INTERVALS IN SUBGROUP LATTICES OF COUNTABLE LOCALLY FINITE GROUPS

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Abstract. We prove that every algebraic lattice with at most countably many compact elements is isomorphic to an interval in the subgroup lattice of a countable locally finite group.

Introduction

The problem of representing finite lattices as congruence lattices of finite algebras is one of the most challenging open problems in the area of algebraic representations of lattices. By a well-known and rather surprising result of P. P. Pálfy and P. Pudlák [6], the problem is equivalent to the problem of representing finite lattices as intervals in subgroup lattices of finite groups. More precisely, the following two problems are equivalent.

CFA. *Is every finite lattice isomorphic to the congruence lattice of a finite algebra?*

ISG. *Is every finite lattice isomorphic to an interval in the subgroup lattice of a finite group?*

The infinite versions of both problems had been solved a long time ago. A celebrated theorem by G. Grätzer and E. T. Schmidt [3] says that a lattice is isomorphic to the congruence lattice of a finite algebra if and only if it is algebraic (i.e. complete and compactly generated). For a simple proof see [7]. The second author proved in [10] that algebraicity of a lattice is also equivalent to being isomorphic to an interval in the subgroup lattice of a group. An alternative proof of this result was proposed by the first author in [8].

Although the two problems stated above are equivalent for the class of all finite lattices, they are not equivalent for each particular finite lattice. But there are many finite lattices for which being isomorphic to the congruence lattice of a finite algebra is equivalent to being isomorphic to an interval in the subgroup lattice of a finite group. Typical examples are lattices M_n of length 2 (lattices with a least element, a greatest element and n mutually non-comparable elements in between) that have become a test case for both versions of the problem. For a prime power q,

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the congruence lattice of the collineation group of the two-dimensional affine space over the q-element field is isomorphic to M_{q+1}. Since all the unary operations are permutations, the congruence lattice is isomorphic to the interval between the stabilizer of a point and the whole affine group. For some time the lattices M_{q+1} for a prime power q were the only lattices of length 2 known to be representable as congruence lattices of finite algebras or as intervals in subgroup lattices of finite groups. Later, A. Lucchini [5] proved that for any prime power q the lattices M_{q+2} are also representable as intervals in subgroup lattices of finite groups. In the same paper he also presented another infinite set of lattices M_n that can be found as intervals in subgroup lattices of finite groups, namely those with $n = \frac{q^t+1}{q+1} + 1$, where q is a prime power and t is an odd prime. Earlier, W. Feit in [1] found M_7 and M_{11} as intervals in the subgroup lattice of the alternating group A_{31} over 31 letters. Thus now M_{16} is the smallest lattice of length 2 for which it is not known if it is isomorphic to an interval in the subgroup lattice of a finite group.

In this paper we prove that every finite lattice is isomorphic to an interval in the subgroup lattice of a countable locally finite group. We prove even a stronger result that every algebraic lattice with at most countably many compact elements can be represented as an interval in the subgroup lattice of a countable locally finite group. Since by a result of P. Hall [4] there exists a universal countable locally finite group that contains as subgroups $2^{\aleph_0}$ distinct copies of any countable locally finite group, the subgroup lattice of the Hall's universal countable locally finite group contains $2^{\aleph_0}$ different intervals isomorphic to any given algebraic lattice with at most countably many compact elements.

The results of this paper still leave unanswered the following problem.

Open problem. Is every algebraic lattice isomorphic to an interval in the subgroup lattice of a locally finite group?

Preliminaries

We will use only elementary concepts from group theory. If G is a group and $X \subseteq G$, then the subgroup of G generated by X will be denoted as $\langle X \rangle$. Recall that a group G is *locally finite* if every finite subset of G generates a finite subgroup.

For undefined lattice theoretical concepts see the monograph [2]. By a *semilattice* we always mean a join-semilattice with a least element 0. We also speak about a $\langle \vee, 0 \rangle$-semilattice. An *ideal* in a $\langle \vee, 0 \rangle$-semilattice P is a *non-empty* subset of P closed under smaller elements and joins. The interval between two elements $a < b$ of P will be denoted by $[a, b]$.

If X is a subset of a semilattice P, then we say that X *join-generates* P if every element of P is a join of a (finite) subset of X. If P happens to be a complete lattice then we say that X *completely join-generates* P if every element of P is the join of a (possibly infinite) subset of X.

In [8] the first author introduced the concept of a group valuation. If P is a semilattice and G a group, then a mapping $\gamma : G \to P$ is called a *group valuation* if it satisfies the following three conditions.

(GV 1) $\gamma(1) = 0_P$, where 1 is the unit element of G,
(GV 2) $\gamma(g^{-1}) = \gamma(g)$ for every $g \in G$,
(GV 3) $\gamma(gh) \leq \gamma(g) \vee \gamma(h)$ for every $g, h \in G$.

The concept of a group valuation is in fact dual to the concept of a complete meet-homomorphism from the ideal lattice $J(P)$ of P into the subgroup lattice $\operatorname{Sub} G$ of the group G *via* the concept of adjoint mappings. And the concept of adjoint mappings is nothing but a Galois connection between two complete lattices if we replace one of the lattices by its dual.

If M and L are complete lattices then two mappings $\alpha : M \to L$ and $\beta : L \to M$ are called *adjoint mappings* if for every $x \in M$ and $y \in L$

$$\alpha(x) \leq y \quad \text{if and only if} \quad x \leq \beta(y). \tag{0.1}$$

Note that the definition of adjoint mappings is not symmetric with respect to α and β but depends on their order. It is straightforward to check that if $\alpha : M \to L$ and $\beta : L \to M$ are adjoint mappings, then α preserves arbitrary (even infinite) joins and β preserves arbitrary meets, see [11]. We say that α is a *complete join-homomorphism* and β is a *complete meet-homomorphism*. Observe also that the definition of a complete join-homomorphism means that also the join of the empty set, i.e. the least element, is preserved. Similarly, a complete meet-homomorphism preserves the top element 1. If $\alpha : M \to L$ is a complete join-homomorphism, then there exists a unique $\beta : L \to M$ such that α and β are adjoint. For $y \in L$ the element $\beta(y)$ is defined as the largest element $x \in M$ such that $\alpha(x) \leq y$. We will denote this uniquely defined complete meet-homomorphism by α^*. Similarly, if $\beta : L \to M$ is a complete meet-homomorphism, then there exists a unique $\alpha : M \to L$ such that α and β are adjoint mappings. For $x \in M$, $\alpha(x)$ is the smallest element $y \in L$ such that $x \leq \beta(y)$. The mapping α will be denoted by $\beta^\dagger$. One can easily check that $(\alpha^*)^\dagger = \alpha$ for any complete join-homomorphism $\alpha : M \to L$ and $(\beta^\dagger)^* = \beta$ for any complete meet-homomorphism $\beta : L \to M$.

Now suppose $\gamma : G \to P$ is a group valuation of the elements of a group G by the elements of a $\langle \vee, 0 \rangle$-semilattice P. For any subgroup H of G we denote by $\gamma_e(H)$ the ideal of P generated by the set $\{\gamma(h) : h \in H\}$. This defines a mapping $\gamma_e : \operatorname{Sub} G \to J(P)$. We also define a mapping $\delta : J(P) \to \operatorname{Sub} G$ by setting $\delta(I) = \{g \in G : \gamma(g) \in I\}$ for any ideal I of P. From the properties of group valutations one can check easily that $\delta(I)$ is indeed a subgroup of G for any $I \in J(P)$.

From the definitions of the mappings γ_e and δ one can also easily verify that for any subgroup H of G and any ideal I of P

$$\gamma_e(H) \subseteq I \quad \text{if and only if} \quad H \subseteq \delta(I).$$

Thus the mappings γ_e and δ are adjoint, in particular γ_e is a complete join-homomorphism and δ is a complete meet-homomorphism. We will say that γ_e is the *expanded version* of γ. We will extend the $*$-notation also to group valuations and use γ^* to denote the adjoint mapping $\delta : J(P) \to \operatorname{Sub} G$ of the expanded version γ_e of a group valuation $\gamma : G \to P$. The defining relation for a group valuation $\gamma : G \to P$ and its adjoint $\gamma^* : J(P) \to \operatorname{Sub} G$ is that for every element $g \in G$ and every ideal I of P

$$\gamma(g) \in I \quad \text{if and only if} \quad g \in \gamma^*(I).$$

We will also say that γ^* is the *adjoint of the group valuation* γ.

The original group valuation $\gamma : G \to P$ can be recovered from the complete join-homomorphism $\gamma_e : \operatorname{Sub} G \to J(P)$ since

$$\gamma_e(\langle g \rangle) = [0_P, \gamma(g)]$$

for any $g \in G$. Thus each of the two mappings γ and γ_e determines the other.

In fact, whenever a complete join-homomorphism $\alpha : \operatorname{Sub} G \to J(P)$ that maps compact elements of $\operatorname{Sub} G$ (i.e. finitely generated subgroups of G) to compact elements of $J(P)$ (i.e. to principal ideals of P), is given, then the mapping $\alpha_r : G \to P$ defined by

$$\alpha_r(g) \text{ is the largest element of } \alpha(\langle g \rangle),$$

is a group valuation. We will also say that α_r is the *reduced version* of α. If α happens to be the adjoint of a complete meet-homomorphism $\beta : J(P) \to \operatorname{Sub} G$, then the reduced version of α will be called the *adjoint group valuation* of β. We will denote it also by $\beta^\dagger$, although some caution is needed here and we always have to say if $\beta^\dagger$ denotes the adjoint complete join-homomorphism of β or the adjoint group valuation of β. The defining relation between a complete meet-homomorphism $\beta : J(P) \to \operatorname{Sub} G$ and the adjoint group valuation $\beta^\dagger$ is that for every $g \in G$ and every ideal I of P

$$\beta^\dagger(g) \in I \quad \text{if and only if} \quad g \in \beta(I).$$

1. More on adjoint mappings

It is an interesting game to translate properties of a complete join-homomorphism $\alpha : M \to L$ to the properties of its adjoint complete meet-homomorphism $\alpha^* : L \to M$ and backwards. In this section we will play this game. Our aim is to find conditions on α that are equivalent to α^* being an isomorphism of L onto a (top) interval in M.

In the following three lemmas we always assume that M and L are complete lattices, $\alpha : M \to L$ is a complete join-homomorphism and $\alpha^* : L \to M$ is its adjoint.

Lemma 1.1. *The mapping α^* is injective if and only if the mapping α is onto.*

Proof. Suppose that α^* is injective. Take any $x \in L$. Then $\alpha\alpha^*(x) \le x$ by (0.1), since $\alpha^*(x) \le \alpha^*(x)$. Hence also $\alpha^*\alpha\alpha^*(x) \le \alpha^*(x)$. On the other hand, from $\alpha\alpha^*(x) \le \alpha\alpha^*(x)$ we obtain again by (0.1) that $\alpha^*(x) \le \alpha^*\alpha\alpha^*(x)$. Thus in fact the equality holds and since α^* is injective, we get $x = \alpha\alpha^*(x)$, thus α is onto.

Conversely, suppose that α is onto. Choose two different elements y, z in L. We may assume $y \not\le z$. Since α is onto, there exists $x \in M$ such that $\alpha(x) = y$. But then $x \le \alpha^*(y)$ from (0.1). By the same reason, $x \not\le \alpha^*(z)$, since $\alpha(x) = y \not\le z$. This proves that $\alpha^*(y) \ne \alpha^*(z)$. □

Lemma 1.2. *The complete meet-homomorphism α^* preserves finite joins if and only if*

$$(1.1) \qquad (\forall\, x \in M)\, (\forall\, y, z \in L)\, (\, \alpha(x) \le y \vee z \;\Rightarrow\; x \le \alpha^*(y) \vee \alpha^*(z)\,).$$

Proof. Let α^* preserve joins. If $\alpha(x) \le y \vee z$, then from (0.1) we obtain $x \le \alpha^*(y \vee z) = \alpha^*(y) \vee \alpha^*(z)$.

Conversely, let the condition (1.1) be satisfied. Since α^* is order preserving (since it preserves meets), we have $\alpha^*(y) \vee \alpha^*(z) \le \alpha^*(y \vee z)$ for any $y, z \in L$. To prove the opposite inequality take any $x \in M$ such that $x \le \alpha^*(y \vee z)$. From (0.1) we obtain $\alpha(x) \le y \vee z$ and from the condition of the lemma we get $x \le \alpha^*(y) \vee \alpha^*(z)$. By choosing $x = \alpha^*(y \vee z)$ we obtain the opposite inequality. □

Lemma 1.3. *The mapping* $\alpha^* : L \to M$ *is onto the interval* $[\alpha^*(0_L), 1_M]$ *of* M *if and only if*

$$(\forall\, x, y \in M)\,(\,\alpha(x) \leq \alpha(y) \;\Rightarrow\; x \leq \alpha^*(0_L) \vee y\,). \tag{1.2}$$

Proof. Let $\alpha^* : L \to M$ be onto the interval $[\alpha^*(0_L), 1_M]$ and $\alpha(x) \leq \alpha(y)$ for some $x, y \in M$. We have $\alpha^*(0_L) \leq \alpha^*(0_L) \vee y$, thus there exists $z \in L$ such that $\alpha^*(0_L) \vee y = \alpha^*(z)$. Then $y \leq \alpha^*(z)$, hence by (0.1) $\alpha(y) \leq z$. Thus also $\alpha(x) \leq z$, therefore $x \leq \alpha^*(z) = \alpha^*(0_L) \vee y$.

Conversely, let the condition (1.2) be satisfied. Take any $y \in M$ such that $\alpha^*(0_L) \leq y$. Then $\alpha^*(0_L) \vee y = y \leq \alpha^*\alpha(y)$ (since $\alpha(y) \leq \alpha(y)$). Now take any $x \in M$ such that $x \leq \alpha^*\alpha(y)$. Then $\alpha(x) \leq \alpha(y)$ from (0.1). By the condition (1.2) we get $x \leq \alpha^*(0_L) \vee y = y$. By setting $x = \alpha^*\alpha(y)$ we obtain the opposite inequality $\alpha^*\alpha(y) \leq y$. Hence $y = \alpha^*\alpha(y)$. □

In the next three lemmas we strengthen our assumptions on M and L. Now M and L will be algebraic lattices and X be an arbitrary join-generating subset of the set of compact elements of M (i.e. every compact element of M is a finite join of some subset of X). It follows that X completely join-generates M.

The next lemma is an easy consequence of Lemma 1.1.

Lemma 1.4. *The mapping* α^* *is injective if an only if*

$$\text{the set } \{\alpha(x) : x \in X\} \text{ completely join-generates the lattice } L. \tag{1.3}$$

Proof. Since X completely join-generates M and α is a complete join-homomorphism, the set $\{\alpha(x) : x \in X\}$ completely join-generates $\operatorname{Im}\alpha$. Thus $\{\alpha(x) : x \in X\}$ completely join-generates L if and only if α is onto L and by Lemma 1.1 this is if and only if α^* is injective. □

The following lemma says that the condition of Lemma 1.2 can be relaxed to the set X.

Lemma 1.5. *The mapping* α^* *preserves joins if and only if*

$$(\forall\, x \in X)\,(\forall\, y, z \in L)\,(\,\alpha(x) \leq y \vee z \;\Rightarrow\; x \leq \alpha^*(y) \vee \alpha^*(z)\,). \tag{1.4}$$

Proof. From Lemma 1.2 we know that if the mapping α^* preserves joins then the condition holds for every $x \in M$.

Suppose conversely that the condition (1.4) holds. Take an arbitrary $t \in M$ and express it in the form $t = \bigvee_{i \in I} x_i$ for some elements $x_i \in X$, $i \in I$. It follows that $\alpha(x_i) \leq \alpha(t) \leq y \vee z$ for every $i \in I$. From the condition (1.4) it follows that $x_i \leq \alpha^*(y) \vee \alpha^*(z)$ for every $i \in I$, hence also $t = \bigvee_{i \in I} x_i \leq \alpha^*(y) \vee \alpha^*(z)$. Thus the condition (1.4) holds for every $t \in M$ and the rest follows from Lemma 1.2. □

Lemma 1.6. *Suppose that the mapping* $\alpha : M \to L$ *maps compact elements of* M *to compact elements of* L*. Then the mapping* α^* *is a lattice-homomorphism onto the interval* $[\alpha^*(0_L), 1_M]$ *if and only if the conditions (1.4) and*

$$(\forall\, x, y \in X)\,(\,\alpha(x) \leq \alpha(y) \;\Rightarrow\; x \leq \alpha^*(0_L) \vee y\,) \tag{1.5}$$

hold.

Proof. If α^* is a lattice-homomorphism, then it satisfies the condition (1.4) for all $x \in M$ by Lemma 1.2. If it is onto the interval $[\alpha^*(0_L), 1_M]$, then by Lemma 1.3 it also satisfies the condition (1.5) for every $x, y \in M$.

To prove the converse implication recall first that from the condition (1.4) we get by Lemma 1.5 that α^* is a lattice-homomorphism. In view of Lemma 1.3 it suffices to prove that the condition (1.5) holds for every $x, y \in M$ to get that α^* is onto the interval $[\alpha^*(0_L), 1_M]$.

Take any $y \in X$. By (0.1) we get that $y \leq \alpha^*\alpha(y)$, since $\alpha(y) \leq \alpha(y)$. Then express $\alpha^*\alpha(y)$ as $\bigvee_{i\in I} x_i$, where $x_i \in X$ for every $i \in I$. We can do it since in an algebraic lattice every element is a join of compact elements, and by our assumption on X, every compact element of M is a (finite) join of elements of X. Then for every $i \in I$ we get $x_i \leq \alpha^*\alpha(y)$, hence $\alpha(x_i) \leq \alpha(y)$. By (1.5), $x_i \leq \alpha^*(0_L) \vee y$, hence also $\alpha^*\alpha(y) = \bigvee_{i\in I} x_i \leq \alpha^*(0_L) \vee y$.

Now let $y \in M$ be compact, $x \in X$, and $\alpha(x) \leq \alpha(y)$. Hence $y = y_1 \vee y_2 \vee \cdots \vee y_k$ for some $k \geq 1$ and $y_1, \dots, y_k \in X$. Then

$$\alpha(x) \leq \alpha(y) = \alpha(y_1 \vee \cdots \vee y_k) = \alpha(y_1) \vee \cdots \vee \alpha(y_k).$$

By (0.1) we get

$$x \leq \alpha^*\alpha(y_1 \vee \cdots \vee y_k) = \alpha^*\alpha(y_1) \vee \cdots \vee \alpha^*\alpha(y_k),$$

since α^* preserves (finite) joins. Now $\alpha^*\alpha(y_j) \leq \alpha^*(0_L) \vee y_j$ for every $j = 1, \dots, k$, hence also

$$x \leq \alpha^*(0_L) \vee y_1 \vee \cdots \vee y_k = \alpha^*(0_L) \vee y.$$

Thus the condition (1.5) holds for any $x \in X$ and any compact $y \in M$.

Now suppose that $x \in X$ and $y \in M$ are such that $\alpha(x) \leq \alpha(y)$. Express y in the form $y = \bigvee_{j\in J} y_j$, where y_j is compact for $j \in J$. Since α is a complete join-homomorphism we get $\alpha(x) \leq \alpha(y) = \bigvee_{j\in J} \alpha(y_j)$. But α is also compactness-preserving, thus $\alpha(x)$ is a compact element of L and there exists a finite set $F \subseteq J$ such that

$$\alpha(x) \leq \bigvee_{j\in F} \alpha(y_j) = \alpha(\bigvee_{j\in F} y_j).$$

But $\bigvee_{j\in F} y_j$ is a compact element of M, hence by the previous paragraph $x \leq \alpha^*(0_L) \vee \bigvee_{j\in F} y_j \leq \alpha^*(0_L) \vee y$.

Finally, let $x, y \in M$ be such that $\alpha(x) \leq \alpha(y)$. Now we express x as $\bigvee_{i\in I} x_i$, where $x_i \in X$ for any $i \in I$. Then $\alpha(x_i) \leq \alpha(x) \leq \alpha(y)$. Thus by the previous paragraph, $x_i \leq \alpha^*(0_L) \vee y$ for any $i \in I$. Hence also $x = \bigvee_{i\in I} x_i \leq \alpha^*(0_L) \vee y$. Thus the condition of Lemma 1.3 is satisfied and the mapping α^* is onto the interval $[\alpha^*(0_L), 1_M]$. □

We can summarize the previous three lemmas into the following theorem.

Theorem 1.7. *Let M, L be algebraic lattices, $\alpha : M \to L$ and $\alpha^* : L \to M$ be adjoint mappings. We suppose moreover that α maps compact elements of M to compact elements of L and that $X \subseteq M$ is a join-generating set of compact elements of M. Then the following holds.*
a) The mapping $\alpha^* : L \to M$ is injective if and only if the condition (1.3) holds.
b) The mapping $\alpha^* : L \to M$ is a join-homomorphism if and only if the condition (1.4) is satisfied.
c) Finally, the mapping $\alpha^* : L \to M$ is a lattice homomorphism onto the interval $[\alpha^*(0_L), 1_M]$ if and only if the conditions (1.4) and (1.5) hold.

If $\gamma : G \to P$ is a group valuation, then the lattices $\operatorname{Sub} G$ and $J(P)$ are algebraic and the expanded version of γ, i.e. the complete join-homomorphism

$\gamma_e : \operatorname{Sub} G \to J(P)$, is compactness-preserving. We choose X to consist of the cyclic subgroups of G. Since $\gamma_e(\langle g\rangle) = [0_P, \gamma(g)]$ for any $g \in G$, the inclusion between the principal ideals $\gamma_e(\langle g\rangle)$ of P corresponds to the order between the elements $\gamma(g)$ of P. However, the adjoint mapping γ^* of γ is defined on ideals of P and not on single elements of P. We allow ourselves to abuse notation and write $\gamma^*(z)$ instead of $\gamma^*([0_P, z])$ for $z \in P$. In this way the previous Theorem 1.7 can be reformulated in the language of group valuations as follows.

Corollary 1.8. *Let $\gamma : G \to P$ be a group valuation and $\gamma^* : J(P) \to \operatorname{Sub} G$ the adjoint of (the expanded version $\gamma_e : \operatorname{Sub} G \to J(P)$ of) the group valuation $\gamma : G \to P$. Then the following holds.*
a) The mapping γ^* is injective if and only if

$$\text{(1.6)} \qquad \text{the set } \{\gamma(g) : g \in G\} \text{ join-generates the semilattice } P.$$

b) It is a join-homomorphism if and only if

$$\text{(1.7)} \qquad (\forall\, g \in G)\, (\forall\, y, z \in P)\, (\, \gamma(g) \le y \vee z \;\Rightarrow\; g \in \gamma^*(y) \vee \gamma^*(z)\,).$$

c) It is a lattice homomorphism onto the interval $[\gamma^*(0_L), 1_M]$ of M if and only if the conditions (1.7) and

$$\text{(1.8)} \qquad (\forall\, g, h \in G)\, (\, \gamma(g) \le \gamma(h) \;\Rightarrow\; g \le \gamma^*(0_L) \vee \langle h\rangle\,)$$

hold.

Proof. Only the *if*-part of b) has to be proved. Take any two ideals I, J of P. From $\gamma(g) \in I \vee J$ it follows that there are $y \in I$ and $z \in J$ such that $\gamma(g) \le y \vee z$. Hence by (1.7), $g \in \gamma^*(y) \vee \gamma^*(z) \subseteq \gamma^*(I) \vee \gamma^*(J)$. Thus condition (1.5) is satisfied and γ^* is join-preserving by Theorem 1.7.b. □

The following lemma considers the conditions (1.6), (1.7) and (1.8) in the union of a countable chain of subgroups.

Lemma 1.9. *Let*

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_i \subseteq G_{i+1} \subseteq \cdots$$

be a countable chain of groups and let P be a $\langle \vee, 0\rangle$-semilattice. Let $\gamma_i : G_i \to P$ be group valuations such that γ_{i+1} extends γ_i for every $i = 0, 1, \ldots$. Let moreover $G = \bigcup_{i=0}^{\infty} G_i$ and $\gamma : G \to P$ be the common extension of all γ_i's. Then

1. *$\gamma : G \to P$ is a group valuation,*
2. *if one of the group valuations γ_i satisfies the condition (1.6), then γ also satisfies the condition (1.6) and the adjoint mapping γ^* of γ is injective,*
3. *if the group valuations $\gamma_i : G_i \to P$ satisfy the condition (1.7) for every $i \ge 1$, then also γ satisfies the condition (1.7) and the adjoint mapping γ^* preserves joins,*
4. *if for every $i = 0, 1, \ldots,$ the group valuations γ_i satisfy the condition*

$$\text{(1.9)} \qquad (\forall\, g, h \in G_i)\, (\, \gamma_i(g) \le \gamma_i(h) \;\Rightarrow\; g \in \gamma_{i+1}^*(0_L) \vee \langle h\rangle\,),$$

then the valuation γ satisfies the condition (1.8).

Proof. Statements (1) and (2) are obvious. To prove the statement (3), let $g \in G$ and $y, z \in P$ be such that $\gamma(g) \le y \vee z$. Then $\gamma(g) = \gamma_i(g)$ for some $i \ge 1$, and since every γ_i, $i \ge 1$, satisfies the condition (1.7), we get

$$g \in \gamma_i^*(y) \vee \gamma_i^*(z) \subseteq \gamma^*(y) \vee \gamma^*(z).$$

And finally, to prove the statement (4), take any $g, h \in G$ such that $\gamma(g) \leq \gamma(h)$. There exists an i such that $g, h \in G_i$, hence $\gamma_i(g) \leq \gamma_i(h)$. By (1.9),

$$g \in \gamma_{i+1}^*(0_L) \vee \langle h \rangle \subseteq \gamma^*(0_L) \vee \langle h \rangle.$$

□

In the case of algebraic lattices with countably many compact elements we will also need the following stronger version of Lemma 1.9.

Lemma 1.10. *Let*

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_i \subseteq G_{i+1} \subseteq \cdots$$

be a countable chain of groups such that G_i is a subgroup of G_{i+1} for any $i = 0, 1, \ldots$ and let

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_i \subseteq P_{i+1} \subseteq \cdots$$

be a sequence of finite semilattices such that every P_i is a $\langle \vee, 0 \rangle$-subsemilattice of P_{i+1} for $i = 0, 1, \ldots$. Let $\gamma_i : G_i \to P_i$ be a group valuation such that γ_{i+1} extends γ_i for every $i = 0, 1, \ldots$. Set $G = \bigcup_{i=0}^{\infty} G_i$, $P = \bigcup_{i=0}^{\infty} P_i$ and let $\gamma : G \to P$ be the common extension of all γ_i's. Then

1. *$\gamma : G \to P$ is a group valuation,*
2. *if all the group valuations γ_i satisfy the condition (1.6), then γ also satisfies the condition (1.6) and the adjoint mapping γ^* of γ is injective,*
3. *if the group valuations $\gamma_i : G_i \to P_i$ satisfy the condition (1.7) for every $i \geq 1$, then also γ satisfies the condition (1.7) and the adjoint mapping γ^* preserves joins,*
4. *if the group valuations γ_i satisfy the condition (1.9) for every $i = 0, 1, \ldots$, then the valuation γ satisfies the condition (1.8).*

Proof. The proof is just a minor modification of the proof of Lemma 1.9 and is left to the reader as an exercise. □

2. Regraphs

In this section all the structures are finite. We will construct some group valuations on a group G with values in a semilattice P as the (reduced versions of) adjoints of complete meet-homomorphisms from $J(P)$ to $\operatorname{Sub} G$. Hence in this section we summarize methods and results of the paper of P. Pudlák and the second author [9] proving that every finite lattice can be embedded into a finite partition lattice. We reformulate some of the definitions and results of that paper to suit more our current purposes. We also introduce a new notion of a product of regraphs that will enable us to formulate in one step the whole inductive construction of an embedding of a given finite lattice into a finite partition lattice presented in [9].

If M is a lattice and $u < v$ two elements of M, then the set

$$M_{u,v} = \{x \in M : \text{ either } x \ngeq u \text{ or } x \geq v\}$$

is a meet-subsemilattice of M. Since also $1_M \in M_{u,v}$, the set $M_{u,v}$ is in fact a complete meet-subsemilattice of M.

The following lemma is straightforward.

Lemma 2.1. *If M is a lattice and K a complete meet-subsemilattice of M, then there exists a natural number k and a sequence*

$$K = M_k \subset M_{k-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that for every $i = 0, 1, \ldots, k-1$, $M_{i+1} = (M_i)_{u,v}$ for suitable $u < v$ in M_i.

Proof. Indeed, if L is a proper complete meet-subsemilattice of M, take any $u \in M \setminus L$ and set v to be the smallest element of L greater than u. Then $u < v$ and $L \subseteq M_{u,v} \subset M$. □

Since $M_{u,v}$ is a complete meet-subsemilattice of M, the inclusion mapping $\iota : M_{u,v} \to M$ is a complete meet-homomorphism. Thus the adjoint mapping $\iota^\dagger : M \to M_{u,v}$ of ι exists and is a complete join-homomorphism. From (0.1) we obtain that for $x \in M$, $\iota^\dagger(x)$ is the smallest element of $M_{u,v}$ greater than or equal to x. In particular, $\iota^\dagger$ is idempotent and its restriction to $M_{u,v}$ is the identity mapping. We will need the following simple property of the mapping $\iota^\dagger$.

Lemma 2.2. *If M and L are lattices, $u < v$ are two elements of M, $\iota^\dagger$ is the adjoint of the inclusion mapping $\iota : M_{u,v} \to M$ and $\varphi : M \to L$ is a join-homomorphism such that $\varphi(u) = \varphi(v)$, then* $\operatorname{Ker} \iota^\dagger \subseteq \operatorname{Ker} \varphi$.

Proof. If $x \in M \setminus M_{u,v}$, then $x \geq u$ and $\iota^\dagger(x) = x \vee v$, since $x \vee v$ is the smallest element of $M_{u,v}$ greater than x. Thus if x, y are two different elements of M such that $\iota^\dagger(x) = \iota^\dagger(y)$, it follows that $\iota^\dagger(x) = \iota^\dagger(y) \geq v$. Hence $x, y \geq u$ and $x \vee v = y \vee v$. But then $\varphi(x) = \varphi(x \vee u) = \varphi(x) \vee \varphi(u) = \varphi(x) \vee \varphi(v) = \varphi(x \vee v)$ and also $\varphi(y) = \varphi(y \vee v)$. Thus $(x, y) \in \operatorname{Ker} \varphi$. □

The following definition is taken from [9].

Definition 2.3. By a *regraph valued by a set A* we mean a triple $\mathbf{W} = (W, T, \tau)$, where W is a non-empty set, T is a symmetric and irreflexive relation on W and $\tau : T \to A$ is a mapping.

Regraphs are a tool to modify mappings from lattices into partition lattices. The partition lattice over a set A is denoted by $\Pi(A)$. The next definition is also taken from [9]

Definition 2.4. Let $\varphi : M \to \Pi(A)$ be a mapping and $\mathbf{W} = (W, T, \tau)$ be a regraph valued by A. Then for every $x \in M$ we define an equivalence relation $\psi(x)$ on the set $A \times W$ as follows. Two elements $(a, u), (b, v) \in A \times W$ are equivalent in $\psi(x)$ if there exists a sequence of elements of $A \times W$

$$(a, u) = (a_0, u_0), (a_1, u_1), \ldots, (a_{2n}, u_{2n}), (a_{2n+1}, u_{2n+1}) = (b, v)$$

such that for every even $i = 0, 2 \ldots, 2n$, $u_i = u_{i+1}$ and $(a_i, a_{i+1}) \in \varphi(x)$, and for every odd $i = 1, 3 \ldots, 2n-1$, $(u_i, u_{i+1}) \in T$ and $a_i = \tau(u_i, u_{i+1})$, $a_{i+1} = \tau(u_{i+1}, u_i)$. The mapping $\psi : M \to \Pi(A \times W)$ is called the $\mathbf{W}$-*power* of φ.

One readily verifies that $\psi(x)$ is indeed an equivalence relation on the set $A \times W$ for every $x \in M$ and that a regraph power of a join-preserving mapping is again a join-preserving mapping. Moreover, the $\mathbf{W}$-power ψ of φ preserves the largest element if both φ does and the regraph $\mathbf{W}$ is connected, i.e. the relation T on the set W is connected.

The proof that every finite lattice can be embedded into a finite partition lattice in [9] is by an induction in the class of all finite lattices based on Lemma 2.1.

Here we recall without proofs major steps of the proof of the finite partition lattice representation theorem. The interested reader is reffered to [9] for the details. To formulate the induction step we need the following definition. We say that a lattice embedding preserving the largest element $\varphi : M \to \Pi(A)$ satisfies the *density condition at an element* $u \in M$ if

$$(\forall a \in A)(\exists b \in A)(\forall z \in M)((a,b) \in \varphi(z) \Leftrightarrow z \geq u). \tag{2.1}$$

We say that $\varphi : M \to \Pi(A)$ satisfies the *density condition at a subset* $X \subseteq M$ if the condition (2.1) holds for every $u \in X$.

The following condition is used to verify the density condition for regraph powers. Let $\varphi : M \to \Pi(A)$ be a complete meet-embedding, $\mathbf{W} = (W, T, \tau)$ be a regraph valued by A and let $\psi : M \to \Pi(A \times W)$ be the $\mathbf{W}$-*power* of φ. We say that the pair $\mathbf{W}, \varphi$ satisfies the *stability condition* if

$$(\forall x \in M)(\forall u \in U)(\forall a, b \in A)(((a,u),(b,u)) \in \psi(x) \Leftrightarrow (a,b) \in \varphi(x)). \tag{2.2}$$

The induction step then starts with a lattice embedding $\varphi : M \to \Pi(A)$ satisfying the *density condition at an element* $u \in M$ and constructs an embedding $\chi : M_{u,v} \to \Pi(C)$.

The embedding $\chi : M_{u,v} \to \Pi(C)$ is constructed in two stages, both stages are regraph powers of previously constructed lattice embeddings of M into a partition lattice satisfying the density condition at the element $u \in M$. The first stage starts with a lattice embedding $\varphi : M \to \Pi(A)$ and constructs a connected regraph $\mathbf{U}$ and the $\mathbf{U}$-power $\psi : M \to \Pi(A \times U)$ of φ that is again a lattice embedding of M. The second stage constructs another connected regraph $\mathbf{V}$ and the $\mathbf{V}$-power $\chi : M \to \Pi(A \times U \times V)$ of ψ such that the restriction $\chi_{|M_{u,v}}$ of χ to the meet-subsemilattice $M_{u,v}$ is a lattice embedding. In the first stage the pair $\mathbf{U}, \varphi$ satisfies the stability condition 2.2. It follows from Basic Lemma 6.2 (the construction of $\mathbf{U}$), Lemma 4.3.a and Corollary 5.2 of [9]. So we obtain the following proposition.

Proposition 2.5. *In the first stage of the induction step of the proof of the finite partition lattice representation theorem in* [9]*, if the lattice embedding* $\varphi : M \to \Pi(A)$ *satisfies the density condition (2.1) at a subset* $X \subseteq M$*, then the regraph power* $\psi : M \to \Pi(A \times U)$ *also satisfies the density condition (2.1) at the same subset* $X \subseteq M$*.*

In the second stage, the pair $\mathbf{V}, \psi_{|M_{u,v}}$ satisfies the stability condition 2.2. To formulate explicitly the stability condition in the second stage we set $A \times U = B$. Then

$$(\forall x \in M_{u,v})(\forall v \in V)(\forall c, d \in B)(((c,v),(d,v)) \in \chi(x) \Leftrightarrow (c,d) \in \psi(x)). \tag{2.3}$$

It follows from Construction 7.1 (the construction of $\mathbf{V}$), the proof of Claim 7.7, and also from Lemma 4.3.a and Corollary 5.2 of [9]. So we obtain the following proposition.

Proposition 2.6. *In the second stage of the induction step of the proof of the finite partition lattice representation theorem in* [9]*, if the lattice embedding* $\psi : M \to \Pi(B)$ *satisfies the density condition (2.1) at a subset* $X \subseteq M$*, then the restriction* $\chi_{|M_{u,v}} : M_{u,v} \to \Pi(B \times V)$ *of the regraph power* $\chi : M \to \Pi(B \times V)$ *satisfies the density condition (2.1) at the subset* $\iota^{\dagger}(X) \subseteq M_{u,v}$*.*

Proof. We only need to prove that $\chi_{|M_{u,v}} : M_{u,v} \to \Pi(B \times V)$ satisfies the density condition 2.1 at $\iota^\dagger(x)$ for every $x \in X$. Take $x \in X$ and $(c, v) \in B \times V$. Since the embedding $\psi : M \to \Pi(B)$ satisfies the density condition (2.1) at x, there exists $d \in B$ such that for any $z \in M$, $(c, d) \in \psi(z)$ if and only if $x \leq z$. In particular, $(c, d) \in \psi(\iota^\dagger(x))$. By (2.3), $((c, v), (d, v)) \in \chi(\iota^\dagger(x))$. Now if $z \in M_{u,v}$ is such that $((c, v), (d, v)) \in \chi(z)$, then again by (2.3), $(c, d) \in \psi(z)$. Since ψ satisfies the density condition at x, we get $x \leq z$. But since $z \in M_{u,v}$, we obtain $\iota^\dagger(x) \leq z$. This proves that $\chi_{|M_{u,v}}$ satisfies the density condition at $\iota^\dagger(x)$. □

To avoid repeated use of regraph powers we introduce here a definition of a product of regraphs and prove that the two steps can be joined into one step based on the product of the two regraphs used in these two steps.

Definition 2.7. Let $\mathbf{U} = (U, R, \rho)$ be a regraph valued by a set A and $\mathbf{V} = (V, S, \sigma)$ be a regraph valued by $A \times U$. By the *product of regraphs* $\mathbf{U}$ and $\mathbf{V}$ we mean the reagraph $\mathbf{U} \times \mathbf{V} = \mathbf{W} = (U \times V, T, \tau)$ valued by the set A, where $((s, t), (u, v)) \in T$ if and only if one of the following two conditions holds.

(1) Either $t = v$ and $(s, u) \in R$. In this case $\tau((s, t), (u, v)) = \rho(s, u)$ (thus also $\tau((u, v), (s, t)) = \rho(u, s)$).
(2) Or $(t, v) \in S$, $\sigma(t, v) = (a, s)$ and $\sigma(v, t) = (b, u)$ for some $a, b \in A$. In this case, $\tau((s, t), (u, v)) = a$ (thus also $\tau((u, v), (s, t)) = b$).

One checks readily that $\mathbf{U} \times \mathbf{V}$ is indeed a regraph valued by A.

The following technical lemma is a key part in our approach transferring the induction in the proof of the finite partition lattice representation theorem to the construction of regraphs.

Lemma 2.8. *Let $K \subseteq L$ be two complete meet-subsemilattices of a finite lattice M. Denote by $\alpha : M \to L$ the adjoint of the inclusion mapping $L \hookrightarrow M$ and by $\gamma : L \to K$ the adjoint of the inclusion mapping $K \hookrightarrow L$. Let $\varphi : M \to \Pi(A)$ be a lattice embedding preserving the largest element, let $\mathbf{U} = (U, R, \rho)$ be a regraph valued by A and $\psi : M \to \Pi(A \times U)$ the $\mathbf{U}$-power of φ. Let us suppose further that ψ preserves the largest element,* $\operatorname{Ker}\psi = \operatorname{Ker}\alpha$ *and the restriction $\psi_{|L}$ of ψ to L is a lattice embedding. Let $\mathbf{V} = (V, S, \sigma)$ be a regraph valued by $A \times U$ and $\chi : L \to \Pi(A \times U \times V)$ be the $\mathbf{V}$-power of $\psi_{|L}$. Let χ preserve the largest element,* $\operatorname{Ker}\chi = \operatorname{Ker}\gamma$*, and let the restriction $\chi_{|K}$ be a lattice embedding. Let us denote by $\mu : M \to \Pi(A \times U \times V)$ the $(\mathbf{U} \times \mathbf{V})$-power of φ. Then*

a) $\mu(x) = \chi(\alpha(x))$ *for every $x \in M$,*
b) $\operatorname{Ker}\mu = \operatorname{Ker}\gamma\alpha$ *and μ preserves the largest element,*
c) $\mu_{|K} : K \to \Pi(A \times U \times V)$ *is a lattice embedding.*

Proof. To prove a) we start with the proof of the inclusion $\chi(\alpha(x)) \subseteq \mu(x)$ for any $x \in M$. Since $(x, \alpha(x)) \in \operatorname{Ker}\alpha = \operatorname{Ker}\psi$, we get $\psi(x) = \psi(\alpha(x))$. In the forthcoming part of the proof we will use the associativity of the Cartesian product of sets and write (a, u, v) instead of either $((a, u), v)$ or $(a, (u, v))$.

Let $((a, s, t), (b, u, v)) \in \chi(\alpha(x))$. Then for a natural number n there exists a sequence

$$(a, s, t) = (a_0, u_0, v_0), (a_1, u_1, v_1), \ldots, (a_{2n+1}, u_{2n+1}, v_{2n+1}) = (b, u, v)$$

such that $v_i = v_{i+1}$ and $((a_i, u_i), (a_{i+1}, u_{i+1})) \in \psi(\alpha(x))$ for every even $i = 0, 2 \ldots, 2n$, and $(v_i, v_{i+1}) \in S$, $\sigma(v_i, v_{i+1}) = (a_i, u_i)$, $\sigma(v_{i+1}, v_i) = (a_{i+1}, u_{i+1})$ for any odd $i = 1, \ldots, 2n-1$.

Since $\psi(\alpha(x)) = \psi(x)$, there exists for every even $i = 0, 2 \ldots, 2n$ a natural number m_i and a sequence

$$(a_i, u_i) = (a_{i,0}, u_{i,0}), (a_{i,1}, u_{i,1}), \ldots, (a_{i,2m_i+1}, u_{i,2m_i+1}) = (a_{i+1}, u_{i+1})$$

such that $u_{i,j} = u_{i,j+1}$ and $(a_{i,j}, a_{i,j+1}) \in \varphi(x)$ for every even $j = 0, 2, \ldots, 2m_i$, and $(u_{i,j}, u_{i,j+1}) \in R$, $a_{i,j} = \rho(u_{i,j}, u_{i,j+1})$ and $a_{i,j+1} = \rho(u_{i,j+1}, u_{i,j})$ for every odd $j = 1, 3 \ldots, 2m_i - 1$.

Then the sequence

$$\begin{aligned}
&(a, s, t) = (a_0, u_0, v_0), (a_{0,1}, u_{0,1}, v_0), \ldots, (a_{0,2m_0+1}, u_{0,2m_0+1}, v_0) = (a_1, u_1, v_1),\\
&(a_2, u_2, v_2) = (a_{2,0}, u_{2,0}, v_2), \ldots, (a_{2,2m_2+1}, u_{2,2m_2+1}, v_2) = (a_3, u_3, v_3),\\
&\qquad\qquad\qquad\qquad\vdots\\
&(a_{2n}, u_{2n}, v_{2n}) = (a_{2n,0}, u_{2n,0}, v_{2n,0}), \ldots, (a_{2n,2m_{2n}+1}, u_{2n,2m_{2n}+1}, v_{2n}) = (b, u, v)
\end{aligned}$$

witnesses the fact that $((a, s, t), (b, u, v)) \in \mu(x)$, since for every odd integer $j = 1, 3, \ldots, 2n-1$, $((u_i, v_i), (u_{i+1}, v_{i+1})) \in T$, and also $\tau((u_i, v_i), (u_{i+1}, v_{i+1})) = a_i$ and $\tau((u_{i+1}, v_{i+1}), (u_i, v_i)) = a_{i+1}$ by the Definition 2.7.

Conversely, let $((a, s, t), (b, u, v)) \in \mu(x)$. Then there exists a sequence

$$(a, s, t) = (a_0, u_0, v_0), (a_1, u_1, v_1), \ldots, (a_{2n+1}, u_{2n+1}, v_{2n+1}) = (b, u, v)$$

such that for every even $i = 0, 2, \ldots, 2n$ $(u_i, v_i) = (u_{i+1}, v_{i+1})$ and $(a_i, a_{i+1}) \in \varphi(x)$, and for every odd $i = 1, 3, \ldots, 2n-1$, $((u_i, v_i), (u_{i+1}, v_{i+1})) \in T$ and also $\tau((u_i, v_i), (u_{i+1}, v_{i+1})) = a_i$, $\tau((u_{i+1}, v_{i+1}), (u_i, v_i)) = a_{i+1}$.

Let i_0 be the maximal index such that $v_0 = v_1 = \cdots = v_{i_0}$. Then i_0 is odd and $i_0 = 2m_0+1$ for some natural number m_0. From the definition of the $(\mathbf{U}\times\mathbf{V})$-power of φ we get that for every odd $j < 2m_0 + 1$ we have $((u_j, v_j), (u_{j+1}, v_{j+1})) \in T$ and $a_j = \tau((u_j, v_j), (u_{j+1}, v_{j+1}))$ and $a_{j+1} = \tau((u_{j+1}, v_{j+1}), (u_j, v_j))$. But since $v_j = v_{j+1}$ we get from the definition of the product of regraphs that $(u_j, u_{j+1}) \in R$, $a_j = \rho(u_j, u_{j+1})$ and $a_{j+1} = \rho(u_{j+1}, u_j)$. Then the sequence

$$(a, s) = (a_0, u_0), (a_1, u_1), \ldots, (a_{2m_0}, u_{2m_0}), (a_{2m_0+1}, u_{2m_0+1}) = (a_{i_0}, u_{i_0})$$

proves that $((a_0, u_0), (a_{i_0}, u_{i_0})) \in \psi(x) \subseteq \psi(\alpha(x))$.

If $i_0 = 2n+1$, then $(a_{i_0}, u_{i_0}) = (b, u)$ and since $t = v_0 = v_{i_0} = v_{2n+1} = v$, we get that also $((a, s, t), (b, u, v)) \in \chi(\alpha(x))$.

Now suppose that $i_0 < 2n+1$. Since i_0 is odd and $v_{i_0} \neq v_{i_0+1}$, by the second part of Definition 2.7 we get $(v_{i_0}, v_{i_0+1}) \in S$ and $\sigma(v_{i_0}, v_{i_0+1}) = (a_{i_0}, u_{i_0})$, $\sigma(v_{i_0+1}, v_{i_0}) = (a_{i_0+1}, u_{i_0+1})$. But then the sequence

$$(a, s, t) = (a, u_0, v_0), (a_1, u_1, v_1), \ldots, (a_{i_0}, u_{i_0}, v_{i_0}), (a_{i_0+1}, u_{i_0+1}, v_{i_0+1})$$

proves that $((a, s, t), (a_{i_0+1}, u_{i_0+1}, v_{i_0+1})) \in \chi(\alpha(x))$. By a simple induction on n we get that also $((a, s, t), (b, u, v)) \in \chi(\alpha(x))$. This concludes the proof of a).

To prove b), let $(x, y) \in \operatorname{Ker}\mu$ for some $x, y \in M$. This is if and only if $\mu(x) = \chi(\alpha(x)) = \chi(\alpha(y)) = \mu(y)$ if an only if $(\alpha(x), \alpha(y)) \in \operatorname{Ker}\chi = \operatorname{Ker}\gamma$. And this is if and only if $\gamma(\alpha(x)) = \gamma(\alpha(y))$, or stating otherwise, $(x, y) \in \operatorname{Ker}\gamma\alpha$. Moreover, since both χ and α preserve the largest element, μ preserves it as well.

And c) follows immediately from a), since for every $x \in K \subseteq L$ we get $\mu(x) = \chi(\alpha(x)) = \chi(x)$, and $\chi_{|K}$ is a lattice embedding by the assumption. □

By applying the previous lemma to the two regraphs constructed in the two stages of the induction step in Sections 6 and 7 of [9] we get the following corollary. Its last claim follows from our Propositions 2.5 and 2.6.

Corollary 2.9. *Let M be a finite lattice, $u < v$ two elements of M and $\varphi : M \to \Pi(A)$ a lattice emebedding preserving the largest element of M and satisfying the density condition (2.1) at a subset $X \subseteq M$ containing the element u. Then there exists a regraph $\mathbf{U} = (U, R, \rho)$ over A such that for the $\mathbf{U}$-power $\psi : M \to \Pi(A \times U)$ of φ we get $\operatorname{Ker}\psi = \operatorname{Ker}\iota^\dagger$, where $\iota^\dagger : M \to M_{u,v}$ is the adjoint of the inclusion embedding $\iota : M_{u,v} \to M$, and its restriction $\psi_{|M_{u,v}}$ to $M_{u,v}$ is a lattice embedding preserving the largest element. The lattice embedding $\psi_{|M_{u,v}} : M_{u,v} \to \Pi(A \times U)$ satisfies the density condition at the subset $\iota^\dagger(X) \subseteq M_{u,v}$ and the pair $\mathbf{U}, \varphi_{|M_{u,v}}$ satisfies the stability condition (2.3).*

And another straightforward induction gives the following "one step" version of the finite partition lattice representation theorem.

Corollary 2.10. *Let N be a finite lattice and let $K \subseteq N$ be its complete meet-subsemilattice. Let $\varphi : N \to \Pi(A)$ be a lattice embedding preserving the largest element and satisfying the density condition (2.1) at a join-generating subset $X \subseteq N$. Then there exists a regraph $\mathbf{W} = (W, T, \tau)$ over A such that the $\mathbf{W}$-power $\mu : N \to \Pi(A \times W)$ of φ has the property $\operatorname{Ker}\mu = \operatorname{Ker}\iota^\dagger$, where $\iota^\dagger : N \to K$ is the adjoint of the inclusion embedding $\iota : K \to N$, and its restriction $\mu_{|K}$ to K is a lattice embedding preserving the largest element. Moreover, the pair $\mathbf{W}, \varphi_{|K}$ satisfies the stability condition (2.2).*

Proof. We will proceed by induction on the set of all complete meet-subsemilattices M of N containing K and ordered by inclusion. We prove that if $\varphi : M \to \Pi(A)$ is a lattice embedding preserving the largest element and satisfying the density condition (2.1) at a subset $X \subseteq M$, then there exists a a regraph $\mathbf{W} = (W, T, \tau)$ such that the $\mathbf{W}$-power $\chi : M \to \Pi(A \times W)$ of φ has the property $\operatorname{Ker}\chi = \operatorname{Ker}\iota^\dagger$, where $\iota^\dagger : M \to K$ is the adjoint of the inclusion embedding $\iota : K \to M$, and its restriction $\chi_{|K}$ to K is a lattice embedding preserving the largest element.

If $M = K$, then we can take any one-element set W and empty T and τ.

Now suppose that $K \neq M$ and that the theorem holds for any proper meet-subsemilattice L of M containing K. Thus the induction hypothesis is that for any proper meet-subsemilattice L of M containing K and any lattice embedding $\psi : L \to \Pi(B)$ preserving the largest element and satisfying the density condition (2.1) at a join-generating subset $Y \subseteq L$ there exists a regraph $\mathbf{V} = (V, S, \sigma)$ valued by B such that the $\mathbf{V}$-power $\chi : L \to \Pi(B \times V)$ of ψ has the property that $\operatorname{Ker}\chi = \operatorname{Ker}\gamma$, where γ is the adjoint of the inclusion mapping $K \hookrightarrow L$ and its restriction $\chi_{|K}$ to K is a lattice embedding preserving the largest element. Moreover, the pair $\mathbf{V}, \psi|\ K$ satisfies the stability condition (2.2).

Since $K \neq M$, there exists some $z \in M \setminus K$. Let ϵ be the adjoint of the inclusion mapping $K \hookrightarrow M$. Then $z < \epsilon(z)$. Since X is a join-generating subset of M and ϵ is join-preserving, there exists $u \in X$ such that $u \leq z$ and $\epsilon(u) \not\leq z$. We set $v = \epsilon(u)$. Then $u < v$ and $K \subseteq M_{u,v} \subset M$.

By Corollary 2.9 there exists a regraph $\mathbf{U} = (U, R, \rho)$ valued by A such that for the $\mathbf{U}$-power $\psi : M \to \Pi(A \times U)$ of φ we get $\operatorname{Ker}\psi = \operatorname{Ker}\alpha$, where $\alpha : M \to M_{u,v}$ is the adjoint of the inclusion mapping $M_{u,v} \hookrightarrow M$. Moreover, the restriction $\psi_{|M_{u,v}} : M_{u,v} \to \Pi(A \times U)$ is a lattice embedding preserving the largest element and the pair $\mathbf{U}, \varphi_{|M_{u,v}}$ satisfies the stability condition (2.2). Finally, the lattice embedding $\psi_{|M_{u,v}} : M_{u,v} \to \Pi(A \times U)$ satisfies the density condition (2.1) at the subset $Y = \alpha(X)$ of $M_{u,v}$. Since α preserves joins and is onto $M_{u,v}$, and X is a join-generating subset of M, the set $Y = \alpha(X)$ is a join-generating subset of $M_{u,v}$.

By the induction hypothesis applied to $M_{u,v}$ and $\psi_{|M_{u,v}}$ there exists a regraph $\mathbf{V} = (V, S, \sigma)$ valued by $A \times U$ such that the $\mathbf{V}$-power $\chi : M_{u,v} \to \Pi(A \times U \times V)$ of $\psi_{|M_{u,v}}$ preserves the largest element, $\operatorname{Ker} \chi = \operatorname{Ker} \gamma$, where γ is the adjoint of the inclusion mapping $K \hookrightarrow M_{u,v}$, the pair $\mathbf{V}, \psi_{|K}$ satisfies the stability condition (2.2), and the restriction $\chi_{|K}$ is a lattice embedding preserving the largest element.

By Lemma 2.8, the $(\mathbf{U} \times \mathbf{V})$-power $\mu : M \to \Pi(A \times U \times V)$ has the property that $\mu_{|K} : K \to \Pi(A \times U \times V)$ is a lattice embedding preserving the largest element and $\operatorname{Ker} \mu = \operatorname{Ker} \gamma\alpha$. But $\gamma\alpha$ is the adjoint of the inclusion mapping $K \hookrightarrow M$.

Finally, to prove that the pair $\mathbf{U} \times \mathbf{V}, \varphi_{|K}$ satisfies the stability condition (2.2) let $x \in K$, $(u, v) \in \mathbf{U} \times \mathbf{V}$ and $((a, u, v), (b, u, v)) \in \mu(x)$. Since $x \in K$ we get $\alpha(x) = x$. By Lemma 2.8.a), $\mu(x) = \chi(\alpha(x))$. But the pair $\mathbf{V}, \psi_{|K}$ satisfies the stability condition, hence $((a, u), (b, u)) \in \psi(x)$. But also the pair $\mathbf{U}, \varphi_{|M_{u,v}}$ staisfies the stability condition. It follows that $(a, b) \in \varphi(x)$. This proves that the pair $\mathbf{U} \times \mathbf{V}, \varphi_{|K}$ satisfies the stability condition (2.2). □

3. The extension lemma and the main results

The partition lattice $\Pi(A)$ on a set A can be canonically embedded into the subgroup lattice of the full symmetric group $\operatorname{Sym} A$ on A by assigning to every partition $\pi \in \Pi(A)$ the subgroup

$$\Theta_A(\pi) = \{p \in \operatorname{Sym} A : (p(a), a) \in \pi \text{ for every } a \in A\}. \tag{3.1}$$

The embeddeding Θ_A preserves both the least and the greatest elements of $\Pi(A)$.

The following lemma proves the crucial extension property.

Lemma 3.1. *Let G be a finite group and K a complete meet-subsemilattice of $\operatorname{Sub} G$. Denote by $\alpha : G \to K$ the group valuation that is (the reduced version of) the adjoint $\iota^\dagger : \operatorname{Sub} G \to K$ of the inclusion mapping $\iota : K \to \operatorname{Sub} G$. Then there exists a finite extension H of the group G and a group valuation $\gamma : H \to K$ extending α and satisfying*

$$(\forall\, g, h \in G)\, (\, \alpha(g) \leq \alpha(h) \;\Rightarrow\; g \in \gamma^*(0_K) \vee \langle h \rangle\,). \tag{3.2}$$

Moreover, the group valuation $\gamma : H \to K$ satisfies the conditions (1.7) and (1.6).

Proof. We set $N = \operatorname{Sub} G$, thus 0_N is the one-element subgroup of G. We define an embedding $\varphi : N \to \Pi(G \times \mathbf{2})$, where $\mathbf{2} = \{0, 1\}$, by

$$((f, i), (g, j)) \in \varphi(F) \text{ if and only if } fg^{-1} \in F$$

for any $F \in \operatorname{Sub} G$, $f, g \in G$ and $i, j \in \mathbf{2}$. One readily verifies that φ is a lattice embedding preserving the largest element.

Next we prove that φ satisfies the density condition (2.1) at every cyclic subgroup $\langle h \rangle$ of G. Take any $(f, i) \in G \times \mathbf{2}$. Then for any $F \in \operatorname{Sub} G$ we obtain that $((f, i), (hf, i)) \in \varphi(F)$ if and only if $h^{-1} \in F$, which is if and only if $\langle h \rangle \subseteq F$. Since the set X of all cyclic subgroups of G join-generates the subgroup lattice $\operatorname{Sub} G$, the embedding φ satisfies all the assumptions of Corollary 2.10.

Thus there exists a regraph $\mathbf{W} = (W, T, \tau)$ over $G \times \mathbf{2}$ such that the $\mathbf{W}$-power $\mu : N \to \Pi(G \times \mathbf{2} \times W)$ of φ has the property $\operatorname{Ker} \mu = \operatorname{Ker} \iota^\dagger$, its restriction $\mu_{|K}$

to K is a lattice embedding preserving the largest element and the pair $\mathbf{W}, \varphi_{|K}$ satisfies the stability condition (2.2).

Next we define a mapping $\Delta : G \to \mathrm{Sym}(G \times \mathbf{2} \times W)$ by

$$\Delta(g)(f,i,w) = \begin{cases} (gf,i,w) & \text{if } i = 0, \\ (f,i,w) & \text{if } i = 1. \end{cases} \tag{3.3}$$

One again checks readily that this is an embedding of G into $\mathrm{Sym}(G \times \mathbf{2} \times W)$. For a fixed $w \in W$ it acts as a regular representation of G on the elements $(h,0,w)$ and as the identity on the elements $(h,1,w)$. By setting $H = \mathrm{Sym}(G \times \mathbf{2} \times W)$ and identifying every element $g \in G$ with $\Delta(g) \in H$ we get that H is a finite extension of G.

Next we set $\chi = \Theta_{G\times\mathbf{2}\times W} \circ \mu_{|K}$. In what follows we will omit the subscript in $\Theta_{G\times\mathbf{2}\times W}$ and write only Θ. Since χ is an embedding of K into $\mathrm{Sub}\,H$ preserving the largest element, it is in particular a complete meet-preserving homomorphism. Thus there exists a group valuation $\gamma : H \to K$ that is (the reduced version of) the adjoint of χ. Hence $\gamma^* = \chi = \Theta \circ \mu_{|K}$.

Since the mapping χ is an injective lattice homomorphism, the conditions (1.7) and (1.6) are satisfied by Corollary 1.8. Next we are going to prove the property (3.2). We need to prove

$$(\forall\, g,h \in G)\,(\,\alpha(g) \le \alpha(h) \;\Rightarrow\; \Delta(g) \in \Theta\,\mu(0_K) \vee \langle \Delta(h)\rangle\,). \tag{3.4}$$

So let us fix $g,h \in G$ such that $\alpha(g) \le \alpha(h)$. As the first step towards proving (3.4) we prove the following claim.

Claim 1.

$$\Theta\,\mu(\langle h\rangle) \subseteq \Theta\,\mu(0_K) \vee \langle \Delta(h)\rangle. \tag{3.5}$$

Proof of Claim. We start with several simple observations. For every $f \in G$ we have $((f,0),(f,1)) \in \varphi(0_M)$, hence $((f,0,w),(f,1,w)) \in \mu(0_M)$ for every $w \in W$. Since $0_M \le 0_K$, we get also $((f,0,w),(f,1,w)) \in \mu(0_K)$ and thus the transposition of $(f,0,w)$ and $(f,1,w)$ belongs to $\Theta\,\mu(0_K)$ for every $f \in G$ and $w \in W$.

From the definition of φ we get that $((e,i),(f,j)) \in \varphi(\langle h\rangle)$ if an only if $e = h^k f$ for an integer k. Since $\varphi(\langle h\rangle)$ is a symmetric relation, we may assume $k \ge 0$. If we conjugate the transposition of $(f,0,w)$ and $(f,1,w)$ by $\Delta(h)$, we obtain the transposition of $(hf,0,w)$ and $(f,1,w)$ for any $w \in W$. Thus the transposition of any two subsequent elements in the sequence

$$(f,0,w),(f,1,w),(hf,0,w),(hf,1,w),(h^2f,0,w),\dots,(h^kf,0,w),(h^kf,1,w)$$

belongs to $\Theta\,\mu(0_K) \vee \langle \Delta(h)\rangle$. Hence also the transposition of any two elements in the sequence, not necessarily two subsequent ones, belongs to $\Theta\,\mu(0_K) \vee \langle \Delta(h)\rangle$. This proves that whenever $((e,i),(f,j)) \in \varphi(\langle h\rangle)$, the transposition of (e,i,w) and (f,j,w) belongs to $\Theta\,\mu(0_K) \vee \langle \Delta(h)\rangle$ for any $w \in W$.

Now take two elements $(e,i,u),(f,j,v) \in G\times\mathbf{2}\times W$ such that $((e,i,u),(f,j,v)) \in \mu(\langle h\rangle)$. By the Definition 2.4 of regraph powers there is a sequence

$$(e,i,u) = (e_0,i_0,u_0),(e_1,i_1,u_1),\dots,(e_{2n+1},i_{2n+1},u_{2n+1}) = (f,j,v) \tag{3.6}$$

such that for every even $k = 0,2,\dots,2n$, $u_k = u_{k+1}$ and $((e_k,i_k),(e_{k+1},i_{k+1})) \in \varphi(\langle h\rangle)$, and for every odd $k = 1,3,\dots,2n-1$, $(u_k,u_{k+1}) \in T$, $\tau(u_k,u_{k+1}) = (e_k,i_k)$, $\tau(u_{k+1},u_k) = (e_{k+1},i_{k+1})$. Thus for every odd $k = 1,3,\dots,2n-1$, $((e_k,i_k,u_k),(e_{k+1},i_{k+1},u_{k+1})) \in \mu(0_M) \subseteq \mu(0_K)$, hence the transposition of the two elements (e_k, i_k, u_k) and $(e_{k+1}, i_{k+1}, u_{k+1})$ belongs to $\Theta\,\mu(0_K)$. By the previous paragraph, for every even $k = 0, 2, \dots, 2n$, the transposition of the elements (e_k, i_k, u_k) and $(e_{k+1}, i_{k+1}, u_k) = (e_{k+1}, i_{k+1}, u_{k+1})$ belongs to $\Theta\,\mu(0_K) \vee \langle\Delta(h)\rangle$. Hence the transposition of any two subsequent elements of the sequence (3.6) belongs to $\Theta\,\mu(0_K) \vee \langle\Delta(h)\rangle$, thus also the transposition of (e, i, u) and (f, j, v) does.

Now by (3.1), a permutation $p \in \mathrm{Sym}(G \times \mathbf{2} \times W)$ belongs to $\Theta\,\mu(\langle h\rangle)$ if and only if $(p(e, i, w), (e, i, w)) \in \mu(\langle h\rangle)$ for every $(e, i, w) \in G \times \mathbf{2} \times W$. Thus p has to permute every block of $\mu(\langle h\rangle)$. By the previous paragraph, if (e, i, u) and (f, j, v) belong to the same block of $\mu(\langle h\rangle)$, then their transposition belongs to $\Theta\,\mu(0_K) \vee \langle\Delta(h)\rangle$. Hence also every $p \in \Theta\,\mu(\langle h\rangle)$ belongs to $\Theta\,\mu(0_K) \vee \langle\Delta(h)\rangle$. This completes the proof of (3.5). □ Claim 1.

Since $\mathrm{Ker}\,\mu = \mathrm{Ker}\,\iota^\dagger$ and $(\iota^\dagger(F), F) \in \mathrm{Ker}\,\iota^\dagger$ for every $F \in \mathrm{Sub}\,G$, we get $\mu(\iota^\dagger(\langle h\rangle)) = \mu(\langle h\rangle)$. But we assume $\alpha(g) \leq \alpha(h)$, hence also $\iota^\dagger(\langle g\rangle) \leq \iota^\dagger(\langle h\rangle)$ and $\langle g\rangle \subseteq \iota^\dagger(\langle g\rangle) \subseteq \iota^\dagger(\langle h\rangle)$. Thus we get that also

$$\Theta\,\mu(\langle g\rangle) \subseteq \Theta\,\mu(\langle \iota^\dagger(h)\rangle) = \Theta\,\mu(\langle h\rangle) \subseteq \Theta\,\mu(0_K) \vee \langle\Delta(h)\rangle.$$

Now for every $(f, 0, w) \in G \times \mathbf{2} \times W$ we have $\Delta(g)(f, 0, w) = (gf, 0, w)$ and since $gff^{-1} = g \in \langle g\rangle$, we get $(\Delta(g)(f, 0, w), (f, 0, w)) \in \mu(\langle g\rangle)$. For $(f, 1, w) \in G \times \mathbf{2} \times W$ we have $\Delta(g)(f, 1, w) = (f, 1, w)$, hence again that $(\Delta(g)(f, 1, w), (f, 1, w)) \in \mu(\langle g\rangle)$. This proves

$$\Delta(g) \in \Theta\,\mu(\langle g\rangle) \subseteq \Theta\,\mu(\langle h\rangle) \subseteq \Theta\,\mu(0_K) \vee \langle\Delta(h)\rangle.$$

The final part of the proof is that the group valuation γ extends the group valuation α. Recall that $\alpha(g) = \iota^\dagger(\langle g\rangle)$ and $\gamma(\Delta(g)) = \chi^\dagger(\langle\Delta(g)\rangle)$ for every $g \in G$. Hence to prove that $\gamma(\Delta(g)) = \alpha(g)$ we have to prove

$$\chi^\dagger(\langle\Delta(g)\rangle) = \iota^\dagger(\langle g\rangle)$$

for every $g \in G$.

Take any subgroup F of G that belongs to K. Then $\iota^\dagger(\langle g\rangle) \leq F$ if and only if $\langle g\rangle \leq \iota(F) = F$, or equivalently, if and only if $g \in F$.

On the other hand, $\chi^\dagger(\langle\Delta(g)\rangle) \leq F$ if and only if $\langle\Delta(g)\rangle \leq \chi(F) = \Theta(\mu(F))$. The last inequality holds if and only if $(\Delta(g)(f, i, w), (f, i, w)) \in \mu(F)$ for any $f \in G$, $i \in \mathbf{2}$ and $w \in W$. Since $\Delta(g)(f, 1, w) = (f, 1, w)$ for any $f \in G$ and $w \in W$, this happens if and only if $(\Delta(g)(f, 0, w), (f, 0, w)) = ((gf, 0, w), (f, 0, w)) \in \mu(F)$ for any $f \in G$ and $w \in W$. By the stability condition (2.3) for the pair $\mathbf{W}, \varphi_{|K}$, this is if and only if $((gf, 0), (f, 0)) \in \varphi(F)$ and this is if and only if $g \in F$.

Thus for every $F \in K$, we get that $\iota^\dagger(\langle g\rangle) \leq F$ if and only if $\chi^\dagger(\langle\Delta(g)\rangle) \leq F$. This proves that $\chi^\dagger(\langle\Delta(g)\rangle) = \iota^\dagger(\langle g\rangle)$ holds for every $g \in G$ and thus γ extends α. □

Before we prove our main results, we state the following simple lemma. The easy proof is left to the reader.

Lemma 3.2. *Let G, H be groups, P a semilattice, and let $\alpha : G \to P$ and $\gamma : H \to P$ be group valuations. Then the mapping $\varepsilon : G \times H \to P$ defined by*

$$\varepsilon(g, h) = \alpha(g) \vee \gamma(h)$$

is also a group valuation.

We call the group valuation ε the *product* of the group valuations α and γ.

Theorem 3.3. *Every finite lattice is isomorphic to an interval in the subgroup lattice of a countable locally finite group.*

Proof. Let L be a finite lattice and denote by $\mathrm{J}(L)$ the set of all join-irreducible elements of L. For every join-irreducible element $i \in \mathrm{J}(L)$ we define a group valuation $\alpha_i : Z_2 \to L$ on the two-element (additive) group $Z_2 = \{0, 1\}$ by setting $\alpha_i(0) = 0_L$ and $\alpha_i(1) = i$. Then we set G_0 as the product of $|\,\mathrm{J}(L)|$-many copies of Z_2 and γ_0 as the product of the group valuations α_i, $i \in \mathrm{J}(L)$. Then $i \in \operatorname{Im} \gamma_0$ for every $i \in \mathrm{J}(L)$, thus $\operatorname{Im} \gamma_0$ join-generates L, hence it satisfies the condition (1.6).

Now suppose that for some $i \geq 0$ group valuations $\gamma_j : G_j \to L$ have been already defined for all $j \leq i$, they satisfy the condition (1.6), γ_j extends γ_{j-1} and satisfies the condition (1.7) for every $j = 1, \ldots, i$, and finally

$$(\forall\, g, h \in G_{j-1})\,(\,\gamma_{j-1}(g) \leq \gamma_{j-1}(h) \;\Rightarrow\; g \in \gamma_j^*(0_L) \vee \langle h \rangle\,),$$

also for every $j = 1, \ldots, i$.

Since γ_i satisfies the condition (1.6), the adjoint $\gamma_i^* : L \to \operatorname{Sub} G_i$ is injective by Corollary 1.8. Thus we may identify every $x \in L$ with the subgroup $\gamma_i^*(x)$ of G_i and consider the lattice L to be a complete meet-subsemilattice of $\operatorname{Sub} G_i$. After the indentification, γ_i^* is the inclusion mapping $\iota : L \to \operatorname{Sub} G_i$ and the group valuation γ_i becomes (the reduced version of) the adjoint of ι.

By Lemma 3.1, there exists an extension G_{i+1} of G_i, a group valuation $\gamma_{i+1} : G_{i+1} \to L$ extending γ_i, satisfying the conditions (1.6) and (1.7), and such that

$$(\forall\, g, h \in G_i)\,(\,\gamma_i(g) \leq \gamma_i(h) \;\Rightarrow\; g \in \gamma_{i+1}^*(0_L) \vee \langle h \rangle\,).$$

Thus we have constructed by induction on i a sequence of groups

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_i \subseteq G_{i+1} \subseteq \cdots$$

and group valuations $\gamma_i : G_i \to L$ satisfying all the assumptions of Lemma 1.9. We set $G = \bigcup_{i=0}^{\infty} G_i$. The common extension $\gamma : G \to L$ satisfies all the three conditions (1.6), (1.7) and (1.8) by Lemma 1.9. Thus by Corollary 1.8, the adjoint $\gamma^* : L \to \operatorname{Sub} G$ of (the extended version of) γ is an isomorphism between L and the interval $[\gamma^*(0_L), G]$ in the subgroup lattice of G. Since the group G is a union of an increasing countable chain of finite groups, it is countable and locally finite. $\square$

Theorem 3.4. *Every algebraic lattice with at most countably many compact elements is isomorphic to an interval in the subgroup lattice of a countable locally finite group.*

Proof. Let L be an algebraic lattice with countably many compact elements. Denote by P the semilattice of compact elements of L and let

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_i \subseteq P_{i+1} \subseteq \cdots$$

be a sequence of finite $\langle \vee, 0 \rangle$-subsemilattices of P such that $P = \bigcup_{i=0}^{\infty} P_i$.

For every $i \geq 0$ we construct a finite group H_i and a group valuation $\beta_i : H_i \to P_i$ satisfying the condition (1.6) as we did in the first paragraph of the proof of Theorem 3.3.

We set $G_0 = H_0$ and $\gamma_0 = \beta_0$. Suppose that for some $i \geq 0$ we have already constructed finite groups G_j, group valuations $\gamma_j : G_j \to P_j$ satisfying the condition (1.6) for every $j = 0, 1, \ldots, i$, and such that γ_j extends γ_{j-1} and satisfies the condition (1.7) for every $j = 1, \ldots, i$, and finally,

$$(\forall\, g, h \in G_{j-1})\,(\,\gamma_{j-1}(g) \leq \gamma_{j-1}(h) \;\Rightarrow\; g \in \gamma_j^*(0_L) \vee \langle h \rangle\,),$$

also for every $j = 1, \dots, i$.

To construct γ_{i+1}, first define a group valuation $\delta_{i+1} : G_i \times H_{i+1} \to P_{i+1}$ as the product of γ_i and β_{i+1}. The valuation δ_{i+1} satisfies the condition (1.6) since β_{i+1} does.

Since δ_{i+1} satisfies the condition (1.6), the adjoint mapping $\delta^*_{i+1} : P_{i+1} \to \operatorname{Sub}(G_i \times H_{i+1})$ is injective by Corollary 1.8. Thus we may identify every $x \in P_{i+1}$ with the subgroup $\delta^*_{i+1}(x)$ of $G_i \times H_{i+1}$ and consider the lattice P_{i+1} to be a complete meet-subsemilattice of $\operatorname{Sub}(G_i \times H_{i+1})$. After the indentification, δ^*_{i+1} is the inclusion mapping $\iota : P_{i+1} \to \operatorname{Sub}(G_i \times H_{i+1})$ and the group valuation δ_{i+1} becomes the (reduced version of the) adjoint of ι.

By Lemma 3.1, there exists an extension G_{i+1} of $G_i \times H_{i+1}$, a group valuation $\gamma_{i+1} : G_{i+1} \to P_{i+1}$ extending δ_{i+1}, satisfying the conditions (1.6) and (1.7), and such that for every $(g_1, g_2), (h_1, h_2) \in G_i \times H_{i+1}$,

$$\delta_{i+1}(g_1, g_2) \leq \delta_{i+1}(h_1, h_2) \Rightarrow (g_1, g_2) \in \gamma^*_{i+1}(0_L) \vee \langle (h_1, h_2) \rangle.$$

By the natural assignement $g \mapsto (g, 1)$ we get that G_i is a subgroup of G_{i+1}. For every $g \in G_i$, $\gamma_{i+1}(g) = \delta_{i+1}(g, 1) = \gamma_i(g) \vee \beta_{i+1}(1) = \gamma_i(g) \vee 0_{P_{i+1}}$, hence γ_{i+1} extends γ_i.

If $g, h \in G_i$ are such that $\gamma_i(g) \leq \gamma_i(h)$, then also $\delta_{i+1}(g, 1) \leq \delta_{i+1}(h, 1)$, hence $(g, 1) \in \gamma^*_{i+1}(0_L) \vee \langle (h, 1) \rangle$. Thus also $g \in \gamma^*_{i+1}(0_L) \vee \langle h \rangle$.

It follows that the sequence of group valuations $\gamma_i : G_i \to P_i$ satisfies all the assumptions of Lemma 1.10, hence their common extension $\gamma : G \to P$, where $G = \bigcup_{i=0}^{\infty} G_i$, satisfies the conditions (1.6), (1.7) and (1.8), and the adjoint $\gamma^* : J(P) \to \operatorname{Sub} G$ is an isomorphism between $J(P)$ and the interval $[\gamma^*(0_P), G]$ in $\operatorname{Sub} G$. Thus also the lattice L is isomorphic to an interval in the subgroup lattice of a countable locally finite group. □

In [4], P. Hall constructed a universal countable locally finite group that contains $2^{\aleph_0}$ different copies of every countable locally finite group. Since the intervals we constructed in the proof of Theorems 3.3 and 3.4 always contain the whole group, we get the following theorem.

Theorem 3.5. *The subgroup lattice of Hall's universal countable locally finite group contains $2^{\aleph_0}$ different intervals isomorphic to any algebraic lattice with at most countably many compact elements.*

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