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Permutation representation of 3 and 4-homogenous latin bitrades

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Abstract. Cavenagh, Donovan and Drápal (2005) gave a construction for 3-homogeneous latin bitrades which was proved by Cavenagh (2006) to give every 3-homogeneous latin bitrade. A corollary is that every 3-homogenous latin bitrade may be partitioned into three disjoint transversals. Using a permutation representation for latin bitrades developed by Drápal, we provide a shorter proof of these results, and then give a construction for a family of separated 4-homogeneous latin bitrades.

Keywords: Latin trade; latin bitrade; homogeneous.

1. Introduction

We define a *partial latin square* P to be a subset of $N_1 \times N_2 \times N_3$, where N_1, N_2, and N_3 are finite sets, which satisfies the following condition: given any elements $r \in N_1$, $c \in N_2$ and $e \in N_3$, there is at most one triple $(r, c, e') \in P$, one triple $(r, c', e) \in P$, and one triple $(r', c, e) \in P$. Often

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$N_1 = N_2 = N_3 = \{0, 1, \dots, n-1\}$, although N_1, N_2 and N_3 may not all be the same size. Partial latin squares are commonly represented as partly filled $n \times n$ arrays, with the triple (r, c, e) corresponding to the entry e placed in row r and column c. Based on this representation, we often refer to the elements of N_1, N_2 and N_3 as rows, columns and entries respectively.

A *latin bitrade* is an ordered pair $(T^\circ, T^\star)$ of partial latin squares such that for each $(i, j, k) \in T^\circ$ (respectively, $T^\star$), there exists:

- $i' \neq i$ such that $(i', j, k) \in T^\star$ (respectively, T°),
- $j' \neq j$ such that $(i, j', k) \in T^\star$ (respectively, T°) and
- $k' \neq k$ such that $(i, j, k') \in T^\star$ (respectively, T°).

We say that a partial latin square T° is a *latin trade* if there exists a latin bitrade $(T^\circ, T^\star)$; $T^\star$ is the *disjoint mate* of T° (not necessarily unique) and is itself a latin trade. A latin trade is k-homogeneous if each row and each column contains exactly k entries, and each entry occurs exactly k times. If a latin trade T° is k-homogenous, then its disjoint mate $T^\star$ is also k-homogenous (from the bitrade definition), and so we may also say that the bitrade $(T^\circ, T^\star)$ is k-homogenous. A k-homogeneous latin bitrade has an equal number of rows, columns and entries; this quantity is the *order* of the latin bitrade. This paper is concerned with 3 and 4-homogeneous latin bitrades; see [3], [4], [1] and [2] for recent results.

A latin bitrade $(T^\circ, T^\star)$ is *connected* if there is no latin bitrade $\{U^\circ, U^\star\}$ with $U^\circ \subset T^\circ$ and $U^\star \subset T^\star$.

In the following results, and throughout this paper, the composition of functions is calculated from the left; thus $\tau_1\tau_2\tau_3(x) = \tau_3(\tau_2(\tau_1(x)))$.

Drápal ([5], [6]) has shown that if τ_1, τ_2 and τ_3 are permutations on a finite set X which satisfy the conditions

(P1) a cycle of τ_r has at most one point in common with a cycle of τ_s, whenever $1 \leq r < s \leq 3$, and

(P2) all permutations τ_i, $1 \leq i \leq 3$ are fixed point free and $\tau_1\tau_2\tau_3$ is the identity,

then the permutations τ_1, τ_2 and τ_3 can be used to define two partial latin squares which, together, form a latin bitrade.

Lemma 1.1. Let τ_i, $1 \leq i \leq 3$, be permutations of a finite set X that satisfy (P1) and (P2). For $1 \leq i \leq 3$, denote by A_i the set of cycles of the permutation τ_i. Define $T^\circ, T^\star \subseteq A_1 \times A_2 \times A_3$, where

- $(\rho_1, \rho_2, \rho_3) \in T^\circ$ if and only if all ρ_i, $1 \leq i \leq 3$, are incident with some $x \in X$, and

- $(\rho_1, \rho_2, \rho_3) \in T^\star$ if and only if there exists an element $x \in X$ such that ρ_1 acts on x, ρ_2 acts on $\rho_1(x)$, and ρ_3 acts on $\rho_2(\rho_1(x))$.

Then $(T^\circ, T^\star)$ is a latin bitrade. Furthermore, every latin bitrade can be produced from a latin bitrade of this form by a surjective relabelling of A_1, A_2 and A_3 (so possibly two or more permutation cycles correspond to a single row, column or entry).

If a latin bitrade $(U^\circ, U^\star)$ can be produced from a latin bitrade $(T^\circ, T^\star)$ by consistent bijective relabellings of the rows, columns and entries, we say that $(U^\circ, U^\star)$ and $(T^\circ, T^\star)$ are *isotopic*, and for the purposes of this paper we consider them as essentially the same. Consider three permutations τ_1, τ_2 and τ_3 which satisfy (P1) and (P2), and let $(T^\circ, T^\star)$ be the latin bitrade defined by Lemma 1.1. Then we say that $(T^\circ, T^\star)$ corresponds to the triple (τ_1, τ_2, τ_3) (and vice-versa). By Lemma 1.1, every latin bitrade $(U^\circ, U^\star)$ can be produced from some such $(T^\circ, T^\star)$ by a surjective relabelling of A_1, A_2 and A_3. If the relabelling is bijective, then $(U^\circ, U^\star)$ is isotopic to $(T^\circ, T^\star)$. We say that $(U^\circ, U^\star)$ also corresponds to the triple (τ_1, τ_2, τ_3), since we are not concerned with the particular choice of labels, and we say that any such latin bitrade is *separated*. Thus a separated latin bitrade has precisely one permutation cycle corresponding to each row, column and entry. It follows that in the permutations corresponding to a separated k-homogeneous latin bitrade, all permutation cycles will be of length k.

In the following section we will define three permutations satisfying (P1) and (P2) on the points of a finite integer lattice, with the property that every permutation cycle has length three. These permutations correspond to a separated 3-homogeneous latin bitrade. In fact every 3-homogeneous latin bitrade is separated (since each row, column and entry occurs three times and the permutations cannot contain fixed points, each row, column and entry must correspond to a single permutation cycle). We go on to prove that every 3-homogeneous latin bitrade corresponds to this construction, for a suitable choice of parameters. We will then briefly show how this result proves that every 3-homogenous latin bitrade consists of the union of three transversals (a transversal is a subset of a partial latin square in which each row, column and entry occurs exactly once). This is essentially a shorter proof of the main results in [2], and demonstrates the usefulness of this representation in analysing the structure of latin bitrades.

In Section 3 we define another set of three permutations on the points of the integer lattice, in this case corresponding to a separated 4-homogeneous latin bitrade (four filled cells per row, four filled cells per column and each symbol occurring in precisely four cells). The resulting latin bitrade is similar to the latin bitrade constructed in [3], except that the latin bitrade presented here is separated. Although we provide an infinite class of separated 4-homogenous latin bitrades, we do not prove that every such latin bitrade has this form.

Before starting on the constructions, we define a generalisation of the set $\mathbb{Z}_m \times \mathbb{Z}_n$, which will form the underlying set for both constructions.

Definition: Given $m, n, d \in \mathbb{Z}$ satisfying $m, n > 0$ and $0 \leq d < m$, define $V(m, n, d)$ to be the set of equivalence classes of $\mathbb{Z} \times \mathbb{Z}$ under the equivalence relation $(x+ma, y+nb) \equiv (x+db, y)$, $\forall\ x, y, a, b \in \mathbb{Z}$, and let (x, y) denote the equivalence class (element of $V(m, n, d)$) containing (x, y) (considered as an element of $\mathbb{Z} \times \mathbb{Z}$).

If $d = 0$, then $V(m, n, d)$ is just $\mathbb{Z}_m \times \mathbb{Z}_n$, which we can visualise as an m by n array with horizontal and vertical "wrap around" rules; $(m, y) = (0, y)$ and $(x, n) = (x, 0)$. If $d \neq 0$, then we simply add a horizontal translation to the vertical "wrap around" rule; $(x, n) = (x + d, 0)$ (this creates a dependency between the two components which does not exist in $\mathbb{Z}_m \times \mathbb{Z}_n$).

2. 3-homogenous latin bitrades

We will define three permutations which correspond (by Lemma 1.1) to a 3-homogenous latin bitrade, and then show that every 3-homogenous latin bitrade (up to isotopism) is generated by this construction.

Definition: Let m, n be positive integers, and let $0 \leq d \leq 3m-1$, such that $d \equiv n \pmod 3$. For all $(x, y) \in V(m, n, (d-n)/3)$, define the following permutations on $V(3m, n, d)$:

$$\begin{aligned} \sigma_1[x,y] &= ((3x-y,y)\ (3x-y+1,y)\ (3x-y+1,y+1)), \\ \sigma_2[x,y] &= ((3x-y+2,y)\ (3x-y+3,y)\ (3x-y+3,y+1)), \\ \sigma_3[x,y] &= ((3x-y+1,y)\ (3x-y+2,y)\ (3x-y+2,y+1)). \end{aligned}$$

Note that by the definition of the underlying set $V(3m, n, d)$, $\sigma_i[x+ma, y+nb] = \sigma_i[x+b(d-n)/3, y]$; thus this is a coherent definition of $\sigma_i[x, y]$ for $(x, y) \in V(m, n, (d-n)/3)$. Let τ_i be the composition (in any order) of the mn cycle permutations $\{\sigma_i[x, y] \mid (x, y) \in V(m, n, (d-n)/3)\}$. We define $P(m, n, d)$ to be the ordered triple (τ_1, τ_2, τ_3).

Lemma 2.1. Let m, n be positive integers and let $0 \leq d \leq 3m-1$, such that $d \equiv n \pmod 3$. Suppose further that if $n = 1$, then $d \neq 1, 3m-2, (3m-1)/2$, and hence $m \geq 4$; while if $n = 2$, then $d \neq 3m-1$ and hence $m \geq 2$. Then $P(m, n, d)$ satisfies the conditions of Lemma 1.1, and the corresponding latin bitrade is 3-homogenous of size mn.

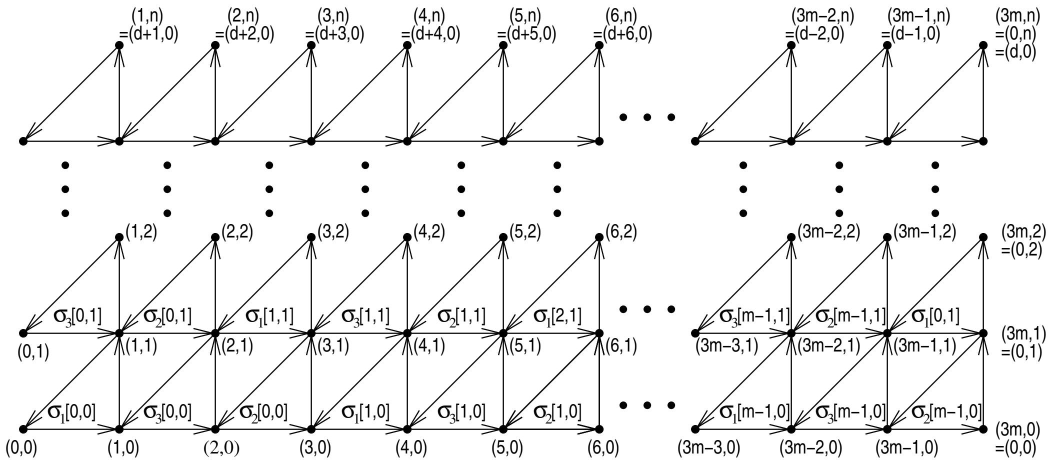

Figure 1. A graphical representation of $P(m, n, d)$.

Proof. Firstly, note that for a point $(x, y) \in V(3m, n, d)$, the congruence class of $x + y$ modulo 3 is well defined (it is preserved by the equivalence $(x + 3ma, y + nb) \equiv (x + db, y)$, since $d \equiv n \pmod 3$). Thus we can consider the points $(x, y) \in V(3m, n, d)$ in 3 cases based on $x + y \pmod 3$; that is, every point in $V(3m, n, d)$ has the form $(3x-y, y)$, $(3x-y+1, y)$, or $(3x-y+2, y)$. Fixing $k \in \{1, 2, 3\}$, and considering the points in these three cases, we see that every point occurs in exactly one cycle $\sigma_k[x, y]$.

It follows that, for each k, the definition of τ_k above is unambiguous (since the permutation cycles are disjoint, the order of composition is irrelevent); τ_k will be a permutation on $V(3m, n, d)$ which is fixed point free, and the set of permutation cyles of τ_k will be $\{\sigma_k[x, y] \mid 0 \leq x \leq m-1,\ 0 \leq y \leq n-1\}$. Again considering the points (x, y) in three cases based on $x + y$ (mod 3), we can verify that $\tau_1\tau_2\tau_3$ is the identity. Thus (P2) holds.

We now check that (P1) holds, by comparing arbitrary cycles of τ_1 and τ_2 and checking that there is at most one common point (the proofs of the other cases are very similar). Without loss of generality, let the cycles be $\sigma_1[0, 0]$ and $\sigma_2[x, y]$. We have $\sigma_1[0, 0] = ((0, 0)\ (1, 0)\ (1, 1))$ and $\sigma_2[x, y] = ((3x - y + 2, y)\ (3x - y + 3, y)\ (3x - y + 3, y + 1))$. If these cycles have any points in common, then either $y = 0$, $y = 1$ or $y = -1$.

Firsly consider the case $y = 0$, so $\sigma_2[x, y] = ((3x + 2, 0)\ (3x + 3, 0)\ (3x + 3, 1))$. Provided that $n \geq 2$, the only possible common point is $(3x+3, 0) = (0, 0)$, so we are done. But if $n = 1$ then $(\alpha, 1) = (\alpha+d, 0)$, so $\sigma_1[0, 0] = ((0, 0)\ (1, 0)\ (1+d, 0))$ and $\sigma_2[x, y] = ((3x+2, 0)\ (3x+3, 0)\ (3x+3+d, 0))$, where $d \equiv 1$ (mod 3). Given the congruences modulo 3, there are exactly three ways in which these two cycles could share two points; $(0, 0) = (3x+3, 0)$ and $(1, 0) = (3x+3+d, 0)$, $(0, 0) = (3x+3, 0)$ and $(1 + d, 0) = (3x + 2, 0)$, or $(1, 0) = (3x + 3 + d, 0)$ and $(1 + d, 0) = (3x + 2, 0)$. The first two cases correspond to $x = m - 1$ and $d = 1, 3m - 2$. The third case is more difficult; we have $3x + 3 + d \equiv 1$ (mod $3m$) and $3x+2 \equiv d+1$ (mod $3m$), giving $2d+1 \equiv 0$ (mod $3m$) and $6x+3 \equiv 0$ (mod $3m$). Since $0 \leq d \leq 3m-1$ and $0 \leq x \leq m-1$, the only possibility is $2d+1 = 3m$ and $6x+3 = 3m$. Hence the third case corresponds to $x = (m - 1)/2$ and $d = (3m - 1)/2$ (clearly this can only occur when m is odd). Since these values of d are excluded when $n = 1$, we are done.

Now consider $y = \pm 1$. If $n \geq 3$ then we can see that the cycles share at most one point by just considering the second coordinate. If $n = 1$ then $y = \pm 1$ reduces to the case $y = 0$. This leaves the case $n = 2$ and $y = 1$, since $y = 1$ and $y = -1$ are equivalent when $n = 2$. Our cycles are $\sigma_1[0, 0] = ((0, 0)\ (1, 0)\ (1, 1))$ and $\sigma_2[x, y] = ((3x + 1, 1)\ (3x + 2, 1)\ (3x + 2 + d, 0))$, where $d \equiv 2$ (mod 3). These cycles share two points only in the case $x = 0$ and $d = 3m - 1$; since this case is excluded when $n = 2$, we have proved (P1).

Finally, note that every permutation cycle in τ_1, τ_2 and τ_3 is of length three. It follows from the construction in Lemma 1.1 that the corresponding latin bitrade is 3-homogeneous.

Let τ_1, τ_2 and τ_3 be permutations on a finite set X which satisfy (P1) and (P2). In the proof of the following lemma, we represent this permutation structure as a digraph, with three classes of edge. The vertices of the digraph correspond to the elements of X, and for each $x \in X$ and $k \in \{1, 2, 3\}$ we have an edge $(x, \tau_k(x))$ (directed from x to $\tau_k(x)$). We say this edge is of type k, but we will usually refer to an edge of type 1, 2 or 3 as a row, column or entry edge respectively. The permutation cycles of τ_k are represented by (directed) cycles in the digraph, in which all edges are of type k. We refer to a directed cycle in which all edges are of type k as a cycle of type k; as with edges, we will generally refer to a cycle of type 1, 2 or 3 as a row, column or entry cycle respectively. All cycles and circuits are assumed to be directed (note that a circuit may repeat vertices but not edges, while a cycle may not repeat edges or vertices, except that the first and last vertices are the same). We refer to a cycle (circuit) as an RCE cycle (circuit) if the edges are successively of type 1, 2 and 3 (following the edge direction around the cycle or circuit); we refer to it as an REC cycle (circuit) if the edges are successively of type 1, 3 and 2. The condition that τ_1, τ_2 and τ_3 are permutations, together with the conditions (P1) and (P2), is equivalent to the following:

(D1) Each vertex of X occurs in exactly one row cycle, one column cycle and one entry cycle. That is, there is exactly one edge of each type directed into each vertex and one edge of each type directed out of each vertex.

(D2) There are no loops.

(D3) Two distinct row, column or entry cycles have at most one vertex in common.

(D4) All RCE cycles are of length three.

In addition to these conditions, if the permutations correspond to a separated k-homogenous latin bitrade, then all the row, column and entry cycles will be of length k; thus we have:

(D5) All row cycles, column cycles and entry cycles are of length three (3-homogenous case only).

Note that by (D1), there is exactly one RCE and one REC circuit containing a given edge. If the given edge is (a, b), there will be exactly one edge (b, c) of the required type (for example, if (a, b) is a column edge and we are interested in the REC circuit, the next edge must be a row edge). Continuing along the directed circuit, we will eventually repeat an edge, thus completing the circuit. Since two edges must be of the same type to match, the length of every REC circuit is divisible by 3 (by (D3), we know the RCE circuits are of length 3, and are thus cycles).

In the following lemma we will use these properties to deduce the digraph structure when the original latin bitrade is 3-homogenous.

Lemma 2.2. Every connected 3-homogenous latin bitrade is equivalent to some $P(m, n, d)$ (with the equivalence given by Lemma 1.1).

Proof. Consider an arbitrary connected 3-homogenous latin bitrade, and let τ_1, τ_2 and τ_3 be the corresponding permutations. We consider the triple (τ_1, τ_2, τ_3) as a digraph, as explained above. This digraph must satisfy (D1) - (D5), and we use these conditions to show that it has the form of $P(m, n, d)$ for some m, n and d. Consider an arbitrary REC circuit; say that it has length $3m$. Label this (directed) circuit $(v_{0,0}, v_{1,0}, \ldots, v_{3m-1,0})$, where $(v_{0,0}, v_{1,0})$ is a row edge, $(v_{1,0}, v_{2,0})$ is an entry edge, $(v_{2,0}, v_{3,0})$ is a column edge, and so on. The first component of the subscript is taken modulo $3m$, so $v_{3m,0} = v_{0,0}$. By (D5), all row, column and entry cycles are of length 3. For each edge $(v_{x,0}, v_{x+1,0})$ in the REC circuit ($x \in \mathbb{Z}_{3m}$), let $v_{x+1,1}$ be the third vertex in the row, column or edge cycle containing $(v_{x,0}, v_{x+1,0})$; see Figure 2. Note that the $6m$ points $\{v_{x,y} \mid x \in \mathbb{Z}_{3m},\ y \in \{0, 1\}\}$ are not necessarily distinct.

For each $\alpha \in \{0, 1, \ldots, m-1\}$, we have a row edge $(v_{3\alpha+1,1}, v_{3\alpha,0})$ and a column edge $(v_{3\alpha,0}, v_{3\alpha,1})$. By (D4), it follows that there exists an entry edge $(v_{3\alpha,1}, v_{3\alpha+1,1})$. Similarly there is a column edge $(v_{3\alpha+1,1}, v_{3\alpha+2,1})$ and a row edge $(v_{3\alpha+2,1}, v_{3\alpha+3,1})$. Thus we have deduced the existence of an REC circuit $(v_{0,1}, v_{1,1}, \ldots, v_{3m-1,1})$, except that we need to show that no edge is repeated. To prove this, assume that $(v_{3\alpha+\gamma,1}, v_{3\alpha+\gamma+1,1}) = (v_{3\beta+\gamma,1}, v_{3\beta+\gamma+1,1})$, say, where $\gamma \in \{0, 1, 2\}$ (the difference

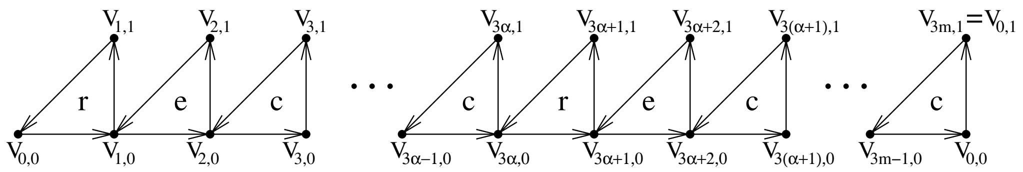

Figure 2. An arbitrary REC circuit with associated row, column and entry cycles.

in position must be a multiple of 3 for the edge type to match). The row, column or entry cycle $(v_{3\alpha+\gamma+1,1}, v_{3\alpha+\gamma,0}, v_{3\alpha+\gamma+1,0})$ must be of the same type as $(v_{3\beta+\gamma+1,1}, v_{3\beta+\gamma,0}, v_{3\beta+\gamma+1,0})$, and thus is identical (by (D1) and the fact that $v_{3\alpha+\gamma+1,1} = v_{3\beta+\gamma+1,1}$). This implies that there is a repeated edge in the original circuit $(v_{0,0}, v_{1,0}, \dots, v_{3m-1,0})$, so we have the required contradiction. If the two REC circuits $(v_{0,0}, v_{1,0}, \dots, v_{3m-1,0})$ and $(v_{0,1}, v_{1,1}, \dots, v_{3m-1,1})$ have a common edge, then by (D1) they must be identical (with some translation, so $v_{x,1} = v_{x+d,0}$). Otherwise, the two circuits are edge disjoint, and we repeat the process with the new circuit $(v_{0,1}, v_{1,1}, \dots, v_{3m-1,1})$, and deduce a third REC circuit $(v_{0,2}, v_{1,2}, \dots, v_{3m-1,2})$. For each REC circuit $(v_{0,\beta}, v_{1,\beta}, \dots, v_{3m-1,\beta})$ deduced in this way, we have associated row, column and entry cycles $\{(\alpha,\beta),(\alpha+1,\beta),(\alpha+1,\beta+1) \mid \alpha \in \mathbb{Z}_{3m}\}$. Inductively, we see that $(\alpha,\beta),(\alpha+1,\beta),(\alpha+1,\beta+1)$ is respectively a row, entry or column cycle when $\alpha+\beta$ is respectively congruent to 0, 1 or 2 (mod 3). We continue this process until we have a repeated circuit. Let n be the number of distinct REC circuits obtained in this way; the $(n+1)$th circuit, $(v_{0,n}, v_{1,n}, \dots, v_{3m-1,n})$, must be some translation of $(v_{0,0}, v_{1,0}, \dots, v_{3m-1,0})$. Say that $v_{x,n} = v_{x+d,0}$. The first component of the subscript is taken modulo $3m$, so we have the general equivalence $v_{x+3ma,y+nb} = v_{x+db,y}$; thus we can regard $v_{x,y}$ as being defined for $(x,y) \in V(3m,n,d)$. The row, column and entry cycles deduced correspond to the permutation cycles of σ_1, σ_2 and σ_3 respectively (the reader may wish to refer to Figure 1). By (D1), there are no further row, column or entry cycles that contain any of the points $\{v_{x,y} \mid (x,y) \in V(3m,n,d)\}$. Since the latin bitrade is assumed to be connected, the digraph has no other components (components of the digraph must correspond to sub-bitrades), and so we have accounted for all vertices and edges in the digraph. Thus we have shown that (τ_1,τ_2,τ_3) has the structure of $P(m,n,d)$, with one caveat: although by definition the edges $\{(v_{x,y}, v_{x+1,y}) \mid (x,y) \in V(3m,n,d)\}$ are all distinct, we have not shown that the points $\{v_{x,y} \mid (x,y) \in V(3m,n,d)\}$ are all distinct (this is why we denote the points by $v_{x,y}$ rather than just (x,y)). We now complete the proof by proving that these points are in fact distinct.

Suppose that two points with different labels are equal (we seek a contradiction). Say $v_{x,y} = v_{i,j}$, where (x,y) and (i,j) are distinct members of $V(3m,n,d)$. There are exactly three edges directed into each vertex and three edges directed out of each vertex (by (D1)), and each edge is contained in exactly one REC circuit, so there are three REC circuits passing through each vertex. From the structure deduced above, the three REC circuits containing (i,j) are $(v_{0,j}, v_{1,j}, \dots, v_{3m-1,j})$, $(\dots, v_{i,j-1}, v_{i,j}, v_{i,j+1}, \dots)$, and $(\dots v_{i+1,j+1}, v_{i,j}, v_{i-1,j-1}, \dots)$. Since $v_{x,y} = v_{i,j}$, one of these circuits must be equal to $(v_{0,y}, v_{1,y}, \dots, v_{3m-1,y})$. We cannot have $(v_{0,y}, v_{1,y}, \dots, v_{3m-1,y}) = (v_{0,j}, v_{1,j}, \dots, v_{3m-1,j})$. The proofs in the other two case are essentially the same, so assume that $(v_{0,y}, v_{1,y}, \dots, v_{3m-1,y}) = (\dots, v_{i,j-1}, v_{i,j}, v_{i,j+1}, \dots)$. It follows that the edge $(v_{i,y}, v_{i,y+1})$ is equal to one of the edges of the circuit $(v_{0,y}, v_{1,y}, \dots, v_{3m-1,y})$; say $(v_{i,y}, v_{i,y+1}) = (v_{w,y}, v_{w+1,y})$. Assume without loss of generality that this is a row edge, and $w = y = 0$. Thus $(v_{i,0}, v_{i,1}) = (v_{0,0}, v_{1,0})$, and $i \equiv 1 \pmod 3$. The entry cycles $(v_{i,1}, v_{i,2}, v_{i-1,1})$ and $(v_{1,0}, v_{2,0}, v_{2,1})$ must be equal (since $v_{i,1} = v_{1,0}$), therefore $v_{i-1,1} = v_{2,1}$. It follows that the row edges $(v_{i-1,1}, v_{i-1,2})$ and $(v_{2,1}, v_{3,1})$ are equal. Relabelling rows and columns, we obtain $(v_{i-3,0}, v_{i-3,1}) = (v_{0,0}, v_{1,0})$. Since $i \equiv 1 \pmod 3$, repeating this process eventually gives $(v_{1,0}, v_{1,1}) = (v_{0,0}, v_{1,0})$, and hence $v_{0,0} = v_{1,0} = v_{1,1}$, contradicting (D2).

We can translate $P(m,n,d)$ into standard latin bitrade form by using the equivalence in Lemma 1.1. For brevity, we let $r = (d-n)/3 \pmod m$, and relabel $\sigma_i[x,y]$ as x_y (i is redundant in the ordered triple).

Corollary 2.1. Let m, n be positive integers, and let $0 \le r \le m-1$. Define T° and $T^\star$ as follows:

$$\begin{aligned} T^\circ &= \{(x_y, (x-1)_y, (x-1)_{y-1}), (x_y, (x-1)_{y-1}, x_y), \\ &\quad (x_y, x_{y+1}, x_{y+1}) \mid (x,y) \in V(m,n,r)\}, \\ T^\star &= \{(x_y, (x-1)_{y-1}, (x-1)_{y-1}), (x_y, x_{y+1}, x_y), \\ &\quad (x_y, (x-1)_y, x_{y+1}) \mid (x,y) \in V(m,n,r)\}. \end{aligned}$$

Then $(T^\circ, T^\star)$ is a connected 3-homogeneous latin bitrade, and furthermore every connected 3-homogeneous latin bitrade is isotopic to this construction for some choice of m, n, r.

The latin trade T° defined in Corollary 2.1 can be partitioned into the transversals $\{(x_y, (x-1)_y, (x-1)_{y-1}) \mid (x,y) \in V(m,n,r)\}$, $\{(x_y, (x-1)_{y-1}, x_y) \mid (x,y) \in V(m,n,r)\}$ and $\{(x_y, x_{y+1}, x_{y+1}) \mid (x,y) \in V(m,n,r)\}$. Thus we have the following result, which is also given in [2].

Corollary 2.2. Any 3-homogeneous latin trade can be partitioned into three transversals.

3. A construction for separated 4-homogenous latin bitrades

The following construction uses the set $V(m,n,d) \times \mathbb{Z}_2$. For brevity, we write $((x,y),i) = (x,y,i)$, for each $((x,y),i) \in V(m,n,d) \times \mathbb{Z}_2$.

Definition: Let m, n be positive integers, and let $0 \le d \le m-1$. For each $(x,y,i) \in V(m,n,d) \times \mathbb{Z}_2$, we define the following permutation cycles on $V(2m,2n,2d) \times \mathbb{Z}_2$:

$$\begin{aligned} \sigma_1[x,y,i] &= ((2x,2y,i)\ (2x,2y+1,i)\ (2x+1,2y+1,i)\ (2x+1,2y,i)), \\ \sigma_2[x,y,i] &= ((2x,2y,i)\ (2x,2y-1,i)\ (2x-1,2y-1,1-i)\ (2x-1,2y,i)), \\ \sigma_3[x,y,0] &= ((2x,2y,0)\ (2x-1,2y+1,0)\ (2x,2y,1)\ (2x-1,2y+1,1)), \\ \sigma_3[x,y,1] &= ((2x,2y-1,0)\ (2x+1,2y,0)\ (2x,2y-1,1)\ (2x+1,2y,1)). \end{aligned}$$

Note that by the definition of $V(2m,2n,2d) \times \mathbb{Z}_2$, for $k \in \{1,2,3\}$ we have $\sigma_k[x+ma, y+nb, i] = \sigma_k[x+db, y, i]$, so σ_k is well-defined on $V(m,n,d) \times \mathbb{Z}_2$.

Each point in the set $V(2m,2n,2d) \times \mathbb{Z}_2$ can be expressed in the form $(2x+a, 2y+b, c)$ for a unique choice of $a, b, c \in \{0,1\}$. Thus we can partition the points of $V(2m,2n,2d) \times \mathbb{Z}_2$ into 8 classes based on this representation. Fixing $k \in \{1, 2, 3\}$, and considering the points in each of these 8 cases, we see that each point in $V(2m, 2n, 2d) \times \mathbb{Z}_2$ occurs in exactly one permutation cycle $\sigma_k[x, y, i]$. Let $A_k = \{\sigma_k[x, y, i] \mid (x, y, i) \in V(m, n, d) \times \mathbb{Z}_2\}$, and define the permutation τ_k to be the composition of the cycle permutations in A_k. Since these cycles are disjoint, the order of composition doesn't matter, and since every point occurs in one cycle of A_k, τ_k will have no fixed points (in fact the set of permutation cycles of τ_k will simply be A_k). Let $Q(m, n, d)$ be the triple (τ_1, τ_2, τ_3).

We represent the permutations graphically in Figure 3. The underlying set $V(2m, 2n, 2d) \times \mathbb{Z}_2$ may be thought of as two "layers", each of which is a copy of $V(2m, 2n, 2d)$. Exploiting the symmetry between the two layers, we represent the points $(a, b, 0)$ and $(a, b, 1)$ with a single vertex labelled (a, b, i), and use a dotted line to indicate an edge between the layers. Thus a solid edge from (a, b, i) to (c, d, i) in Figure 3 represents edges from $(a, b, 0)$ to $(c, d, 0)$ or $(a, b, 1)$ to $(c, d, 1)$, while a dotted edge from (a, b, i) to (c, d, i) represents edges from $(a, b, 0)$ to $(c, d, 1)$ or $(a, b, 1)$ to $(c, d, 0)$.

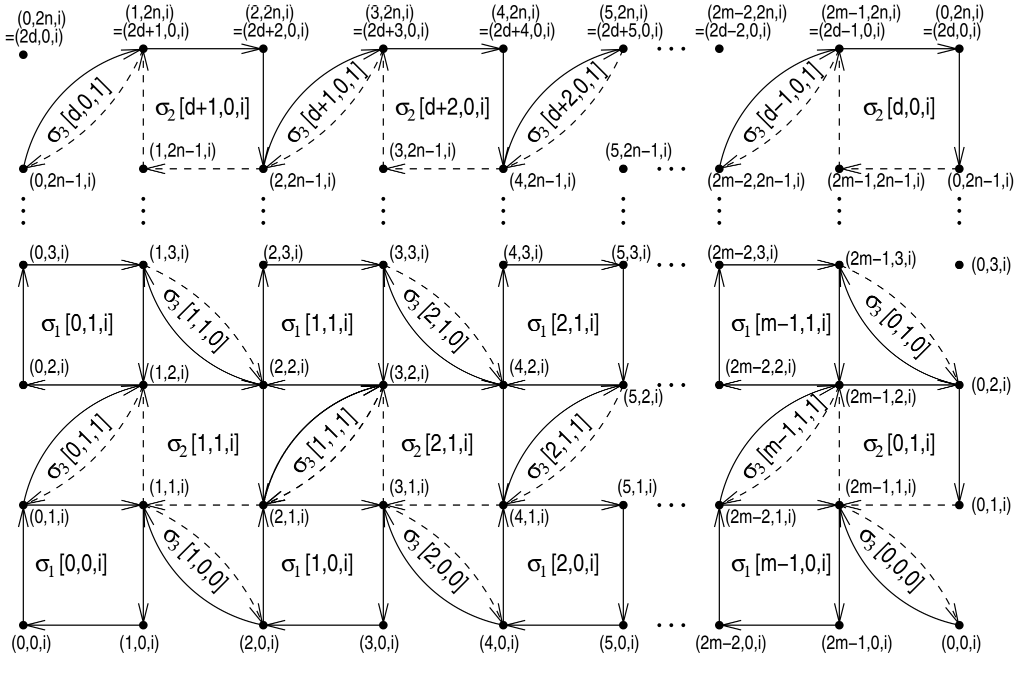

Figure 3. Q(m,n,d)

The action of these permutations on each point is described in Table 1. In Table 2, we list the three cycles (from A_1, A_2 and A_3 respectively) which intersect a given point. These cycles are unique since the cycles of A_k are disjoint, for each $k \in \{1, 2, 3\}$.

If two cycles intersect in a given point, then they both appear in the corresponding row of Table 2. In order to check that (P1) holds for the permutations τ_1, τ_2, τ_3, we need to verify that no pair of cycles can appear on the same row of Table 2 more than once (alternatively, the reader may wish to verify (P1) using Figure 3). For instance, if $\sigma_2[x, y, i] = \sigma_2[x + 1, y, i]$, then the cycles $\sigma_1[x, y, i]$ and $\sigma_2[x, y, i]$ intersect in the points $(2x, 2y, i)$, from row 1, and $(2x + 1, 2y, i)$, from row 3, thus contradicting (P1).

	$\tau_1((a,b,i))$	$\tau_2((a,b,i))$	$\tau_3((a,b,i))$
a even, b even	$(a,b+1,i)$	$(a,b-1,i)$	$(a-1,b+1,i)$
a even, b odd	$(a+1,b,i)$	$(a-1,b,1-i)$	$(a+1,b+1,i)$
a odd, b even	$(a-1,b,i)$	$(a+1,b,i)$	$(a-1,b-1,1-i)$
a odd, b odd	$(a,b-1,i)$	$(a,b+1,1-i)$	$(a+1,b-1,1-i)$

Table 1. The action of τ_1, τ_2 and τ_3 on a point in $V(2m,2n,2d)\times\mathbb{Z}_2$.

(a,b,i)	Cycles which act on (a,b,i)		
$(2x,2y,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x,y,i]$,	$\sigma_3[x,y,0]$
$(2x,2y+1,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x,y+1,i]$,	$\sigma_3[x,y+1,1]$
$(2x+1,2y,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x+1,y,i]$,	$\sigma_3[x,y,1]$
$(2x+1,2y+1,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x+1,y+1,1-i]$,	$\sigma_3[x+1,y,0]$

Table 2. The permutation cycles intersecting a point in $V(2m,2n,2d)\times\mathbb{Z}_2$.

In fact this occurs whenever $m=1$, so if (P1) holds then $m\geq 2$. On the other hand, if $m\geq 2$ and $n\geq 2$, then it is easily verified from Table 2 that (P1) holds. If $m=2$ and $n=1$, then by considering the various ways in which two cycles could intersect in two points, we see that (P1) holds if and only if $d\neq 0,1,m-1$.

Table 3 is similar to Table 2, but instead of listing the three cycles which act on the given point (a,b,i), we give three cycles which act consecutively on the point or the image of the point; that is, we give $\rho_1\in A_1$, $\rho_2\in A_2$ and $\rho_3\in A_3$ such that ρ_1 acts on (a,b,i), ρ_2 acts on $\rho_1((a,b,i))$, and ρ_3 acts on $\rho_2(\rho_1((a,b,i)))=(\rho_1\rho_2)((a,b,i))$. In fact, in each case we have $(\rho_1\rho_2\rho_3)((a,b,i))=(a,b,i)$, which implies that $(\tau_1\tau_2\tau_3)((a,b,i))=(a,b,i)$, and hence that $\tau_1\tau_2\tau_3$ is the identity permutation.

(a,b,i)	Cycles which act consecutively on (a,b,i)		
$(2x,2y,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x,y+1,i]$,	$\sigma_3[x,y,0]$
$(2x,2y+1,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x+1,y+1,1-i]$,	$\sigma_3[x,y+1,1]$
$(2x+1,2y,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x,y,i]$,	$\sigma_3[x,y,1]$
$(2x+1,2y+1,i)$	$\sigma_1[x,y,i]$,	$\sigma_2[x+1,y,i]$,	$\sigma_3[x+1,y,0]$

Table 3. The permutation cycles acting consecutively on a point in $V(2m,2n,2d)\times\mathbb{Z}_2$.

Since τ_1, τ_2 and τ_3 are fixed point free and $\tau_1\tau_2\tau_3$ is the identity permutation, (P2) holds. Thus we have the following lemma:

Lemma 3.1. Let m,n,d be integers satisfying $m\geq 2$, $n\geq 1$, $0\leq d\leq m-1$, and $d\neq 0,1,m-1$

if $n = 1$ (which implies that $m \geq 4$ if $n = 1$). Then $Q(m, n, d)$ corresponds to a connected, separated 4-homogeneous latin bitrade with $2mn$ rows, $2mn$ columns and $2mn$ entries.

The corresponding latin bitrade $(T^{\circ}, T^{\star})$ is defined by Lemma 1.1. The triples of T° and $T^{\star}$ can be read off Tables 2 and 3 respectively, allowing (x, y, i) to take any value in $V(m, n, d) \times \mathbb{Z}_2$. The trades produced by this construction have a more complicated representation than necessary, but this can easily be simplified by relabelling the row, column and entry cycles.

Example: We give the 4-homogenous latin bitrade construction with $m = 3$ and $n = 2$; this produces a latin bitrade of order 12. We label $\sigma_k[x, y, i]$ by the integer $2nx + 2y + i$, and then use a and b to represent 10 and 11. Note that we now use the same set of labels for the rows, columns and entries, but these are distinguished by context.

$\circ$	0	1	2	3	4	5	6	7	8	9	a	b
0	0		3		1			4				
1		0		3		1	4					
2	1		2			6	3					
3		1		2	6			3				
4					4		7		5			8
5						4		7		5	8	
6					5		6			a	7	
7						5		6	a			7
8	9			0					8		b	
9		9	0							8		b
a		2	b						9		a	
b	2			b						9		a

$\star$	0	1	2	3	4	5	6	7	8	9	a	b
0	1		0		4			3				
1		1		0		4	3					
2	2		3			1	6					
3		2		3	1			6				
4					5		4		8			7
5						5		4		8	7	
6					6		7			5	a	
7						6		7	5			a
8	0			b					9		8	
9		0	b							9		8
a		9	2						a		b	
b	9			2						a		b

3.1. A variation on the 4-homogenous construction

The permutations comprising $Q(m, n, d)$ are defined on the underlying set $V(2m, 2n, 2d) \times \mathbb{Z}_2$. This set has the equality $(2m, y, i) = (0, y, i)$. If we replace this with the equality $(2m, y, i) = (0, y, 1 - i)$, the two layers of the set are joined into one, although the structure of the construction is otherwise the same. Since this variation is very similar to $Q(m, n, d)$, we just briefly write down the permutations without going into any details.

Definition: Let m, n be positive integers, and let $0 \leq d \leq 2m - 1$. For each $(x, y) \in V(2m, n, d)$, we define the following permutation cycles on $V(4m, 2n, 2d)$:

$$\begin{aligned}
\rho_1[x,y] &= ((2x,2y)\ (2x,2y+1)\ (2x+1,2y+1)\ (2x+1,2y)), \\
\rho_2[x,y] &= ((2x,2y)\ (2x,2y-1)\ (2x-1+2m,2y-1)\ (2x-1,2y)), \\
\rho_3[x,y] &= \begin{cases} ((2x,2y)\ (2x-1,2y+1)\ (2x+2m,2y)\ (2x-1+2m,2y+1)), \\ \qquad \text{if } 0 \leq x \leq 2m-1, \\ ((2x,2y-1)\ (2x+1,2y)\ (2x+2m,2y-1)\ (2x+1+2m,2y)), \\ \qquad \text{if } 2m \leq x \leq 4m-1. \end{cases}
\end{aligned}$$

This system of permutations will satisfy the conditions of Lemma 1.1 provided that $m \geq 2$, $n \geq 1$, and if $n = 1$ then $d \neq 0, 1, 2m-1$. We do not prove this, but it can be verified by writing out the latin bitrade that the construction is valid for $(m,n,d) = (2,1,2), (3,1,2)$. Thus this variation gives latin bitrades of orders 4 and 6, which the original construction $Q(m,n,d)$ does not; with these examples and $Q(m,n,d)$, we can construct connected separated 4-homogenous latin bitrades of every even order greater than 2.

4. A generalisation to infinite latin bitrades

A partial latin square may be regarded as a finite partial quasigroup. By removing the requirement that the number of rows, columns and entries is finite, the definition of a latin bitrade may be immediately generalised to the infinite case (so the generalised latin bitrade will consist of a pair of partial quasigroups rather than partial latin squares). Lemma 1.1 and the definitions of k-homogenous and separated latin bitrades may also be immediately generalised.

Our constructions for separated 3 and 4-homogeneous latin bitrades may be generalised by allowing m and n to take the value ∞. We generalise $V(m,n,d)$ in the natural way: $V(\infty,n,d)$ is the set of equivalence classes of $\mathbb{Z} \times \mathbb{Z}$ under the equivalence relation $(x, y+nb) \equiv (x+db, y)$, $V(m,\infty,d)$ is the set of equivalence classes of $\mathbb{Z} \times \mathbb{Z}$ under the equivalence relation $(x+ma, y) \equiv (x,y)$, while $V(\infty,\infty,d)$ is just $\mathbb{Z} \times \mathbb{Z}$ (clearly d is redundant when $n = \infty$). In this way the constructions in this paper give an infinite family of generalised separated 3 and 4-homogeneous latin bitrades. In fact the reasoning in the proof of Lemma 2.2 holds in this generalisation to the infinite case, and so every infinite 3-homogeneous latin bitrade must have the form $P(\infty,n,d)$, $P(m,\infty,d)$, or $P(\infty,\infty,d)$.

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