Fair-sized projective modules

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Abstract

We investigate a condition on paritcular chains of ideals that can help us to understand infinitely generated modules over noetherian rings a bit. The results are applied to semilocal noetherian rings, integral group rings of finite groups and universal enveloping algebras of solvable Lie algebras of finite dimension.

Key words: Projective modules, idempotent ideals, noetherian rings.

1 Introduction

This paper is devoted to the study of infinitely generated projective modules over some associative unital rings. Specifically, we are interested in cases when the ring possesses projective modules that are not direct sums of finitely generated modules. Some general results and examples of rings possessing such modules were given in [9]. Our motivation was to find a technique that can be used to prove that a superdecomposable projective module over a semilocal ring exists.

Let us briefly explain the idea. According to the theorem of Kaplansky, any projective right module over a ring R is a direct sum of countably generated right modules, so we are left to investigate countably generated projectives, that is direct summands of a countably generated free right module $F = R^{(\mathbb{N})}$. Suppose that $P \oplus P' = F$. The canonical projection $\pi\colon F \to P$ is given by a column-finite $\mathbb{N} \times \mathbb{N}$ idempotent matrix A. We say that A represents P (observe that the columns of A generate P). Suppose that I_n is an ideal of R generated by those entries of A that are below the n-th row. Clearly, P is finitely generated if and only if there exists $k \in \mathbb{N}$ such that $I_l = 0$ for any $l \geq k$. The other extreme is when $I_1 = I_2 = \ldots = R$. It is a well-known

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result [3, Theorem 3.1] of Bass that in this case $P \simeq F$ provided $R/J(R)$ is right noetherian. In this paper we focus on the case when the sequence $I_1 \supseteq I_2 \supseteq \dots$ terminates at an ideal I. It is easy to see that I is idempotent. We show that if R is left and right noetherian, and the sequence $I_1 \supseteq I_2 \supseteq \cdots$ terminates at I, then P contains any countably generated projective module having its trace ideal in I as a direct summand. This can be seen as a version of [3, Theorem 3.1]. It is easy to give a condition assuring that any sequence of ideals derived from any idempotent column finite matrix terminates: If $I_1, I_2, \dots$ is a sequence of ideals such that $I_{k+1}I_k = I_{k+1}$ for any $k \in \mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \dots$. We shall call this condition (*).

In section 2 we show that over a left and right noetherian ring R with (*), the theory of projective modules "reduces" to the theory of idempotent ideals in R and the theory of finitely generated projective modules over factors of R modulo idempotent ideals. This justifies a bit the despect to infinitely generated projective modules indicated in the introduction of [3].

Further sections are devoted to some examples. We prove (*) for semilocal noetherian rings, integral group rings of a finite group and universal enveloping algebras of finite solvable Lie algebras over a field of characteristic zero. Then we derive the following statements:

(i) There exists a semilocal noetherian ring possessing a superdecomposable projective module.
(ii) Any indecomposable projective module over an integral group ring of a finite group is finitely generated.
(iii) Any infinitely generated projective module over a finite dimensional solvable Lie algebra over a field of characteristic zero is free.

The second statement should give an answer to [7, Problem 8.34].

Let us briefly recall some notions and notation. By a ring we always mean an associative ring with a unit. A module stands for a unital right module over a ring. If M is a module over R, then $\sum_{f \in \mathrm{Hom}_R(M,R)} f(M)$ is an ideal called the *trace ideal of* M. We denote this ideal by $\mathrm{Tr}(M)$. If P is a projective module over R, then $\mathrm{Tr}(P)$ is the least element of $\{X \subseteq R \mid PX = P\}$, so clearly $\mathrm{Tr}(P)$ is an idempotent ideal (i.e., $\mathrm{Tr}(P)^2 = \mathrm{Tr}(P)$). Let us recall an important result of Whitehead:

Fact 1.1 [14, Corollary 2.7] *Let I be an idempotent ideal of R finitely generated on the left. Then there exists a countably generated projective right R-module P such that* $\mathrm{Tr}(P) = I$.

To avoid the confusion we call rings which have all left or right ideals finitely generated left and right noetherian rings, although they are often called noetherian rings. Finally by infinitely generated projective module we mean a countably but not finitely generated projective module.

2 I-big modules

Let P be a countably generated projective module and let I be an ideal. We say that P is I-big if for any countably generated projective module Q of the trace ideal contained in I there exists an epimorphism from P onto Q (and hence P contains a direct summand isomorphic to Q).

Remark 2.1 (Eilenberg's trick) Let I be an ideal and let P be an I-big projective module. If Q is a countably generated projective module of the trace ideal contained in I, then $P \oplus Q \simeq P$. This is because $Q^{(\omega)}$ is a direct summand of P.

Lemma 2.2 *Let I be an idempotent ideal that is finitely generated as a left ideal. Then there exists an I projective module P such that* $\mathrm{Tr}(P) = I$. *Such a module is unique up to isomorphism.*

PROOF. By [14, Corollary 2.7] there exists a countably generated projective module P such that $\mathrm{Tr}(P) = I$. Of course, $\mathrm{Tr}(P^{(\omega)}) = I$. If Q is a countably generated projective module having the trace ideal contained in I, then $QI = Q$ and Q is a factor of $P^{(\omega)}$. Let P_1, P_2 be I-big modules having trace ideal I. By Remark 2.1, $P_1 \oplus P_2 \simeq P_1$. Similarly, $P_1 \oplus P_2 \simeq P_2$, therefore $P_1 \simeq P_2$.

Remark 2.3 Observe that we proved that for any ideal I that is a trace ideal of a countably generated projective module there exists a unique I-big module. However, our intention is to use I-big modules over left and right noetherian rings.

We shall say that a ring R satisfies the condition (*) if for any sequence $I_1, I_2, \ldots$ of ideals in R such that $I_{k+1}I_k = I_{k+1}, k \in \mathbb{N}$ (observe that the sequence is in fact a descending chain) there exists $n \in \mathbb{N}$ such that $I_k = I_n$ for any $n \leq k \in \mathbb{N}$.

We shall use this condition in the following context: Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be an idempotent column-finite matrix over R and let $I_k = \sum_{k \leq i \in \mathbb{N}, j \in \mathbb{N}} Ra_{i,j}R$ for any $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ there exists n_k such that $a_{i,j} = 0$ for any $i \geq n_k$ and $j < k$. Obviously, it is possible to choose $k < n_k$. Since A is idempotent, we have $I_{n_k}I_k = I_{n_k}$. If R satisfies (*), then all but finitely many I_k's are equal to an idempotent ideal.

Lemma 2.4 *Let R be a ring satisfying (*) and let P be a countably generated projective module over R. The set of ideals $I(P) = \{I \subseteq R \mid P/PI$ is finitely generated $\}$ contains the least element I_0, which is an idempotent ideal.*

PROOF. Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be an idempotent column finite matrix representing P and $I_k, k \in \mathbb{N}$ ideals defined above. Let I_0 be the least element of $\{I_k \mid k \in \mathbb{N}\}$. As remarked above I_0 is idempotent. Let $\{e_i \mid i \in \mathbb{N}\}$ be the canonical free base of $R^{(\mathbb{N})}$ and suppose that $I_0 = I_m = I_{m+1} = \ldots$. Then $\sum_{i=1}^{m} Ae_iR + PI_0 = P$, so P/PI_0 is finitely generated. Suppose that K is an ideal such that P/PK is a finitely generated module. Imagine $P = AR^{(\mathbb{N})} \subseteq R^{(\mathbb{N})}$. The elements of PK are exactly the elements of P whose components are given by elements of K. If P/PK is finitely generated, then there exists $k \in \mathbb{N}$ such that $a_{i,j} \in K, i \geq k, j \in \mathbb{N}$, therefore $I_0 \subseteq K$.

Thus if R satisfies (*), then a countably generated projective module gives a pair (I, P'), where I is an idempotent ideal and P' is a finitely generated projective R/I-module.

Lemma 2.5 *Let I be an idempotent ideal such that I is finitely generated as a left module and as a right module. If P, Q are I-big projective modules satisfying $P/PI \simeq Q/QI$, then $P \simeq Q$.*

PROOF. Let B be the unique I-big projective module having the trace ideal I. Observe that $P \oplus B^{(\omega)} \simeq P$ according to Remark 2.1. If $f: P \to Q$ induces an isomorphism $P/PI \to Q/QI$, then $f(P) + QI = Q$. Since QI is countably generated and $\mathrm{Tr}(B) = I$, we obtain an epimorphism $h: P \oplus B^{(\omega)} \to Q$ such that $h|_P = f$. Since f induces a monomorphism $P/PI \to Q/QI$ and $h(B^{(\omega)}) \subseteq QI$, $X = \mathrm{Ker}\ h \subseteq PI \oplus B^{(\omega)}$. Thus X is a projective module having its trace contained in I, so $Q \oplus X \simeq Q$ since Q is I-big. Finally, $Q \oplus X \simeq P \oplus B^{(\omega)} \simeq P$ and $Q \simeq P$ follows.

Lemma 2.6 *Let I be a proper idempotent ideal that is finitely generated as a left ideal of a ring R. If P' is a finitely generated projective module over R/I, then there exists an I-big projective module P such that $P/PI \simeq P'$.*

PROOF. By Lemma 2.2 it is enough to find a countably generated projective module P such that $P/PI \simeq P'$.

Suppose that P' is given by a $n \times n$ matrix X that is idempotent modulo I. This means that the R-matrix X is a lift of an idempotent R/I-matrix $\overline{X}$. Let $I = Ii_1 + \cdots + Ii_l, i_1, \ldots, i_l \in I$. Consider a sequence of matrices $A_1, A_2, \ldots$ given inductively: A_1 has $c_1 = n$ columns and $r_1 = ln + n$ rows.

The square matrix given by the first n rows of A_1 is exactly X and $(A_1)_{i,j} = 0$ if $n < i \le n+(j-1)l$ or $i > n+jl$ and the remaining entries in each column are filled by generators $i_1, \dots, i_l$. (So in each column we have an "independent" set of generators for I.)

If A_k, r_k, c_k have been defined, then A_{k+1} has $c_{k+1} = r_k$ columns and $r_{k+1} = r_k + lr_k$ rows. The $n \times n$ top left corner of A_{k+1} is given by the matrix X and all the other entries in the first r_k rows are zero. The remaining lr_k rows contains $i_1, \dots, i_l$ placed in each column in the same fashion as described for A_1.

Now we claim that for any $k \in \mathbb{N}$ there is a $c_{k+1} \times r_{k+1}$ matrix B_k such that $B_k A_{k+1} A_k = A_k$. Observe that the $c_k \times c_k$ matrix given by the first c_k rows of A_k is idempotent modulo I. We can find a $r_k \times r_k$ matrix C_k such that $C_k A_k = A_k$: The top left corner of C_k is given by X, the other entries in the first c_k columns are zero and the matrix is completed by elements of I since $I = Ii_1 + \cdots + Ii_l$ and $i_1, \dots, i_l$ are placed independently in the bottom part of A_k. This matrix C_k can be written as $D_k A_{k+1}$, where D_k is a suitable $r_k \times r_{k+1}$ matrix (again we place X to the top left corner of D_k, the other entries in the first r_k columns will be zero and the remaining entries can be completed because the generators for I are placed independently in A_{k+1}.) Now, since $A_k = C_k A_k = D_k A_{k+1} A_k$, we put $B_k = D_k$.

Let $F_k = R^{c_k}$ and let $f_k \colon F_k \to F_{k+1}$ be given by A_k. By [14, Theorem 2.1], the colimit of the direct system induced by f_k's is a projective module P. Applying the functor $- \otimes_R R/I \colon \mathrm{Mod}-R \to \mathrm{Mod}-R/I$ we see that P/PI is an R/I-module isomorphic to the colimit of the system $(R/I)^n \xrightarrow{\overline{X}} (R/I)^n \xrightarrow{\overline{X}} \dots$ that is easily seen to be $\overline{X}(R/I)^n \simeq P'$. Therefore $P/PI \simeq P'$.

The following lemma is in some sense a clone of [3, Theorem 3.1]. We give a brief but complete proof of a statement for left and right noetherian rings rather than specifying what should be modified in the proof of [3, Theorem 3.1] to get a real generalization.

Lemma 2.7 *Let R be a left and right noetherian ring. Let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be an idempotent column-finite matrix. Let $I_k = \sum_{i \ge k, j \in \mathbb{N}} Ra_{i,j}R$. If there exists $n \in \mathbb{N}$ such that $I_m = I_n$ for any $m \ge n$, then the module $P = AR^{(\mathbb{N})} \subseteq R^{(\mathbb{N})}$ is I_n-big.*

PROOF. Put $I = I_n$ and observe that I is finitely generated as a left ideal. Let us denote a_i the i-th column of A. We shall construct a sequence $p_1, p_2, \dots$ of elements in P, let us denote $p_i = (c_{j,i})_{j \in \mathbb{N}}$, such that there exist integers $1 = i_1 < i_2 < \dots$ such that for any $k \in \mathbb{N}$ $I \subseteq Rc_{i_k,k} + \cdots + Rc_{i_{k+1}-1,k}$ and

$c_{l,k} = 0$ for any $l \geq i_{k+1}$. Let us describe how we can find p_1 (an element of P such that the left ideal generated by its components contains I). Put $s_1 = 1, s'_1 = 1 \in \mathbb{N}, x_1 = a_{1,1}, t_1 = 1 \in R$ and let $m_1 \in \mathbb{N}$ be such that $a_{1,1}R + \cdots + a_{1,m_1}R = \sum_{j\in\mathbb{N}} a_{1,j}R$ (we are using R right noetherian here). Since A is column finite, there exists $s'_1 < m'_1 \in \mathbb{N}$ such that $a_{i,j} = 0$ for any $i \geq m'_1, j \leq m_1$. If $I \subseteq Rx_1$, we finish (and put $p_1 = a_1$). Suppose the opposite. Since $I \subseteq I_{m'_1}$, there exist $s_2, s'_2 \in \mathbb{N}$, $s'_2 \geq m'_1$ and $t_2 \in R$ such that $a_{s'_2,s_2}t_2 \notin Rx_1$. Put $x_2 = a_{s'_2,s_2}t_2$. Again, there is $m_2 \in \mathbb{N}$ such that $a_{s'_2,1}R + \cdots + a_{s'_2,m_2}R = \sum_{j\in\mathbb{N}} a_{s'_2,j}R$. Next we find m'_2 such that $a_{i,j} = 0$ for any $i \geq m'_2, j \leq m_2$. If $I \not\subseteq Rx_1 + Rx_2$ we continue inductively. Observe that $Rx_1 \subsetneq Rx_1 + Rx_2$ Since R is left noetherian, the process has to stop at some $f \in \mathbb{N}$, that is $I \subseteq Rx_1 + \cdots + Rx_f$. Now p_1 is the right R-combination of $a_1, \ldots, a_{m_f}$ such that the s'_i-th component of this combination is x_i for $1 \leq i \leq f$. We find $i_2 > i_1$ such that the k-th component of p_1 is zero whenever $k \geq i_2$.

Suppose that $p_1, \ldots, p_k$ have been constructed. Find $i_{k+1} > i_k$ such that $c_{j,k} = 0$ for any $j \geq i_{k+1}$. Then p_{k+1} is constructed in the same way as p_1 but we start on the i_{k+1}-th row of A (i.e., we start with $x_1 = a_{i_{k+1},1}, s_1 = 1, s'_1 = i_{k+1}$).

Now, let Q be a countably generated projective module of the trace ideal contained in I given by a column finite idempotent matrix B over R (again, we consider Q as a submodule of $R^{(\mathbb{N})}$). Since the trace ideal of Q lies in I, all entries of B are in I. Let C be a matrix such that columns of C are given by $p_1, p_2, \ldots$. The shape of C guarantees the existence of a column finite matrix D having all entries in $\mathrm{Tr}(Q)$ such that $DC = B$ (it is important to realize that elements of D can be chosen in I). Now, let $f\colon R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ be given by D. Observe, that $Q \subseteq f(P)$ and if $\pi\colon R^{(\mathbb{N})} \to Q$ is a projection, then $\pi f|_P$ is an epimorphism of P onto Q, so P is I-big.

Remark 2.8 Imitating the proof [3, Theorem 3.1] precisely, we could get the following. Let R be a ring such that $R/J(R)$ is right noetherian. Let P, I_k be as above and suppose that $I = I_n = I_{n+1} = \ldots$ is a finitely generated left ideal such that $I \cap J(R) = J(R)I$. Then P is I-big. (For $I = R$ we get the Bass' big projectives theorem). Also we could leave out the assumption (*) and prove that P is $\cap_{n\in\mathbb{N}} I_n$-big but we do not have an application for this version of Lemma 2.7.

Let R be a ring and let $V_r(R)$ be a representative set of finitely generated projective right R-modules, $V_l(R)$ be a representative set of finitely generated projective left R-modules, $V_r(R)^*$ be a representative set of right countably generated projective modules and let $V_l(R)^*$ be a representative set of countably generated projective left R-modules.

Theorem 2.9 *Let R be a left and right noetherian ring satisfying (*). Let*

Id(R) *be the set of its idempotent ideals and let* $\mathcal{S}$ *be the disjoint union* $\cup_{I\in \mathrm{Id}(R)} V_r(R/I)$. *Then there is a bijection* $\varphi\colon V_r(R)^* \to \mathcal{S}$. *Moreover, there is a bijection between* $V_r(R)^*$ *and* $V_l(R)^*$ *extending the classical bijection between* $V_r(R)$ *and* $V_l(R)$ *induced by* $\mathrm{Hom}_R(-, R_R)$.

PROOF. By Lemma 2.7 any countably generated P projective right module is I-big, where I is the least ideal such that and P/PI is finitely generated. As we know I is idempotent. This gives a map from $V_r(R)^*$ to $\mathcal{S}$. This map is a bijection by Lemma 2.5 and Lemma 2.6. The bijection between $V_r^*(R)$ and $V_l^*(R)$ then follows from dualities between finitely generated projective left and right R/I-modules, where I varies Id(R).

Now we should demonstrate the technique in an example. The following one is from [4, Example 4.4] and in our context it has been already studied in [9].

Example 2.10 Let $R = \begin{pmatrix} \mathbb{Z} & 4\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ be a subring of 2×2 matrix ring over $\mathbb{Z}$. Let us denote $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $P_1 = e_1R$ and $P_2 = e_2R$ are finitely generated projective modules. According to [9, Lemma 10.4], the only nontrivial idempotent ideals of R are $A = \mathrm{Tr}(P_1) = Re_1R, B = \mathrm{Tr}(P_2) = Re_2R$. By [9, Proposition 9.5] any projective right module is a generator, $P_1^{(X)}$ or $P_2^{(Y)}$. To finish the classification of infinitely generated projective modules, we just need to prove that R satisfies (*). If I is an ideal of R, then there are $n_1, n_2, n_3, n_4 \in \mathbb{Z}$ such that $I = \begin{pmatrix} n_1\mathbb{Z} & n_2\mathbb{Z} \\ n_3\mathbb{Z} & n_4\mathbb{Z} \end{pmatrix}$. Observe that $n_1 \in n_3\mathbb{Z}$, $4n_3 \in n_1\mathbb{Z}$, $n_2 \in n_1\mathbb{Z}$, $4n_1 \in n_2\mathbb{Z}$, $n_4 \in n_3\mathbb{Z}$ and $4n_3 \in n_4\mathbb{Z}$. Suppose that $KI = K$ for a nonzero ideal K. Then $n_3\mathbb{Z} = \mathbb{Z}$. Then there are only finitely many ideals containing I, so R satisfies (*).

Having proved this we are left to check finitely generated projective modules over R/A and R/B. Both these rings are isomorphic to a local ring $\mathbb{Z}_4$, so finitely generated modules over R/A and R/B are free. Now let P be a countably generated projective R-module. Suppose that P is A-big and $P/PA \simeq R/A^k, k \in \mathbb{N}_0$. But observe that $Q = R^k \oplus e_1^{(\mathbb{N})}$ is also an A-big module such that and $Q/QA \simeq R/A^k$. Similar statement applies for B-big modules, so we have that any infinitely generated projective R-module is isomorphic to direct sum of copies of R, P_1, P_2. This answers [9, Question 10.10]. Observe that we do not say anything about finitely generated projectives.

Remark 2.11 Observe that if R is a left and right noetherian having (*), then any indecomposable projective module is finitely generated. Although we think that (*) is a very particular property (see [6]), it seems that it may occure quite often in natural examples of left and right noetherian rings.

3 Semilocal noetherian rings

Recall that a ring R is said to be *semilocal*, if $R/J(R)$ is semisimple. If P, Q are projective modules, then $P/PJ(R) \simeq Q/QJ(R)$ if and only if $P \simeq Q$ ([10, Theorem 1.3]). In this section we show that any semilocal left and right noetherian ring satisfies (*), so any countably generated projective module over such a ring is I-big. Further we show a connection of the pair $(I, P/PI)$ studied in the previous section and the semisimple module $P/PJ(R)$. Finally, an example of a superdecomposable projective module over a semilocal noetherian ring is given.

Lemma 3.1 *Let R be a left noetherian semilocal ring. Then R has only finitely many idempotent ideals.*

PROOF. By [14, Corollary 2.7], any idempotent ideal of R is a trace ideal of a countably generated projective module P, hence it is also a trace ideal of $P^{(\omega)}$. But since R is semilocal, there exist only finitely many non-isomorphic countably generated projective modules of type $P^{(\omega)}$ according to [10, Theorem 1.3]. Therefore, there exist also only finitely many distinct idempotent ideals.

Corollary 3.2 *Let R be a semilocal left and right noetherian ring. Then R satisfies (*).*

PROOF. Let $\pi\colon R \to R/J(R)$ be the natural projection. Consider a descending sequence of ideals (in R) such that $I_{k+1}I_k = I_{k+1}$. Since $\pi(I_1), \pi(I_2), \dots$ gives a descending sequence in an artinian ring $R/J(R)$, there exists $k_0 \in \mathbb{N}$ such that $\pi(I_k) = \pi(I_{k_0})$ for any $k \geq k_0$. Then $I_{k+1} = I_{k+1}(I_{k+1} + J(R))$ for any $k \geq k_0$. By Nakayama lemma we see that I_k is idempotent for any $k > k_0$. Now we conclude by Lemma 3.1.

Lemma 3.3 *Let P be a projective module of the trace ideal I and let S be a simple module. The following conditions are equivalent.*

(i) S is a factor of I
(ii) S is a factor of P

(iii) $SI = S$

PROOF. (i)⇒(ii) Suppose first that $f\colon I \to S$ is nonzero. Then $f(i) \neq 0$ for some $i \in I$. Since I is the trace ideal of P, there are $g_1, \dots, g_k\colon P \to I$ and $p_1, \dots, p_k \in P$ such that $g_1(p_1) + \cdots + g_k(p_k) = i$. Therefore $fg_j \neq 0$ for some $1 \leq j \leq k$. (Observe that we did not use P projective for this implication.)

(ii)⇒(iii) Follows from $PI = P$.

(iii)⇒(i) Let $f\colon R \to S$ be nonzero. Since $SI = S$, $f(I) = S$.

Proposition 3.4 *Let R be a semilocal left and right noetherian ring. Suppose that P is a countably generated projective module. Then there exists the least ideal I in R such that P/PI is finitely generated. Moreover, let $\{S_1, \dots, S_k\}$ be a set of representatives of simple modules indexed such that $P/PJ(R) \simeq S_1^{n_1} \oplus \cdots \oplus S_l^{n_l} \oplus S_{l+1}^{(\omega)} \oplus \cdots \oplus S_k^{(\omega)}$, $n_1, \dots, n_l \in \mathbb{N}_0$, $0 \leq l \leq k$. Then P is I-big, $S_i = S_i I$ if and only if $i > l$ and $P/PI/\operatorname{rad}(P/PI) \simeq S_1^{n_1} \oplus \cdots \oplus S_l^{n_l}$*

PROOF. By Corollary 3.2, R satisfies (*). By Lemma 2.4, there exists I such that P/PI is finitely generated and I is contained in any ideal K such that P/PK is finitely generated. The definition of I in the proof of Lemma 2.4 together with Lemma 2.7 give that P is I-big. Since I is finitely generated as a left ideal, there exists a unique I-big projective module B of the trace ideal I and $P \oplus B^{(\omega)} \simeq P$ according to Remark 2.1. By Lemma 3.3, if S is a simple module, then $S^{(\omega)}$ is a factor of P (and hence of $P/PJ(R)$) whenever $SI = S$. So we choose the enumeration for simple modules such that $S_1, \dots, S_l$ are annihilated by I and $S_{l+1}, \dots, S_k$ are factors of I. Let $0 \leq \lambda_1, \dots, \lambda_k \leq \infty$ be such that $P/PJ(R) \simeq S_1^{(\lambda_1)} \oplus \cdots \oplus S_k^{(\lambda_k)}$. As remarked above $\lambda_{l+1} = \cdots = \lambda_k = \infty$. On the other hand, $S_1^{(\lambda_1)} \oplus \cdots \oplus S_l^{(\lambda_l)}$ is a factor of P annihilated by I, hence a factor of P/PI, so $\lambda_1, \dots, \lambda_l$ are finite. Suppose that $P/PI/\operatorname{rad}(P/PI) \simeq S_1^{n_1} \oplus \cdots \oplus S_l^{n_l}$. Obviously, $\lambda_i \leq n_i$. On the contrary, $S_1^{n_1} \oplus \cdots \oplus S_l^{n_l}$ is a factor of P, so $n_i \leq \lambda_i$ follows.

Recall that a nonzero module is called *superdecomposable* if it does have no indecomposable direct summand. The following lemma explains our craving for the existence of superdecomposable projectives over semilocal rings.

Lemma 3.5 *Let R be a semilocal ring. Suppose that there is a superdecomposable projective module over R. Then R possesses a nonzero decomposable projective module having all its nonzero direct summands isomorphic.*

PROOF. By the theorem of Kaplansky, if there exists a superdecomposable projective module, then there exists a superdecomposable countably generated projective module. Let $S_1, \ldots, S_k$ be a complete list of representatives of simple R-modules. If P is a countably generated projective module, then we write $\dim(P) = (n_1, \ldots, n_k)$ if $P/PJ(R) \simeq S_1^{(n_1)} \oplus \cdots \oplus S_k^{(n_k)}$, where $0 \le n_i \le \omega$. If P is a superdecomposable module, then there exists a superdecomposable countably generated projective module Q such that $\dim(Q) = (m_1, \ldots, m_k)$, where $m_i = 0$ or $m_i = \omega$ for any $1 \le i \le k$. Let Q' be a superdecomposable module such that $\dim(Q')$ has all components in $\{0, \omega\}$ and the number of nonzero components is as small as possible. Then it is easy to see that $\dim(Q') = \dim(Q'')$ for any nonzero direct summand of Q', so [10, Theorem 1.3] gives that Q' has the required property.

The following example discovered by Puninski [9] shows that a superdecomposable projective module may exist even over a semilocal noetherian ring.

Example 3.6 (cf. [9, Proposition 7.5]) Let $\Sigma = \mathbb{Z} \setminus 2\mathbb{Z} \cup 3\mathbb{Z} \cup 5\mathbb{Z}$ and let $\mathbb{Z}_\Sigma$ be the localization of integers at Σ. Then the group ring $\mathbb{Z}_\Sigma[A_5]$ is a semilocal noetherian ring possessing a superdecomposable projective module.

PROOF. It was already remarked in [9] that $R = \mathbb{Z}_\Sigma[A_5]$ is a semilocal left and right noetherian ring and the augmentation ideal I of the natural epimorphism $R \to \mathbb{Z}_\Sigma$ is idempotent since $[A_5, A_5] = A_5$. It was also said there that any finitely generated projective module is a generator. We can see directly that if P is a finitely generated module, then $\mathrm{Tr}(P)$ cannot be contained in I. Since $\mathbb{Z}_\Sigma$ is a Dedekind ring of zero characteristic, 2,3,5 are not invertible in $\mathbb{Z}_\Sigma$, $P' = P \otimes_{\mathbb{Z}_\Sigma[A_5]} \mathbb{Q}[A_5]$ is a free $\mathbb{Q}[A_5]$-module by [12, Theorem 8.1]. If $\mathrm{Tr}(P) \subseteq I$, then $P'I' = P'$, where I' is an augmentation ideal of $\mathbb{Q}[A_5]$, a contradiction. Let Q be a projective module having the trace ideal I. If Q' is a nonzero direct summand of Q, then Q' cannot be finitely generated, and there is a nonzero idempotent ideal K such that Q' is K-big. Therefore Q' cannot be indecomposable.

Remark 3.7 Over semilocal rings the classification of countably generated projective modules can be given by describing all possible dimensions (if we continue the above notation we ask which elements of $(\omega + 1)^k$ can be obtained as dim of a projective module). Proposition 3.4 gives a way how to do it for semilocal left and right noetherian rings. However, it seems that it can be hard to describe idempotent ideals and finitely generated projectives over corresponding factors. But some partial steps can be quite easy: Let $R = \mathbb{Z}_\Sigma[A_5]$ be the ring from Example 3.6, I the augmentation ideal of R. Let $C(I) = \{P \in \mathrm{Proj} - R \mid P \text{ is } I- \text{ big and } P/PI \text{ is finitely generated } \}$. Suppose that we want to classify modules in $C(I)$. Observe that there are up to isomorphism exactly 3 simple modules annihilated by I say S_1, S_2, S_3 (the simple factors of $\mathbb{Z}_\Sigma$). Let $S_3, \ldots, S_k$ be the list of all remaining representatives of simple modules. Since $\mathbb{Z}_\Sigma$ is an indecomposable semilocal commutative ring, all projective $\mathbb{Z}_\Sigma$-modules are free, therefore if $P \in C(I)$ then $\dim(P) = (k, k, k, \omega, \ldots, \omega)$. Finally, if we observe that the multiplicity of S_1, S_2, S_3 in $R/J(R)$ is 1, then [10, Theorem 1.3] give us that any module of $C(I)$ is a direct sum of a finitely generated free module and the (unique) I-big module of the trace ideal I.

This remark implies the following question: Is any I-big projective module over a semilocal ring a direct sum of a finitely generated module and an I-big module of the trace I?

4 Integral group rings

In this section we apply our approach to integral group algebras for some finite groups. If G is a finite solvable group, then any infinitely generated projective (left or right) $\mathbb{Z}[G]$-module is free according to [13, Theorem 7] and $\mathbb{Z}[G]$ contains only trivial idempotent ideals according to [11]. If G is not solvable but finite, then a non-trivial idempotent ideal exists by [1]. We prove that for any finite group G the ring $\mathbb{Z}[G]$ satisfies (*), so regarding section 2 we can understand infinitely generated projective $\mathbb{Z}[G]$-modules a bit.

Lemma 4.1 *Let G be a finite group. Suppose there is a sequence of ideals $I_1, I_2, \ldots \subseteq \mathbb{Z}[G]$ such that $I_{k+1}I_k = I_{k+1}$. Then there is a descending chain $K_1 \supseteq K_2 \supseteq \cdots$ of idempotent ideals in $\mathbb{Z}[G]$ such that $\cap_{i\in\mathbb{N}}K_i = \cap_{i\in\mathbb{N}}I_i$. Moreover, if $I_1, I_2, \ldots$ contains a strictly descending subsequence, then $K_1, K_2, \ldots$ can be chosen strictly descending.*

PROOF. Let $I_1, I_2, \ldots$ be a sequence of ideals in $\mathbb{Z}[G]$ such that $I_{k+1}I_k = I_{k+1}$. Let I'_k be a subspace of $\mathbb{Q}[G]$ generated by I_k. $I'_1, I'_2, \ldots$ are ideals of $\mathbb{Q}[G]$ such that $I'_{k+1}I'_k = I'_{k+1}$. Now, $\mathbb{Q}[G]$ is semisimple, so there is $m \in \mathbb{N}$ such that $I'_k = I'_m$ for any $k \geq m$. Since I_k's are finitely generated, for any $k \geq m$ there exists $n_k \in \mathbb{N}$ such that $n_k I_k \subseteq I_{k+1}$. Now, I_k/I_{k+1} is a finitely generated $\mathbb{Z}_{n_k}[G]$ (left or right)-module, so the set of ideals $\{I_{k+1} \subseteq I \subseteq I_k \mid I_{k+1}I = I_{k+1}\}$ contains a minimal element K. Observe that $I_{k+1}KK = I_{k+1}K = I_{k+1}$, so $K = K^2$. So we have that if the sequence $I_1, I_2, \ldots$ is strictly descending, then there exists a strictly descending sequence $K_1, K_2, \ldots$ of idempotent ideals in $\mathbb{Z}[G]$ such that $\cap_{k\in\mathbb{N}}I_k = \cap_{k\in\mathbb{N}}K_k$.

Lemma 4.2 *Let R be a ring, N a nilpotent ideal, $\pi\colon R \to R/N$ the natural projection and I, K idempotent ideals. If $\pi(I) = \pi(K)$, then $I = K$.*

PROOF. Let $N^k = 0$. If $I + N = K + N$, then $(I + N)^k = (K + N)^k$ and $I = K$ since $I = I^k$ implies $(I + N)^k = I$.

The following lemma is standard (see [2, Proposition 27.1]).

Lemma 4.3 *Let R be a ring and let N be a nil ideal, $\pi : R \to R/N$ be the canonical projection. If e' is an idempotent in R/N, then there is an idempotent $e \in R$ such that $\pi(e) = e'$. Moreover, if there is an element $x \in Z(R)$ such that $\pi(x) = e'$, then e can be chosen in $Z(R)$.*

PROOF. Let $x \in R$ be such that $\pi(x) = e'$. There is a $k \in \mathbb{N}$ such that $(x^2 - x)^k = 0$. The binomial formula gives a $c \in R$ such that $x^k = x^{k+1}c$. Observe that in any case c commutes with x. Then $x^k = x^{2k}c^k$, and therefore $(x^k c^k)^2 = x^{2k}c^k c^k = x^k c^k$. So $x^k c^k$ is an idempotent lifting e'. If $x \in Z(R)$ then $c \in Z(R)$ and $x^k c^k \in Z(R)$, because $Z(R)$ is a subring of R.

Recall that the center of a group ring over a commutative ring is formed by functions constant on all conjugacy classes of the group. Therefore central idempotents can be lifted in a coefficient reduction if the kernel is nil.

Lemma 4.4 *Let G be a finite group and let p be a prime $p \not| |G|$. If $k \in \mathbb{N}$ and I is an idempotent ideal of $R = \mathbb{Z}_{p^k}[G]$. Then there exists a central idempotent $e \in R$ such that $I = Re$.*

PROOF. Let $R' = \mathbb{Z}_p[G]$ and $\pi = \mathbb{Z}_{p^k}[G] \to \mathbb{Z}_p[G]$ be the coefficient reduction. By Maschke's theorem R' is semisimple, so $\pi(I) = e'R'$ for a central idempotent e'. Since Ker π is nilpotent, by Lemma 4.3 there is a central idempotent e such that $\pi(e) = e'$. Put $K = eR$ and observe that $K = K^2$. Then $\pi(I) = \pi(K)$, so $I = K$ by Lemma 4.2.

Lemma 4.5 *Let G, p, k be as above. If K is an idempotent ideal of $R = \mathbb{Z}_{p^{2k}}[G]$, then $p^k K$ is essential in K.*

PROOF. Let x be an element of K. Let e be a central idempotent such that $K = eR$. If $p^k x = 0$, then $ex = x \in K \cap p^k R = p^k K$ and we are done.

Theorem 4.6 *Let G be a finite group. Then the integral group ring over G satisfies (*).*

PROOF. Let $I_1 \supsetneq I_2 \supsetneq \cdots$ be a chain of idempotent ideals in $R = \mathbb{Z}[G]$. We can suppose that for any $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $n_k I_k \subseteq I_{k+1}$.

We claim that for any $k \in \mathbb{N}$ we find $n \in \mathbb{N}$ such that $I_j + nR = I_i + nR$ implies $I_j = I_i$ whenever $1 \leq j, i \leq k$. Obviously, it is enough to find n such that $I_i \cap nR = I_j \cap nR$ for $1 \leq i, j \leq k$. Let $x_1, x_2, \ldots$ be a sequence of positive integers such that $x_i | x_{i+1}$ and for any $y \in \mathbb{N}$ there exists x_l such that $y | x_l$. For given $p, q \in \mathbb{N}$ consider $A_{p,q} = \{r \in R \mid x_p r \in I_q\}$. $A_{p,q}$ are ideals such that $A_{1,q} \subseteq A_{2,q} \subseteq \ldots$ for any $q \in \mathbb{N}$. Since R is noetherian, this sequence terminates in an ideal A independent of q (this is because $n_q I_q \subseteq I_{q+1}$). Observe that for sufficiently large $l \in \mathbb{N}$ (depending on q) $x_l A = x_l R \cap I_q$. This gives the existence of n and the claim is also true for any multiple of n. We can choose n such that $nI_1 \subseteq I_k$.

Now let us consider the coefficient reduction $\varphi\colon R \to \mathbb{Z}_{n^2}[G]$ suppose that $n = p_1^{i_1} \ldots p_l^{i_l}$, $p_1, \ldots, p_l$ being different primes. Thus $\mathbb{Z}_{n^2}[G] \simeq \mathbb{Z}_{p_1^{2i_1}}[G] \times \cdots \times \mathbb{Z}_{p_l^{2i_l}}[G]$. Suppose that $\varphi(I_1), \ldots, \varphi(I_k)$ is a strictly descending chain of idempotent ideals in $\mathbb{Z}_{n^2}[G]$ all of them are containing $n\varphi(I_1)$. Let us have a look at the component given by p_j if $p_j \not| |G|$: We have idempotent ideals in $\mathbb{Z}_{p_j^{2i_j}}[G]$ contained in the image of I_1 and containing $p_j^{i_j}$ times the image of I_1. So in this component, the projections of considered ideals are the same according to Lemma 4.5 and Lemma 4.4. On the other hand, if p_j divides $|G|$, the number of idempotent ideals in $\mathbb{Z}_{p_j^{2i_j}}[G]$ does not exceed number of idempotent ideals in $\mathbb{Z}_{p_j}[G]$. Therefore we have that the number of idempotent ideals in $\mathbb{Z}_{n^2}[G]$ that are between $\varphi(I_1)$ and $n\varphi(I_1)$ is bounded by a number that is independent of n. This contradiction concludes the proof.

The following corollary answers a question of P. Linnell about indecomposable projectives over $\mathbb{Z}[G]$.

Corollary 4.7 *Let G be a finite group. Then all indecomposable projective modules over $\mathbb{Z}[G]$ are finitely generated.*

PROOF. Let P be a countably but not finitely generated projective module over $R = \mathbb{Z}[G]$. Since R is a left and right noetherian ring satisfying (*), P is I-big, where I is a nonzero idempotent ideal. Therefore P contains a decomposable direct summand (the only I-big module of the trace I).

5 One more application

Finally let us have a look at universal enveloping algebras. Let g be a Lie algebra over a field k and let X be a base of g. A *universal enveloping algebra of* g, denoted by $U(\mathsf{g})$, is a factor of a free k-algebra over X with respect to relations $xy - yx = [x, y], x, y \in X$. If g is a nilpotent Lie algebra of finite dimension, then $U(\mathsf{g})$ is a left and right noetherian AR-domain and it follows that all infinitely generated projective modules are free ([9, Lemma 8.6]). The AR-property does not hold for solvable Lie algebras in general, but it appears that the property (*) does. It enable us to prove that infinitely generated projective modules are free over $U(\mathsf{g})$ if g is a solvable Lie algebra of finite dimension and k has characteristic zero. This concludes the proof of [9, Conjecture 8.5] saying that a finite dimensional Lie algebra over a field of characteristic zero is solvable if and only if any (left and right) projective module over $U(\mathsf{g})$ is a direct sum of finitely generated modules.

Lemma 5.1 *Let* k *be an algebraically closed field of characteristic zero,* g *a solvable Lie algebra over* k *of finite dimension,* h *an ideal in* g *having codimension one. Fix an element* $g \in \mathsf{g} \setminus \mathsf{h}$*. Let* S *be a noetherian domain,* D *a derivation on* S*,* $R = S_D[x]$ *a skew polynomial ring given by* $xs - sx = D(s)$*. Suppose that there exists an epimorphism* $\varphi: U(\mathsf{g}) \to R$ *such that* $\varphi(U(\mathsf{h})) \subseteq S$ *and* $\varphi(g) = x$*. If* X, Y *are nonzero ideals in* R *such that* $XY = X$*, then* $Y \cap S \neq 0$*.*

PROOF. By [8, Theorem 1.2.9] R is a left and right noetherian domain so it has a (left and right) quotient field K. Any nonzero element $p \in R$ can be uniquely written as $\sum_{i=0}^{n} a_i x^i, a_i \in S, a_n \neq 0$, as usually we say that n is the degree of p and a_n is the leading coefficient of p. Observe that $p = \sum_{i=0}^{n} x^i a_i', a_i' \in S, a_n = a_n'$. Let $n_0 = \min\{\deg(p) \mid 0 \neq p \in X\}$, $m_0 = \min\{\deg(p) \mid 0 \neq p \in Y\}$. Let $\psi: \mathsf{g} \to \mathrm{End}_{\mathsf{k}}(R)$ be a representation of g given by $\psi(y)(s) = \varphi(y)s - s\varphi(y)$ for any $y \in \mathsf{g}$. Observe that the elements of X of degree n_0 together with 0 form a space V invariant under $\psi(\mathsf{g})$. Since $U(\mathsf{g})$ has a filtration consisting of finite dimensional $\mathrm{ad} - \mathsf{g}$ submodules, R and therefore V have also filtrations consisting of finite dimensional $\psi-$submodules. By [5, Theorem 1.3.12], it follows that there exists an element $0 \neq p_0 \in V$ and a form $\lambda \in \mathsf{g}^*$ such that $\psi(y)(p_0) = \lambda(y)p_0, y \in \mathsf{g}$. So it follows that there exists an element $p_0 \in X$ of degree n_0 such that $Rp_0 = p_0R$. Similarly, there exists an element $q_0 \in Y$ of degree m_0 such that $Rq_0 = q_0R$ (elements of this property are called normal).

Fix $0 \neq s \in S$, $s^{-1} \in K$. Observe that if $p \in R$ and $s_1 s^{-1} \in S$ for some $0 \neq s_1 \in S$, then $\deg(s_1 s^{-1} p) = \deg(p)$, so we can define $\deg(s^{-1}p) = \deg(p)$ for any $0 \neq s \in S, p \in R$. Using the standard induction on $\deg(s^{-1}p)$ one can prove the following: If $0 \neq s \in S$ and $p \in X$, then $s^{-1}p$ can be expressed as a sum of elements of the form $s_i^{-1}s_i'p_0x^i, s_i \neq 0, s_i' \in S, i \in \mathbb{N}_0$. Similarly for $0 \neq s \in S$ and $q \in Y$, qs^{-1} can be expressed as a sum of $x^iq_0s_i's_i^{-1}$. Now, since $p_0 = \sum_{i=1}^n p_iq_i$, $p_i \in X, q_i \in Y$, there are $0 \neq t_1, t_2 \in S$ such that $t_1p_0t_2$ can be written as a sum of elements $sp_0x^iq_0s', s, s' \in S, i \in \mathbb{N}$. Since p_0, q_0 are normal, there exists $r \in R$ such that $t_1p_0t_2 = p_0q_0r$. It follows that $\deg(q_0) = 0$, so $0 \neq q_0 \in Y \cap S$.

Proposition 5.2 *Let g be a finite dimensional solvable Lie algebra over a commutative field k of characteristic zero. If $I_1, I_2, \ldots$ are nonzero ideals in $U(\mathsf{g})$ such that $I_{k+1}I_k = I_{k+1}, k \in \mathbb{N}$, then $I_k = U(\mathsf{g})$ for any $k \in \mathbb{N}$.*

PROOF. Let us prove the proposition for k algebraically closed. We proceed by induction on $\dim \mathsf{g}$, the case $\dim \mathsf{g} = 1$ is trivial. Since g is completely solvable, there exists h, an ideal of g of codimension 1. Fix some $g \in \mathsf{g} \setminus \mathsf{h}$. Let $R = U(\mathsf{g})$, $R' = U(\mathsf{h})$. Recall that R is a free (left or right) R'-module with a free base $B = \{1, g, g^2, \ldots\}$. If I is an ideal of R let $c(I)$ be the smallest ideal in R' such that $I \subseteq c(I)B$. If $XY = X$ for nonzero ideals in R, then $c(X)c(Y) = c(X)$, so, by induction, $c(I_k) = R'$ for any $k \in \mathbb{N}$.

Recall that any prime ideal in R' or R is completely prime. Now we want to prove the following claim: Let P' be a prime of R' such that $[g, p] := gp - pg \in P'$ for any $p \in P'$, and let $P = RP'R$. If $\pi \colon R \to R/P$ is the natural projection, then $\pi(I_k) = R/P$ for any $k \in \mathbb{N}$. Once we prove this, the proposition will follow if we put $P' = 0$. Let us proceed by induction on the maximal length of a chain of primes in R'/P'. (Recall that this length is bounded by $\dim \mathsf{h}$ by [5, Theorem 3.5.12].) First observe that R/P can be identified with a skew polynomial ring $R'/P'_D[x]$, where $D([r']) = [gr' - r'g], [r'] \in R'$. (Realize that the ring homomorphism $R \to R'/P'_D[x]$ sending $\sum_{i=0}^m s_ig^i$ to $\sum_{i=0}^m [s_i]x^i$ factorizes through π via an isomorphism. Also note that we need the extra property of P' to define D correctly.)

Since $c(I_k) = c(I_{k+1}) = R'$, $\pi(I_k), \pi(I_{k+1}) \neq 0$ and we can apply Lemma 5.1 to see $\pi(I_k) \cap R'/P' \neq 0$. So if R'/P' is a simple ring, $\pi(I_k) = R/P$ and this completes the first step of the induction. Let $P' \subsetneq I'$ be an ideal of R' such that $I'/P' = \pi(I_k) \cap R'/P'$. Observe that $(R/P)(I'/P')(R/P) \subseteq \pi(I_k)$. Let $P_1', \ldots, P_l'$ be the minimal primes of I' in R'. Since I' is stable under $[g, -]$, $P_1', \ldots, P_l'$ are also stable under $[g, -]$ according to [5, Lemma 3.3.3]. Let $P_j = RP_j'R$ and let $\pi_j \colon R \to R/P_j$ be the natural projections, the induction gives $\pi_j(I_k) = R/P_j$. Thus $R/P = (\pi(I_k) + \pi(P_1)) \cdots (\pi(I_k) + \pi(P_l)) \subseteq \pi(I_k) + \pi(P_1 \cdots P_l)$. Observe that any element of $\pi(P_1 \cdots P_l)$ is a polynomial having all its coefficients in $P_1' \cdots P_l' + P'/P'$. Since by [8, Theorem 2.3.7] there exists $n \in \mathbb{N}$ such that $(P_1' \cap \cdots \cap P_l')^n \subseteq I'$, $R/P \subseteq (\pi(I_k) + \pi(P_1 \cdots P_l))^n \subseteq \pi(I_k) + \pi(I_k)$, so $\pi(I_k) = R/P$. This completes the induction.

In general, let $\overline{\mathsf{k}}$ be an algebraic closure of k. Consider $\overline{R} = U(\mathsf{g})\otimes\overline{\mathsf{k}} \simeq U(\mathsf{g}\otimes\overline{\mathsf{k}})$ and the ideals $\overline{I_k} = I_k\otimes\overline{\mathsf{k}}$. It is easy to see that $\overline{I_{k+1}} = \overline{I_{k+1}I_k}$. By the preceding step, $\overline{I_k} = \overline{R}$. But this is possible only if $I_k = U(\mathsf{g})$.

Corollary 5.3 *Let* g *be a finite dimensional solvable Lie algebra over a commutative field of characteristic zero. Then*

(i) Any idempotent ideal of $U(\mathsf{g})$ *is trivial.*
(ii) The universal enveloping algebra of g *satisfies (*).*
(iii) Any projective $U(\mathsf{g})$*-module that is not finitely generated is free.*

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