this groupoid satisfies the equations of $\mathbf{W}_3$. (Observe that in order to check a permutational equation of the form considered here, it is sufficient to interpret its leftmost variable by an arbitrary element and all the remaining variables by variables only.) So, this groupoid is the groupoid $\mathcal{F}_{3,3}$ and the free groupoid has precisely 66 elements.

$\mathcal{F}_{3,3}$	x	y	z
x	x	xy	xz
$xyzx$	$xyzx$	$xyxzy$	$xzxyz$
xy	xyx	xy	xyz
xyx	xyx	$xyxy$	$xyxz$
xyz	$xyzx$	$xyzy$	xyz
$xyxy$	xyx	$xyxy$	$xyxyz$
$xyxyz$	$xyzx$	$xyxzy$	$xyxyz$
$xyxz$	$xyzx$	$xyxzy$	$xyxz$
$xyxzy$	$xyzx$	$xyxzy$	$xzxyz$
$xyzy$	$xyzx$	$xyzy$	$xyzyz$
$xyzyz$	$xyzyzx$	$xyzy$	$xyzyz$
$xyzyzx$	$xyzyzx$	$xyxzy$	$xzxyz$

The groupoid does not belong to $\mathbf{W}_4$, since the element $xyzyzx$ can be reduced to $xyzx$. It easily follows that the groupoid $\mathcal{F}_{3,4}$ (which has to be a factor of $\mathcal{F}_{3,3}$) has 60 elements. We get

Theorem 4.6. *The groupoid $\mathcal{F}_{3,3}$ has 66 elements and its multiplication table can be reconstructed from the above given fragment of 12 rows and 3 columns. The groupoid $\mathcal{F}_{3,4}$ has 60 elements and its multiplication table can be reconstructed from the fragment for $\mathcal{F}_{3,3}$ in which the last row is deleted and the element $xyzyzx$ is replaced with $xyzx$.*

Next we are going to describe the groupoid $\mathcal{F}_{4,5}$. So, in the next lemmas let $\lhd$ stand for $\lhd_{4,5}$. Denote the four generators by x, y, z, u.

Lemma 4.7. $xyxy \lhd \langle xyxyzuzu, xyxyuzuz \rangle$.

Proof. Let t be a $\sim_5$-minimal extension of $xyxy$. Clearly, t cannot start with either $xyxyx$ or $xyxyy$, so (if it is different from $xyxy$) it must start with either $xyxyz$ or $xyxyu$. Each of these words can continue (to remain $\sim_5$-minimal) only in the indicated way. We have $xyxyzuzux \sim_5 xyxyzux \sim_5 xyzux$ and $xyxyzuzuy \sim_5 xyxyzuy$, so that $xyxyzuzu$ has no proper $\sim_5$-minimal extension. □

Lemma 4.8. $xyxzy \lhd \langle xyxzyu \rangle$.

Proof. We cannot continue with z, since $xyxzyz \sim_5 xyzxyz \sim_5 xyzyxz \sim_5 xyyzxz \sim_5 xyzxz$. So, clearly we can continue with u only. It is evident that the word $xyxzyu$ cannot be continued with either x or y or u. It also cannot be continued with z, since $xyxzyuz \sim_5 xyxzuyz \sim_5 xyzuxyz \sim_5 xzyuxyz \sim_5 xzyyuxz \sim_5 xzyuxz$. □

Lemma 4.9. $xyxzu \lhd \langle xyxzuzu, xyxzuy \rangle$.

Proof. Clearly, the word cannot continue with either x or u and if it is continued with z then there is only one possible further continuation, $xyxzuzu$.

For the continuations of $xyxzuy$, consider

$$xyxzuyz \sim_5 xyzuxyz \sim_5 xyzuyxz \sim_5 xyyzuxz \sim_5 xyzuxz \text{ and}$$
$$xyxzuyu \sim_5 xyuzxyu \sim_5 xyuzyxu \sim_5 xyyuzxu \sim_5 xyuzxu.$$

□

Lemma 4.10. $xyxz \lhd \langle xyxzyu, xyxzuzu, xyxzuy \rangle$.

Proof. It follows from 4.8 and 4.9, since clearly the word cannot be continued with either x or z. □

Lemma 4.11. $xyx \lhd \langle xyxyzuzu, xyxyuzuz, xyxzyu, xyxzuzu, xyxzuy, xyxuyz, xyxuzuz \rangle$.

Proof. It follows from 4.7 and 4.10. □

Lemma 4.12. $xyzx \lhd \langle xyzxu \rangle$.

Proof. A continuation of $xyzxy$ (of $xyzxz$, respectively) is $\sim_5$-equivalent to a continuation of $xyxzy$ (of $xzxyz$, respectively, since $xyzxz \sim_5 xzyxz \sim_5 xzxyz$) of the same length and so need not be considered. We have $xyzxuz \sim_5 xyzuxz \sim_5 xzyuxz \sim_5 xzxyuz$, a word starting with xzx. □

Lemma 4.13. $xyzy \lhd \langle xyzyzu, xyzyuz \rangle$.

Proof. It is easy to see that $xyzyz \lhd \langle xyzyzu \rangle$. Since $xyzyuzu \sim_5 xyzuyzu \sim_5 xyuzyzu \sim_5 xyuzzyu \sim xyuzyu$, we have $xyzyu \lhd \langle xyzyuz \rangle$. □

Lemma 4.14. $xyzu \lhd \langle xyzux, xyzuy, xyzuzu \rangle$.

Proof. Since

$$xyzuxy \sim_5 xyxzuy,$$
$$xyzuxz \sim_5 xyzxuz \sim_5 xzyxuz \sim_5 xzxyuz,$$
$$xyzuxu \sim_5 xuyzxu \sim_5 xuxyzu,$$

we have $xyzux \lhd \langle xyzux \rangle$. Since

$$xyzuyz \sim_5 xyzyuz,$$
$$xyzuyu \sim_5 xyuzyu \sim_5 xyuyzu,$$

we have $xyzuy \lhd \langle xyzuy \rangle$. Clearly, $xyzuz \lhd \langle xyzuzu \rangle$. □

From these lemmas it follows that every word in variables x, y, z, u is $\sim_5$-equivalent with at least one word that can be extended to a word similar to one of the words $xyxyzuzu, xyxzyu, xyxzuzu, xyxzuy, xyzxu, xyzyzu, xyzyuz, xyzux, xyzuy, xyzuzu$. It is not difficult to write all such words; their number is 548. Now we know that $\mathcal{F}_{4,5}$ has at most 548 elements and, similarly as in the case of three generators, we can write a fragment of the multiplication table. This fragment that is displayed has 4 columns and 28 representative rows. Each of the first 2 rows can be permuted to 4 different rows, each of the next 7 rows to 12, and each of the last 19 rows to 24.

$\mathcal{F}_{4,5}$	x	y	z	u
x	x	xy	xz	xu
$xyzux$	$xyzux$	$xyxzuy$	$xzxyuz$	$xuxyzu$
xy	xyx	xy	xyz	xyu
xyx	xyx	$xyxy$	$xyxz$	$xyxu$
$xyxy$	xyx	$xyxy$	$xyxyz$	$xyxyu$
$xyzx$	$xyzx$	$xyxzy$	$xzxyz$	$xyzxu$
$xyxzuy$	$xyzux$	$xyxzuy$	$xzxyuz$	$xuxyzu$
$xyzxu$	$xyzux$	$xyxzuy$	$xzxyuz$	$xyzxu$
$xyzuy$	$xyzux$	$xyzuy$	$xyzyuz$	$xyuyzu$
xyz	$xyzx$	$xyzy$	xyz	$xyzu$
$xyxyz$	$xyzx$	$xyxzy$	$xyxyz$	$xyxyzu$
$xyxz$	$xyzx$	$xyxzy$	$xyxz$	$xyxzu$
$xyxzy$	$xyzx$	$xyxzy$	$xzxyz$	$xyxzyu$
$xyzy$	$xyzx$	$xyzy$	$xyzyz$	$xyzyu$
$xyzyz$	$xyzx$	$xyzy$	$xyzyz$	$xyzyzu$
$xyzu$	$xyzux$	$xyzuy$	$xyzuz$	$xyzu$
$xyxyzu$	$xyzux$	$xyxzuy$	$xyxyzuz$	$xyxyzu$
$xyxyzuz$	$xyzux$	$xyxzuy$	$xyxyzuz$	$xyxyzuzu$
$xyxyzuzu$	$xyzux$	$xyxzuy$	$xyxyzuz$	$xyxyzuzu$
$xyxzu$	$xyzux$	$xyxzuy$	$xyxzuz$	$xyxzu$
$xyxzyu$	$xyzux$	$xyxzuy$	$xzxyuz$	$xyxzyu$
$xyxzuz$	$xyzux$	$xyxzuy$	$xyxzuz$	$xyxzuzu$
$xyxzuzu$	$xyzux$	$xyxzuy$	$xyxzuz$	$xyxzuzu$
$xyzyzu$	$xyzux$	$xyzuy$	$xyzyuz$	$xyzyzu$
$xyzyu$	$xyzux$	$xyzuy$	$xyzyuz$	$xyzyu$
$xyzyuz$	$xyzux$	$xyzuy$	$xyzyuz$	$xyuyzu$
$xyzuz$	$xyzux$	$xyzuy$	$xyzuz$	$xyzuzu$
$xyzuzu$	$xyzux$	$xyzuy$	$xyzuz$	$xyzuzu$

One can verify that the groupoid satisfies the equations of $\mathbf{W}_5$ either by an aid of computer or also manually. The result is that the equations are indeed satisfied, and we obtain

Theorem 4.15. *The groupoid $\mathcal{F}_{4,5}$ has 548 elements and its multiplication table can be reconstructed from the fragment of 28 rows and 4 columns.*

It is easy to see that the groupoids $\mathcal{F}_{4,n}$ are infinite for $n \leq 4$. The reason is that the terms $xyxyzuzuxyxyzuzu\dots$ are pairwise $\sim_n$-inequivalent.

Theorem 4.16. *If k is even and $n < 2k-3$ then $\mathcal{F}_{k,n}$ is infinite. If k is odd and $n < 2k-4$ then $\mathcal{F}_{k,n}$ is infinite.*

Proof. Denote the generators by $x_1, \dots, x_k$. For k even the words

$$x_1x_2x_1x_2\dots x_{k-1}x_kx_{k-1}x_kx_1x_2x_1x_2\dots$$

and for k odd the words

$$x_1x_2x_1x_2 \dots x_{k-2}x_{k-1}x_{k-2}x_{k-1}x_kx_1 \dots$$

are pairwise inequivalent with respect to the equations of $\mathbf{W}_n$. □

Theorem 4.17. *Let $k \geq 3$. Then $\mathcal{F}_{k,2k-3}$ is finite.*

Proof. For $k = 3$ it follows from the above theorem. So, let $k \geq 4$. Put $n = 2k - 3$ and denote by $\sim$ the equational theory of $\mathbf{W}_n$. Consider only words in k fixed variables. By a minimal word we will mean a word that it is not $\sim$-equivalent to a shorter word. Clearly, every minimal word is I-reduced.

Suppose that there exists a minimal word $x_1 \dots x_m$ containing at least three occurrences of some variable, and take such a minimal word of minimal possible length. Then $x_1 = x_i = x_m$ for precisely one $i \in \{2, \dots, m-1\}$ and each variable different from x_1 has at most two occurrences in $x_1 \dots x_m$. Consequently, $m \leq 2k + 1$. Since $i - 2 \leq m - 4 \leq 2k - 3 = n$ and $x_1 = x_i$, the variables $x_2, \dots, x_{i-1}$ can be arbitrarily permuted and (consequently) are pairwise different. From the same reason, $x_{i+1}, \dots, x_{m-1}$ can be arbitrarily permuted and are pairwise different. If $x_2, \dots, x_{m-1}$ are pairwise different or if there is at most one pair of equal elements among them then $m - 2 \leq k + 1 \leq 2k - 3 = n$ (since $k \geq 4$), so that the inner variables in $x_1 \dots x_m$ can be arbitrarily permuted; in particular, they can be permuted in such a way that x_i gets to the position with index 2, so that the word starts with two equal variables and can be shortened, a contradiction. Hence there exist four different indexes j, m, r, s with with $x_j = x_m$, $x_r = x_s$, $j < i < m$ and $j < r < i < s$. We can assume that $s < m$, because the two places can be permuted. Take such a quadruple j, m, r, s with the largest possible j. Then $x_{j+1}, \dots, x_{m-1}$ are all different with the only exception $x_r = x_s$, so the length of this sequence is at most k which is less than n, and x_r, x_s can be permuted to become neighbors and then one of them deleted, a contradiction.

So, every minimal word contains at most two occurrences of each of the k variables. There are only finitely many such words and every word in the k variables is $\sim$-equivalent to at least one minimal word. □

Theorem 4.18. *Let $k \geq 3$. If k is odd put $n = 2k - 2$, and if k is even put $n = 2k - 3$. Then $\mathcal{F}_{k,n} = \mathcal{F}_{k,m}$ for all $m \geq n$.*

Proof. It is sufficient to prove for every $m \geq n$ that if $\mathcal{F}_{k,n} \in \mathbf{W}_m$ then $\mathcal{F}_{k,n} \in \mathbf{W}_{m+1}$; the statement will then follow by induction on m. Let $m \geq n$ and $\mathcal{F}_{k,n} \in \mathbf{W}_m$. We need to prove $xy_1 \dots y_{m+1}x = xy_{p(1)} \cdots y_{p(m+1)}x$ in $\mathcal{F}_{k,n}$ for all elements $x, y_1, \dots, y_{m+1}$ of $\mathcal{F}_{k,n}$ and all permutations p of $\{1, \dots, m+1\}$. However, clearly it is sufficient to prove it only in the case when all the elements $x, y_1, \dots, y_{m+1}$ are from the k-element set of generators of $\mathcal{F}_{k,n}$. In order to do this, it is sufficient to prove that $xy_1 \dots y_{m+1}x = xz_1 \dots z_mx$ for some sequence $z_1, \dots, z_m$ such that $\{x, z_1, \dots, z_m\} = \{x, y_1, \dots, y_{m+1}\}$.

If some member of the sequence $x, y_1, \dots, y_{m+1}, x$ is equal to the next following member, we can delete it and the claim is confirmed. So, we can assume that $y_i \neq y_{i+1}$ for all i and $y_1 \neq x \neq y_{m+1}$.

Suppose that $y_i = y_j = y_r$ for some $i < j < r$. Then $xy_1 \dots y_{m+1}x = xy_1 \dots y_i y_j y_{i+1} \dots y_{j-1} y_{j+1} \dots y_r \dots y_{m+1}x = xy_1 \dots y_{j-1} y_{j+1} \dots y_{m+1}x$. So, we can assume that every element occurs at most two times in $y_1, \dots, y_{m+1}$.

Consider first the case when $y_i = y_j = x$ for some $1 \leq i < j \leq m+1$. Then

$$\begin{aligned} xy_1 \dots y_{m+1}x &= xy_i y_1 \dots y_{i-1} y_{i+1} \dots y_j \dots y_{m+1}x \\ &= xy_1 \dots y_{i-1} y_{i+1} \dots y_{m+1}x. \end{aligned}$$

Now let $y_i = x$ for precisely one i. Since the sequence $y_1, \dots, y_{m+1}$ with y_i deleted contains at most $k-1$ different elements and $k < m$, we have $y_j = y_q$ and $y_r = y_s$ for two different pairs $j < q$ and $r < s$. Without loss of generality, $j < r$. If $j < q < i$ then $xy_1 \dots y_{m+1}x = xy_1 \dots y_j y_q y_{j+1} \dots y_{q-1} y_{q+1} \dots y_i \dots y_{n+1}x = xy_1 \dots y_{q-1} y_{q+1} \dots y_{n+1}x$. So, we can assume that $j < i < q$ and, similarly, $r < i < s$. Since y_q, y_s are between y_i and the last occurrence of x, they can be permuted and thus we can suppose that $s < q$. But then the two occurrences of $y_r = y_s$ are between the two occurrences of $y_j = y_q$, can be moved to get one next to the other and then one of them can be deleted. It remains to consider the case when x does not occur in $y_1, \dots, y_{m+1}$.

Let k be odd. The sequence $y_1, \dots, y_{m+1}$ contains at most $k-1$ different elements. If each of them occurs at most twice, we get $m+1 \leq 2k-2 = n$, a contradiction. Thus at least one of these elements occurs at least three times; this case has been handled above.

Let k be even and let us work again under the assumption that no element occurs more than twice in $y_1, \dots, y_{m+1}$. If some of these elements occurs only once, we get $m+1 \leq 2k-3 = n$, a contradiction. Thus every element occurs precisely twice in $y_1, \dots, y_{m+1}$. Clearly, we can assume that there is no quadruple i, j, r, s of indexes with $i < j < r < s$, $y_i = y_s$ and $y_j = y_r$. We are going to prove by induction on $i \geq 0$ that if $4i+1 \leq m+1$ then $4i+4 \leq m+1$ and $xy_1 \dots y_{m+1}x = xz_1 \dots z_{m+1}x$ for some $z_1, \dots, z_{m+1}$ such that $z_{4j+1} = z_{4j+3}$ and $z_{4j+2} = z_{4j+4}$ for all $j \leq i$. Let this be true for all numbers less than i. So, we can suppose that $y_{4j+1} = y_{4j+3}$ and $y_{4j+2} = y_{4j+4}$ for all $j < i$. Since y_{4i+1} does not occur in $y_1, \dots, y_{4i}$, we have $y_{4i+1} = y_q$ for some $q \geq 4i+3$. If $q > 4i+3$ then $y_{4i+2} = y_r$ for some $r > q$ and the variables between y_{4i+2} and y_r can be permuted so that y_q is moved to the position of y_{4i+3}. So, we can assume that $y_{4i+1} = y_{4i+3}$. Since y_{4i+2} does not occur in $y_1, \dots, y_{4i+1}$, we have $y_{4i+2} = y_s$ for some $s \geq 4i+4$. If $s > 4i+4$, then from a similar reason y_s can be moved to the position of y_{4i+4}, and thus we can also suppose that $y_{4i+2} = y_{4i+4}$.

It follows that the number of different elements in $y_1, \dots, y_{m+1}$ is even. But the number is $k-1$, which is odd. So, if k is even, the assumption that no element occurs more than twice in $y_1, \dots, y_{m=1}$ is contradictory. □

The following table summarizes what we know about the cardinalities of $\mathcal{F}_{k,n}$ for $k \leq 6$ and $n \leq 9$.

$\lvert\mathcal{F}_{k,n}\rvert$	2	3	4	5	6	7	8	9
$k=2$	∞	8	8	8	8	8	8	8
$k=3$	∞	66	60	60	60	60	60	60
$k=4$	∞	∞	∞	548	548	548	548	548
$k=5$	∞	∞	∞	∞	?	f_1	f_2	f_2
$k=6$	∞	∞	∞	∞	∞	∞	∞	f_3

Here f_1, f_2 and f_3 are some finite numbers that we did not compute. In particular, we do not know if $f_1 = f_2$. We do not know whether $\mathcal{F}_{5,6}$ is finite.

5. The variety $\mathbf{W}_\infty$

We denote by $\mathbf{W}_\infty$ the intersection of the varieties $\mathbf{W}_n$ $(n = 1, 2, \dots)$. In this section $\sim$ always denotes the equational theory of $\mathbf{W}_\infty$.

Lemma 5.1. *Let $x_1, \dots, x_n$ be variables and $1 \leq i < j < k < m \leq n$ be such that $x_i = x_k$ and $x_j = x_m$. Then*

$$x_1 \dots x_n \sim x_1 \dots x_i x_{p(i+1)} \dots x_{p(m-1)} x_m \dots x_n$$

for any permutation p of $\{i+1, \dots m-1\}$ such that $p(j) < p(k)$.

Proof. x_j can be moved to the position $i+1$ and then x_k can be moved to the position $i+2$. Since the remaining variables of $x_{i+1} \dots x_{m-1}$ are now between two occurrences of the same variable x_m, they can be arbitrarily permuted. Then the variable at position $i+2$ can be moved to an arbitrary place p with $i+2 \leq p < m$ and the variable at position $i+1$ to an arbitrary place q with $i+1 \leq q < p$. □

Let us fix a strict linear ordering $\sqsubset$ of the set of variables. A word $x_1 \dots x_n$ is said to be admissible if

(1) $x_1 \dots x_n$ is I-reduced
(2) every variable has at most two occurrences in $x_1 \dots x_n$
(3) whenever $1 \leq i < j \leq n$ and $x_i = x_j$ then the variables $x_{i+1}, \dots, x_{j-1}$ are pairwise different and if each of them has only one occurrence in $x_1 \dots x_n$ then $x_{i+1} \sqsubset x_{i+2} \sqsubset \dots \sqsubset x_{j-1}$
(4) whenever $1 \leq i < j < k < m \leq n$, $x_i = x_k$ and $x_j = x_m$ then $j = i+1$, $k = i+2$, each of the variables $x_{i+3}, \dots, x_{m-1}$ has only one occurrence in $x_1 \dots x_n$ and $x_{i+3} \sqsubset x_{i+4} \sqsubset \dots \sqsubset x_{m-1}$

Lemma 5.2. *Every word is $\sim$-equivalent with at least one admissible word.*

Proof. It is sufficient to consider a word $x_1 \dots x_n$ that is not $\sim$-equivalent with any shorter word. Clearly, $x_1 \dots x_n$ is I-reduced. If $1 \leq i < j < k \leq n$ and $x_i = x_j = x_k$ then x_j can be moved to position $i+1$ and then, because of the idempotency, deleted. If $1 \leq i < j \leq n$, $x_i = x_j$ and the variables

x_r $(r = i+1, \dots, j-1)$ are pairwise different then these variables can be permuted to obtain $x_{i+1} \sqsubset \dots \sqsubset x_{j-1}$. Let $1 \le i < j < k < m \le n$, $x_i = x_k$ and $x_j = x_m$. By 5.1 we can suppose that $j = i+1$ and $k = i+2$. Suppose $x_c = x_d$ for some $c \in \{i+3, \dots, m-1\}$ and some $d \ne c$. If $i+3 \le d \le m-1$ then x_c and x_d are between the two occurrences of x_m and thus x_d can be deleted. If $d < i$ then x_i and x_{i+2} are between the two occurrences of x_c. If $d > m$ then x_{i+1} and x_{i+2} can be moved to positions $m-2$ and $m-1$ respectively, so that then both occurrences of x_m are between the two occurrences of x_c and the word $x_1 \dots x_n$ can be again shortened. Thus each of the variables $x_{i+3}, \dots, x_{m-1}$ has only one occurrence in $x_1 \dots x_n$. These variables can be permuted to obtain $x_{i+3} \sqsubset \dots \sqsubset x_{m-1}$. □

Lemma 5.3. *Let $x_1 \dots x_n$ and $y_1 \dots y_m$ be two different admissible words. Then the equation $x_1 \dots x_n \approx y_1 \dots y_m$ together with the equations of $\mathbf{W}_\infty$ implies one of the following three equations:*

(1) $xyxy \approx xy$
(2) $yzyzx \approx yzyx$
(3) $xyzyz \approx xzyz$

Proof. By induction on $n+m$. If $x_n \ne y_m$ then $z(x_1 \dots x_n) \approx z(y_1 \dots y_m)$ gives $zx_n \approx zy_m$ which implies $xy \approx x$ and then the equation (1). So, let $x_n = y_m$.

Suppose $\{x_1, \dots, x_n\} \ne \{y_1, \dots, y_m\}$. Without loss of generality, $y_i \notin \{x_1, \dots, x_n\}$ for some i. Substitute y for y_i and x for any other variable. We get one of the equations $x \approx yx$, $x \approx xyx$ and $x \approx yxyx$. Each of these equations implies (1). (In the case of $x \approx yxyx$ take the substitution $x \mapsto yx$.)

If $x_1 \ne y_1$, take a new variable z and substitute zx_1 for x_1. We get $zx_1 \dots zx_n \approx y_1 \dots y_m$ where $\{z, x_1, \dots, x_n\} \ne \{y_1, \dots, y_m\}$ and thus we get the equation (1) as before.

Thus we can assume that $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$, $x_1 = y_1$ and $x_n = y_m$. Since $x_1 \dots x_n \ne y_1 \dots y_m$, we have $n > 1$ and $m > 1$.

Suppose that x_n has only one occurrence in $x_1 \dots x_n$ and only one occurrence in $y_1 \dots y_m$. If $x_{n-1} \ne y_{m-1}$, substitute x for x_n, y for x_{n-1} and z for all other variables. We get that either zyx or $yzyx$ or $zyzyx$ is $\sim$-equivalent with either yzx or $zyzx$ or $yzyzx$. In each of the four cases (the two terms must start with the same variable) we get either (1) or (2). Now let $x_{n-1} = y_{m-1}$. If we substitute x_{n-1} for x_n, we get $x_1 \dots x_{n-1} \sim y_1 \dots y_{m-1}$ where $x_1 \dots x_{n-1}$ and $y_1 \dots y_{m-1}$ are two different admissible terms, so that the induction assumption can be applied.

Suppose that x_n has only one occurrence in $x_1 \dots x_n$ but two occurrences in $y_1 \dots y_m$. Substitute x for x_n and y for all other variables. We get $yx \sim yxyx$, i.e., we get (1).

It remains to consider the case when $x_n = y_m$ has two occurrences in $x_1 \dots x_n$ and two occurrences in $y_1 \dots y_m$. Let $i < n$, $j < m$, $x_i = x_n$ and $y_j = y_m$. Put $C = \{x_{i+1}, \dots, x_{n-1}\}$ and $D = \{y_{j+1}, \dots, y_{m-1}\}$.

Suppose that each variable from C has only one occurrence in $x_1 \dots x_n$ and each variable from D has only one occurrence in $y_1 \dots y_m$. If $C - D \neq \emptyset$ and $D - C = \emptyset$, substitute x for x_n, x for every variable from $C \cap D$ and y for all other variables to obtain $yxyx \sim yx$, i.e., we get (1). If $C - D \neq \emptyset$ and $D - C \neq \emptyset$, substitute x for x_n, x for every variable from C and y for all other variables to obtain that $yx \sim yxyx$. If $C = D$, substitute a variable $x \notin \{x_1, \dots, x_n\}$ for every variable from $\{x_i, \dots, x_n\}$ to obtain $x_1 \dots x_{i-1}x \sim y_1 \dots y_{j-1}x$ where $x_1 \dots x_{i-1}x$ and $y_1 \dots y_{j-1}x$ are two different admissible words, so that the induction assumption can be applied.

Next suppose that each variable from C has only one occurrence in $x_1 \dots x_n$, while $y_{j-1} = y_{j+1}$. If $y_{j+1} \notin C$, substitute x for all variables from $\{x_i, \dots, x_n\}$ and y for all other variables to obtain $yx \sim yxyx$. Let $y_{j+1} \in C$. If $C - D \neq \emptyset$, substitute x for all variables from $\{y_j, \dots, y_m\}$ and y for all other variables to obtain $yxyx \sim yx$. If $D - C \neq \emptyset$, substitute x for all variables from $\{x_i, \dots, x_n\}$ and y for all other variables to obtain $yx \sim yxyx$. If $C = D$, substitute x for all variables from $\{x_i, \dots, x_n\} - \{y_{i+1}\}$, y for y_{i+1} and z for all other variables to obtain $zxyx \sim zyxyx$; we get (3).

Finally, let $x_{i-1} = x_{i+1}$ and $y_{j-1} = y_{j+1}$. If $x_{i+1} = y_{j+1}$, substitute x_n for x_{i+1} to obtain $x_1 \dots x_{i-1}x_{i+2} \dots x_n \sim y_1 \dots y_{j-1}y_{j+2} \dots y_m$ and use the induction assumption. Let $x_{i+1} \neq y_{j+1}$. If $y_{j+1} \notin \{x_i, \dots, x_n\}$, substitute x for every variable from $\{x_i, \dots, x_n\}$ and y for all other variables to obtain $yx \sim yxyx$. If $y_{j+1} \in \{x_i, \dots, x_n\}$ and $x_{i+1} \in \{y_j, \dots, y_m\}$, substitute x for x_n, x for x_{i+1}, z for every variable from $\{x_{i+2}, \dots, x_{n-1}\}$ and y for every other variable to obtain either $zxyx \sim zyxyzx$ or $zxyx \sim zyxyx$ and thus (substitute x for y in the first equation) either (1) or (3). $\square$

Theorem 5.4. *The variety $\mathbf{W}_\infty$ is generated by $\mathcal{F}_{3,4}$ and every word is $\mathbf{W}_\infty$-equivalent with precisely one admissible word.*

Proof. By 5.2, every word is $\mathbf{W}_\infty$-equivalent with at least one admissible word. If two different admissible words are $\mathbf{W}_\infty$-equivalent then $\mathbf{W}_\infty$ satisfies one of the three equations 5.3(1), 5.3(2) and 5.3(3). However, it is easy to check that none of these three equations is satisfied in $\mathcal{F}_{3,4}$. Since $\mathcal{F}_{3,4}$ belongs to $\mathbf{W}_\infty$, it follows that every word is $\mathbf{W}_\infty$-equivalent with precisely one admissible word. If $\mathcal{F}_{3,4}$ satisfies an equation not satisfied by all algebras in $\mathbf{W}_\infty$ then, again by 5.3, it satisfies one of the three equations which is not possible. $\square$

Remark 5.5. The variety $\mathbf{W}_\infty$ is not generated by $\mathcal{F}_{2,2}$. Indeed, $\mathcal{F}_{2,2}$ satisfies $xyzyz \approx xzyz$ and this equation is not satisfied in $\mathbf{W}_\infty$.

Remark 5.6. According to 5.4, the cardinality $C(k)$ of the k-generated free algebra in $\mathbf{W}_\infty$ can be computed in the following way. Denote by S_k the set of finite sequences $\langle n_1, \dots, n_r \rangle$ of positive integers such that $n_1 + \dots + n_r = k$.

Put

$$D(k)=\sum_{\langle n_1,\dots,n_r\rangle\in S_k}\prod_{i=1}^{r}\binom{k-n_1-\dots-n_{i-1}}{n_i}n_i^2$$

Then $C(k)=\sum_{i=1}^{k}\binom{k}{i}D(i)$. In particular,

$$C(2)=8,\ C(3)=60,\ C(4)=548,\ C(5)=6180,\ C(6)=83502.$$

Remark 5.7. The equations of $\mathbf{W}_\infty$ together with $xyxy\approx xy$ imply the equation $xyxz_1\dots z_ny\approx xyz_1\dots z_ny$. Indeed, $xyxz_1\dots z_ny=xyxyz_1\dots z_ny=xyz_1\dots z_ny$.

Remark 5.8. The equations of $\mathbf{W}_\infty$ together with an equation $xyxz_1\dots z_nyu\approx xyxz_1\dots z_nu$ ($n\geq 1$) imply $xyxy\approx xy$. Indeed, take the substitution sending y to x, $z_1,\dots,z_n$ to y and u to y.

Theorem 5.9. *The intersection of* $\mathbf{W}_3$ *with the variety determined by* $xyxy\approx xy$ *is the variety of idempotent slim groupoids satisfying* $xyzu\approx xzyu$.

Proof. Denote by $\sim$ the equational theory of $\mathbf{W}_3$ extended by $xyxy\approx xy$. We have

$$\begin{aligned}
&xyxzy\sim xyxyzy\sim xyzy\\
&xyxzy\sim xyzxy\sim xzyxy\sim xzxyxy\sim xzxy\\
&xyzy\sim xzxy\\
&xzxy\sim xyxzy\sim xzyzy\sim xzy\\
&xyzu\sim xyxzu\sim xyxzxu\sim xzxyxu\sim xzyu
\end{aligned}$$

□

6. The variety $\mathbf{Y}$

Denote by $\mathbf{Y}$ the variety determined by the equations of $\mathbf{W}_\infty$ together with the equations $xyxyz\approx xyxz$ and $zxyxy\approx zyxy$. In this section we denote by $\sim$ the equational theory of $\mathbf{Y}$.

Lemma 6.1. *We have*

(1) $zxyxu\sim zyxyu$
(2) $zxyv_1\dots v_nxu\sim zyxv_1\dots v_nyu$
(3) $xyxu_1\dots u_nyz\sim xyu_1\dots u_nxz$
(4) $zxyxu_1\dots u_ny\sim zyxu_1\dots u_ny$

Proof. (1) $zxyxu\sim zxyxyu\sim zyxyu$.

(2) $zxyv_1\dots v_nxu\sim zxyxv_1\dots v_nxu\sim zyxyv_1\dots v_nxu\sim zyxyv_1\dots v_nxyxu\sim zyxyv_1\dots v_nyxyu\sim zyxv_1\dots v_nyu$.

(3) $xyxu_1\dots u_nyz\sim xyu_1\dots u_nyxyz\sim xyu_1\dots u_nxyxz\sim xyu_1\dots u_nxz$.

(4) $zxyxu_1\dots u_ny\sim zyxyu_1\dots u_ny\sim zyxu_1\dots u_ny$. □

By a 2-admissible word we mean a word $x_1\dots x_n$ such that

(1) $x_1 \dots x_n$ is I-reduced
(2) every variable has at most two occurrences in $x_1 \dots x_n$
(3) whenever $1 \le i < j \le n$ and $x_i = x_j$ then the variables $x_{i+1}, \dots, x_{j-1}$ are pairwise different and if each of them has only one occurrence in $x_1 \dots x_n$ then $x_{i+1} \sqsubset x_{i+2} \sqsubset \dots \sqsubset x_{j-1}$
(4) whenever $1 < i < j < n$ and $x_i = x_j$ then $x_i \sqsubset x_{i+1} \sqsubset \dots \sqsubset x_{j-1}$
(5) whenever $1 \le i < j < k < m \le n$, $x_i = x_k$ and $x_j = x_m$ then $i = 1$, $j = 2$, $k = 3$, $m = n$ and $x_4 \sqsubset x_5 \sqsubset \dots \sqsubset x_{n-1}$

Lemma 6.2. *Let $x_1 \dots x_n$ and $y_1 \dots y_m$ be two different 2-admissible words. Then the equation $x_1 \dots x_n \approx y_1 \dots y_m$ together with the equations of $\mathbf{Y}$ implies $xyxy \approx xy$.*

Proof. By induction on $n + m$. If $x_n \ne y_m$ then $z(x_1 \dots x_n) \approx z(y_1 \dots y_m)$ gives $zx_n \approx zy_m$ which implies $xy \approx x$ and then $xyxy \approx xy$. So, let $x_n = y_m$.

Suppose $\{x_1, \dots, x_n\} \ne \{y_1, \dots, y_m\}$. Without loss of generality, $y_i \notin \{x_1, \dots, x_n\}$ for some i. Substitute y for y_i and x for any other variable. We get one of the equations $x \approx yx$, $x \approx xyx$ and $x \approx yxyx$. Each of these equations implies $xyxy \approx xy$. (In the case of $x \approx yxyx$ take the substitution $x \mapsto yx$.)

If $x_1 \ne y_1$, take a new variable z and substitute zx_1 for x_1. We get $zx_1 \dots zx_n \approx y_1 \dots y_m$ where $\{z, x_1, \dots, x_n\} \ne \{y_1, \dots, y_m\}$ and thus we get the equation $xyxy \approx xy$ as before.

Thus we can assume that $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$, $x_1 = y_1$ and $x_n = y_m$. Since $x_1 \dots x_n \ne y_1 \dots y_m$, we have $n > 1$ and $m > 1$.

Suppose that x_n has only one occurrence in $x_1 \dots x_n$ and only one occurrence in $y_1 \dots y_m$. If $x_{n-1} = y_{m-1}$ then we can we substitute x_{n-1} for x_n to obtain $x_1 \dots x_{n-1} \sim y_1 \dots y_{m-1}$ where $x_1 \dots x_{n-1}$ and $y_1 \dots y_{m-1}$ are two different 2-admissible terms, so that the induction assumption can be applied. Let $x_{n-1} \ne y_{m-1}$. If x_{n-1} has only one occurrence in $x_1 \dots x_n$, substitute x for x_n, x for x_{n-1} and y for all other variables to obtain $yx \approx yxyx$. If y_{m-1} has a single occurrence in $y_1 \dots y_m$, we can proceed similarly. It remains to consider the case when $x_{n-1} = x_i$ and $y_{m-1} = y_j$ for some $i < n - 1$ and $j < m - 1$. We cannot have $x_{n-1} \in \{y_{j+1}, \dots, y_{m-2}\}$ and $y_{m-1} \in \{x_{i+1}, \dots, x_{n-2}\}$ at the same time, since then we would get both $x_{n-1} \sqsubset y_{m-1}$ and $y_{m-1} \sqsubset x_{n-1}$. Let $y_{m-1} \notin \{x_{i+1}, \dots, x_{n-2}\}$ (the other case is similar). Substituting x for $x_i, \dots, x_n$ and y for all other variables we get $yx \approx yxyx$.

Suppose that x_n has only one occurrence in $x_1 \dots x_n$ but two occurrences in $y_1 \dots y_m$. Substitute x for x_n and y for all other variables. We get $yx \sim yxyx$.

It remains to consider the case when $x_n = y_m$ has two occurrences in $x_1 \dots x_n$ and two occurrences in $y_1 \dots y_m$. Let $i < n$, $j < m$, $x_i = x_n$ and $y_j = y_m$. Put $C = \{x_{i+1}, \dots, x_{n-1}\}$ and $D = \{y_{j+1}, \dots, y_{m-1}\}$.

Suppose that each variable from C has only one occurrence in $x_1 \dots x_n$ and each variable from D has only one occurrence in $y_1 \dots y_m$. If $C - D \ne \emptyset$

and $D - C = \emptyset$, substitute x for x_n, x for every variable from $C \cap D$ and y for all other variables to obtain $yxyx \sim yx$. If $C - D \neq \emptyset$ and $D - C \neq \emptyset$, substitute x for x_n, x for every variable from C and y for all other variables to obtain that $yx \sim yxyx$. If $C = D$, substitute a variable $x \notin \{x_1, \dots, x_n\}$ for every variable from $\{x_i, \dots, x_n\}$ to obtain $x_1 \dots x_{i-1}x \sim y_1 \dots y_{j-1}x$ where $x_1 \dots x_{i-1}x$ and $y_1 \dots y_{j-1}x$ are two different admissible words, so that the induction assumption can be applied.

Now consider the case when a variable from D has two occurrences in $y_1 \dots y_m$. Then $y_1 = y_3$, $y_2 = y_m$ and $y_4 \sqsubset \dots \sqsubset y_{m-1}$. If also some variable from C has two occurrences in $x_1 \dots x_n$, we get $x_1 \dots x_n = y_1 \dots x_n$, a contradiction. Thus every variable from C has only one occurrence in $x_1 \dots x_n$. In particular, the variable $x_1 = y_1 = y_3$ does not belong to C. Substitute x for $x_i, \dots, x_n$ and y for all other variables to obtain $yx \approx yxyx$.

Finally, the case when a variable from C has two occurrences in $x_1 \dots x_n$ can be handled similarly. □

The free groupoid in $\mathbf{Y}$ with three generators can be obtained from the groupoid $\mathcal{F}_{3,4}$ if we take its factor by the congruence generated by all pairs $\langle ababc, abac \rangle$ and $\langle abcbc, acbc \rangle$. It is easy to construct the multiplication table of this groupoid. It has 48 elements and we will denote it by $\mathcal{F}_{3,\mathbf{Y}}$. One can easily check that the groupoid does not satisfy $xyxy \approx xy$.

Theorem 6.3. *The variety $\mathbf{Y}$ is generated by $\mathcal{F}_{3,\mathbf{Y}}$ and every word is $\mathbf{Y}$-equivalent with precisely one 2-admissible word.*

Proof. It follows from 6.1 that every word is $\mathbf{Y}$-equivalent with at least one 2-admissible word. If two different 2-admissible words were $\mathbf{Y}$-equivalent then $\mathbf{Y}$ would satisfy $xyxy \approx xy$ by 6.2. However, this equation is not satisfied in $\mathcal{F}_{3,\mathbf{Y}}$. Since $\mathcal{F}_{3,\mathbf{Y}}$ belongs to $\mathbf{Y}$, it follows that every word is $\mathbf{Y}$-equivalent with precisely one 2-admissible word. If $\mathcal{F}_{3,\mathbf{Y}}$ satisfied an equation not satisfied by all algebras in $\mathbf{Y}$ then, again by 6.2, it would satisfy $xyxy \approx xy$ which it does not. □

Theorem 6.4. *The lattice of subvarieties of $\mathbf{Y}$ has six elements: the trivial variety V_0, the variety V_1 of left zero semigroups, the variety V_2 of right zero semigroups, the variety V_3 of rectangular bands, the variety V_4 of idempotent slim groupoids satisfying $xyxy \approx xy$, and itself. The only proper inclusions are $V_0 \subset V_1 \subset V_3 \subset V_4 \subset \mathbf{Y}$ and $V_0 \subset V_2 \subset V_3$.*

Proof. It follows from the above results that every proper subvariety of $\mathbf{Y}$ is contained in V_4. The lattice of subvarieties of V_4 has been described in [1]. □

Theorem 6.5. *The variety $\mathbf{Y}$ is generated by the inherently nonfinitely based four-element groupoid $\mathcal{G}_{4,3}$ introduced in 2.2.*

Proof. It is easy to check that $\mathcal{G}_{4,3}$ satisfies all the equations of $\mathbf{Y}$ but not the equation $xyxy \approx xy$. □

References

[1] S. Crvenković and J. Dudek, *Rectangular groupoids.* Czechoslovak Math. J. **35** (1985), 405–414.

[2] J. A. Gerhard, *The lattice of equational classes of idempotent semigroups.* J. Algebra **15** (1970), 195–224.

[3] E. Jacobs and R. Schwabauer, *The lattice of equational classes of algebras with one unary operation.* Ann. of Math. **71** (1964), 151–155.

[4] J. Ježek, *Slim groupoids.* To appear.

[5] R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties, Volume I.* Wadsworth & Brooks/Cole, Monterey, CA, 1987.

Department of Algebra, Charles University, Sokolovská 83, 18675 Praha 8, Czech Republic

E-mail address: `jezek@karlin.mff.cuni.cz`