Avoidable structures, I: Finite ordered sets, semilattices and lattices

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Abstract. We find all finite unavoidable ordered sets, finite unavoidable semilattices and finite unavoidable lattices.

1. Introduction

If a structure $\mathbf{A}$ cannot be embedded into a structure $\mathbf{B}$, we also say that $\mathbf{B}$ *avoids* $\mathbf{A}$. Two structures are said to be *incomparable* if each of them avoids the other.

Let K be a class of structures of a given signature. A finite structure $\mathbf{A} \in K$ is said to be *avoidable in* K if there exists an infinite set S of pairwise incomparable finite structures from K such that S avoids $\mathbf{A}$ (i.e., every structure from S avoids $\mathbf{A}$).

For example, it is easy to see that in the class of all groupoids, every finite algebra is avoidable. Since a nontrivial finite group is avoided by a set of sufficiently large cyclic groups of prime orders, the only unavoidable finite algebra in the class of groups is the one-element group. The same argument shows that the only unavoidable finite algebra in the class of semigroups is the one-element semigroup.

For other classes K of algebras, the question of determining which finite objects are avoidable in K may be more interesting. In this paper we are going to answer this question for the classes of ordered sets, semilattices, lattices and $0,1$-lattices.

It is easy to see that a finite structure $\mathbf{A} \in K$ is unavoidable in K if and only if the class of finite structures from K avoiding $\mathbf{A}$ is well-quasi-ordered by the relation of embeddability.

A quasi-ordered set $\mathbf{A} = \langle A, \leq \rangle$ is said to be *well-quasi-ordered* if it contains no infinite antichains and no infinite descending chains. By a *bad sequence* for $\mathbf{A}$ we mean an infinite sequence a_n $(n \geq 0)$ of elements of A such that $a_i \nleq a_j$ whenever $i < j$. It is easy to see that a quasi-ordered set is well-quasi-ordered if and only if it has no bad sequence. We will need the following theorem, which is one of the basic well-known results in the theory of well-quasi-orderings.

1991 *Mathematics Subject Classification*: 06A12.

Key words and phrases: ordered set, semilattice, lattice, avoidable.

While working on this paper, the second author (Ježek) and the third author (McKenzie) were supported by US NSF grant DMS-0604065. The second author was also supported by the Grant Agency of the Czech Republic, grant #201/05/0002 and by the institutional grant MSM0021620839 financed by MSMT.

1.1. Theorem. *Let $\mathbf{A} = \langle A, \leq \rangle$ be a quasi-ordered set containing no infinite descending chains. If $\mathbf{A}$ is not well-quasi-ordered, then there exists a bad sequence a_n $(n \geq 0)$ for $\mathbf{A}$ such that the set $\{a \in A : a < a_i$ for some $i\}$ is well-quasi-ordered by $\leq$.*

Proof. Define a_n by induction on n as follows: if a_i has been defined for all $i < n$ then let a_n be a minimal element (with respect to $\leq$) with the property that there exists a bad sequence $b_0, b_1, \dots$ with $b_i = a_i$ for all $i \leq n$. Clearly, a_n $(n \geq 0)$ is a bad sequence. Put $B = \{a \in A : a < a_i \text{ for some } i\}$ and suppose that there is a bad sequence $c_0, c_1, \dots$ of elements of B. Denote by n the least number such that $c_i < a_n$ for some i. Denote by m the least number such that $c_m < a_n$. It is easy to see that the sequence $a_0, \dots, a_{n-1}, c_m, c_{m+1}, \dots$ is bad, contradicting the minimality of a_n. □

A bad sequence a_n with the property formulated in the conclusion of 1.1 will be called a *minimal bad sequence* for $\mathbf{A}$. Observe that an infinite subsequence of a minimal bad sequence is also minimal bad.

It may not be completely clear what we mean by an embedding of a structure into a structure, if the structures are not algebras. By an embedding of an ordered set $\mathbf{A}$ into an ordered set $\mathbf{B}$ we mean an injective mapping f of A into B such that $x \leq y$ in $\mathbf{A}$ if and only if $f(x) \leq f(y)$ in $\mathbf{B}$.

By a semilattice we shall mean a meet-semilattice. The least element of a finite semilattice $\mathbf{A}$ is denoted by $0_{\mathbf{A}}$, or just by 0.

By a *forest* we mean an ordered set every principal order ideal of which is a chain. A *tree* is a forest with a least element. Dual forests and dual trees are defined dually. Observe that every finite tree is a semilattice; finite forests can be considered as partial meet-semilattices, where $a \wedge b$ is defined if and only if the two elements belong to the same tree-component.

We will make use of the following Kruskal's Tree Theorem [3].

1.2. Theorem. *Let $\langle \mathbf{Q}, \leq \rangle$ be a well-quasi-ordered set. Then the class of finite $\mathbf{Q}$-labeled forests $F = \langle F, \leq, \lambda \rangle$ (λ is a mapping of F into Q) is well-quasi-ordered with respect to the quasi-ordering $\leq$ defined as follows: $F_1 \leq F_2$ if and only if there exists an injective mapping h of F_1 into F_2 such that $\lambda(a) \leq \lambda(h(a))$ in $\mathbf{Q}$ for all $a \in F_1$, and $a \wedge b = c$ in F_1 if and only if $h(a) \wedge h(b) = h(c)$ in F_2.*

Sometimes we need to work with proper classes of finite structures as if they were sets. This could be easily avoided, but with technical difficulties, if instead of the structures, their isomorphism types were considered.

This paper is rather self-contained; see [5] for the necessary background in universal algebra and lattice theory. Also, see Kriz and Thomas [2] and Kruskal [4] for the general theory of well-quasi-orderings.

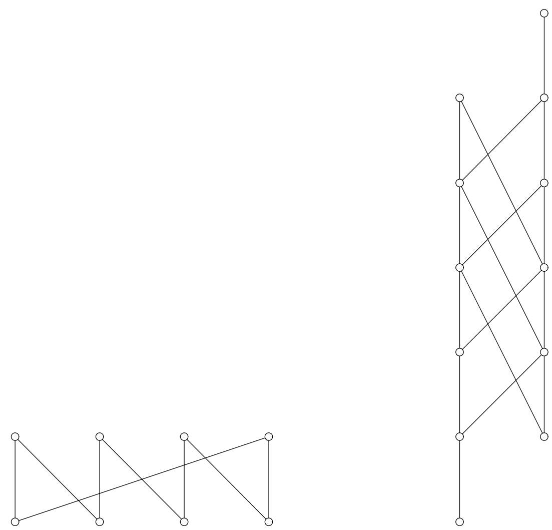
There is a related paper [1], in which we solve similar problems for the class of finite distributive lattices.

2. Ordered sets

For every integer $n \geq 3$ denote by $\mathbf{C}_n$ the ordered set with elements $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ and covers $a_i < b_i$ $(1 \leq i \leq n)$, $a_i < b_{i+1}$ $(1 \leq i \leq n-1)$ and $a_n < b_1$. These ordered sets are called *crowns*.

For every integer $n \geq 4$ denote by $\mathbf{D}_n$ the ordered set with elements $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ and covers $a_i < a_{i+1}$ $(1 \leq i < n)$, $b_i < b_{i+1}$ $(1 \leq i < n)$, $a_i < b_i$ $(1 < i < n)$ and $b_i < a_{i+3}$ $(1 \leq i \leq n-3)$.

For example, the ordered sets $\mathbf{C}_4$ and $\mathbf{D}_6$ are shown in Fig. 1.



$\mathbf{C}_4$ $\mathbf{D}_6$

Fig. 1

2.1. Proposition. *The ordered sets* $\mathbf{C}_n$ $(n \geq 3)$ *are pairwise incomparable. Likewise, the ordered sets* $\mathbf{D}_n$ $(n \geq 4)$ *are pairwise incomparable.*

Proof. For the first sequence of ordered sets the statement is easier, so we will prove it for the second only. Suppose, on the contrary, that there are indexes $4 \leq k < m$ such that there exists an embedding f of $\mathbf{D}_k$ into $\mathbf{D}_m$. In every $\mathbf{D}_n$, the element b_1 is the only non-maximal element which is incomparable with at least three elements, so we must have $f(b_1) = b_1$. The three elements incomparable with b_1 form a chain, so we must have $f(a_i) = a_i$ for $i \leq 3$. The only element incomparable with a_3, except b_1, is the element b_2; so, $f(b_2) = b_2$. The only element incomparable with b_2, except a_3, is the element a_4; so, $f(a_4) = a_4$. We can proceed in this way to conclude that $f(a_i) = a_i$ for all $i \leq k$. But a_k has three incomparable elements in $\mathbf{D}_k$, while only two incomparable elements in $\mathbf{D}_m$, a contradiction. □

We denote by $\mathbf{N}$ the ordered set with four elements a, b, c, d and the only covers $a < b$, $c < b$ and $c < d$.

2.2. Lemma. *Every unavoidable finite ordered set is embeddable into* $\mathbf{N}$.

Proof. It follows from 2.1 that an unavoidable ordered set must be isomorphic to a proper subset of $\mathbf{C}_n$ for some n; so, it must be a subset of a fence. It follows also from 2.1 that an unavoidable ordered set must be properly embeddable into $\mathbf{D}_m$ for some m. However, $\mathbf{D}_m$ has no three-element antichains and so it avoids all fences with more than four elements. Also, $\mathbf{D}_m$ avoids the cardinal sum of two copies of the two-element chain. Thus an unavoidable ordered set must be embeddable into the fence with four elements, i.e., the ordered set $\mathbf{N}$. □

2.3. Theorem. *A finite ordered set is unavoidable if and only if it is embeddable into* $\mathbf{N}$.

Proof. The direct implication follows from 2.2. The converse follows from the result of Pouzet [6] which says that the class of finite ordered sets avoiding $\mathbf{N}$ is well-quasi-ordered by embeddability; see also Thomassé [7] for a more general result. □

We want to thank a referee for providing us the information that the class of finite ordered sets avoiding $\mathbf{N}$ has a well-known and nice characterization (see for example Valdes, Tarjan, and Lawler [8] or Thomassé [7]). The class is the least class of finite ordered sets which contains the one-element ordered set and is closed under both the disjoint unions and ordinal sums of ordered sets.

3. Semilattices

For every integer $n \geq 3$ denote by $\mathbf{C}_n^0$ the crown $\mathbf{C}_n$ with the least element added.

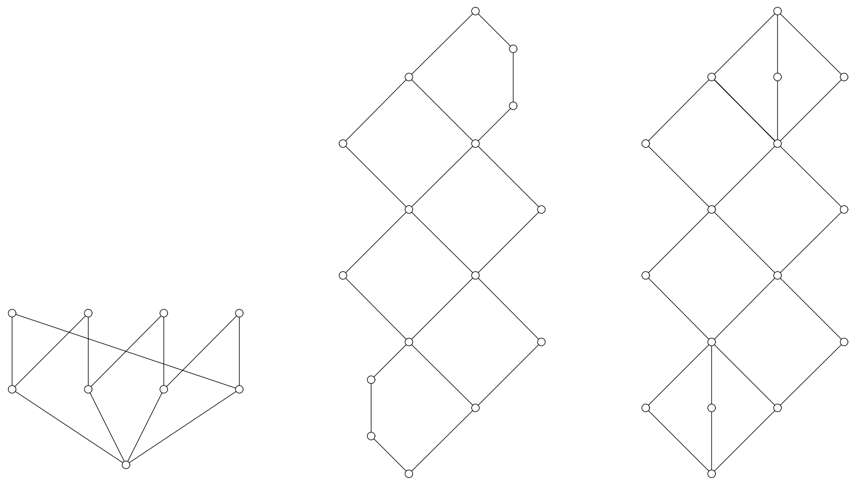
For every integer $n \geq 2$ denote by $\mathbf{P}_n$ and $\mathbf{Q}_n$ the semilattices with elements $0, c, d, 1, e, f$, a_i $(0 \leq i \leq n)$ and b_j $(1 \leq j \leq n-1)$ and covers

1. $a_i < a_{i+1}$ $(0 \leq i \leq n-1)$,
2. $a_i < b_{i+1} < a_{i+2}$ $(0 \leq i \leq n-2)$,
3. $0 < a_0$, $0 < c < d < a_1$, $a_n < 1$, $a_{n-1} < e < f < 1$ for $\mathbf{P}_n$,
4. $0 < a_0$, $0 < c < a_1$, $0 < d < a_1$, $a_n < 1$, $a_{n-1} < e < 1$, $a_{n-1} < f < 1$ for $\mathbf{Q}_n$.

For example, $\mathbf{C}_4^0$, $\mathbf{P}_5$ and $\mathbf{Q}_5$ are pictured in Fig. 2.

3.1. Proposition. *The semilattices* $\mathbf{C}_n^0$ $(n \geq 3)$ *are pairwise incomparable. The semilattices* $\mathbf{P}_n$ $(n \geq 2)$ *are pairwise incomparable. The semilattices* $\mathbf{Q}_n$ $(n \geq 2)$ *are pairwise incomparable. Consequently, every unavoidable semilattice is isomorphic to a proper subsemilattice of* $\mathbf{C}_n^0$ *for some* $n \geq 3$*; it is also isomorphic to a proper subsemilattice of* $\mathbf{P}_m$ *for some* $m \geq 2$ *and to a proper subsemilattice of* $\mathbf{Q}_k$ *for some* $k \geq 2$*. In particular, every unavoidable semilattice* $\mathbf{S}$ *is of height at most* 2*, every element of* $\mathbf{S}$ *has at most two covers and at most two subcovers and* S *has no three-element antichains.*

Proof. It is obvious. □

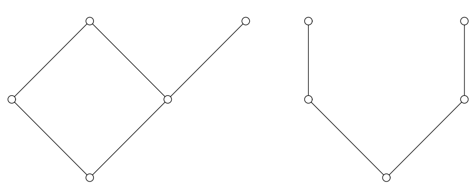


$\mathbf{C}_4^0$ $\mathbf{P}_5$ $\mathbf{Q}_5$

Fig. 2

3.2. Proposition. *Every unavoidable semilattice is isomorphic to a subsemilattice of the semilattice* $\mathbf{N}_0$ *with elements* a, b, c, d, e *and covers* $a < b < c$, $a < d < c$, $d < e$ (*the ordered set* $\mathbf{N}$ *with the least element added*).

Proof. Denote by $\mathbf{U}$ the semilattice with elements a, b, c, d, e and covers $a < b < c$, $a < d < e$. Let $\mathbf{S}$ be an avoidable semilattice. By 3.1, $\mathbf{S}$ is of height at most 2 and $\mathbf{S}$ has no three-element antichains. These two properties of $\mathbf{S}$ imply that $\mathbf{S}$ is either isomorphic to a sublattice of $\mathbf{N}_0$ or $\mathbf{S}$ is isomorphic to $\mathbf{U}$. The conclusion of the proposition now follows from the observation that the semilattices $\mathbf{Q}_n$ $(n \geq 2)$ avoid $\mathbf{U}$. □



$\mathbf{N}_0$ $\mathbf{U}$

Fig. 3

The ordinal sum of two semilattices $\mathbf{A}, \mathbf{B}$ is their disjoint union with the operation defined so that both $\mathbf{A}$ and $\mathbf{B}$ are subsemilattices and $a < b$ whenever $a \in A$ and $b \in B$.

The 0-amalgamated sum of two finite semilattices $\mathbf{A}, \mathbf{B}$ is their almost disjoint union, with only the two least elements identified, such that both $\mathbf{A}$ and $\mathbf{B}$ are subsemilattices and $a \wedge b = 0$ (where 0 is the common least element) whenever $a \in A$ and $b \in B$. (For example, the semilattice $\mathbf{U}$ is the 0-amalgamated sum of two three-element chains.)

3.3. Theorem. *The class of finite semilattices avoiding $\mathbf{N}_0$ is the least class of semilattices containing the one-element semilattice and closed under ordinal and* 0*-amalgamated sums.*

Proof. Denote by K_1 the first class and by K_2 the second. Clearly, $K_2 \subseteq K_1$. We need to prove that $K_1 \subseteq K_2$.

Claim 1. Let a be the meet of all maximal elements of a semilattice $\mathbf{A} \in K_1$. Then a is comparable with every element of A. Suppose that there is an element b incomparable with a. There exist two maximal elements c, d of A such that $b < c$ and $b \nleq d$. One can easily check that the five elements b, c, d, $c \wedge d$ and $b \wedge d = b \wedge (c \wedge d)$ constitute a subsemilattice isomorphic with $\mathbf{N}_0$, a contradiction.

Claim 2. Let $\mathbf{A} \in K_1$ have at least two maximal elements and let the meet of the set M of maximal elements of $\mathbf{A}$ be 0*, the least element. Define a relation $\sim$ on M by $x \sim y$ iff $x \wedge y \neq 0$. Then $\sim$ is an equivalence with at least two blocks.* Clearly, the relation $\sim$ is reflexive and symmetric. Let $a \sim b$ and $b \sim c$, so that $a \wedge b > 0$ and $b \wedge c > 0$. If $a \wedge b$ is incomparable with $b \wedge c$ then the five elements b, c, $a \wedge b$, $b \wedge c$ and $a \wedge b \wedge c$ constitute a subsemilattice isomorphic with $\mathbf{N}_0$, a contradiction. Hence $a \wedge b$, $b \wedge c$ are comparable and $a \wedge b \wedge c$ is the smaller of the two elements. We get $a \wedge c > 0$ and thus $a \sim c$. Thus $\sim$ is an equivalence. Suppose that it has only one block, i.e., $x \wedge y > 0$ for all $x, y \in M$. Let M_0 be a maximal subset of M such that $\bigwedge M_0 > 0$. Then M_0 has at least two elements and there is an element $a \in M - M_0$. Take an element $b \in M_0$. We have $0 < a \wedge b < a$ and $a \wedge \bigwedge M_0 = 0$, so that the five elements a, b, 0, $a \wedge b$ and $\bigwedge M_0$ constitute a subsemilattice isomorphic with $\mathbf{N}_0$, a contradiction.

Suppose that K_1 is not contained in K_2. Let $\mathbf{A}$ be a semilattice of the least cardinality belonging to $K_1 - K_2$. Clearly, A has at least two elements. Denote by M the set of maximal elements of $\mathbf{A}$ and put $a = \bigwedge M$. By Claim 1, a is comparable with every element of A. Thus if $a > 0$ then $\mathbf{A}$ is the ordinal sum of its two proper subsemilattices $A - B$ and B, where $B = \{x \in A : x \geq a\}$; but both $A - B$ and B belong to K_2 by the minimality of $\mathbf{A}$ and then also $\mathbf{A} \in K_2$ (since K_2 is closed under ordinal sums), a contradiction. Hence $\bigwedge M = 0$ and M has at least two elements. Define the equivalence $\sim$ on M as in Claim 2. Denote by $B_1, \dots, B_n$ the blocks of $\sim$, so that $n \geq 2$. For every $i = 1, \dots, n$ denote by F_i the set of the elements $x \in A$ for which there exists an element $y \in B_i$ with $x \leq y$. Then $\mathbf{F}_i$ are subsemilattices of $\mathbf{A}$; by the minimality of $\mathbf{A}$, $\mathbf{F}_i \in K_2$ for all i. Clearly, if $x \in F_i$ and $y \in F_j$ where $i \neq j$, then $x \wedge y = 0$. Thus $\mathbf{A}$ is the 0-amalgamated sum of its subsemilattices $\mathbf{F}_i \in K_2$; since K_2 is closed under 0-amalgamated sums, we get $\mathbf{A} \in K_2$. □

Let $\mathbf{A}$ be a semilattice avoiding $\mathbf{N}_0$. We denote by $T_{\mathbf{A}}$ the set of the elements $x \in A$ such that for every $y \in A$, either $x \wedge y = 0$ or y is comparable with x. Notice that $T_{\mathbf{A}}$ is a subsemilattice of $\mathbf{A}$.

3.4. Theorem. *Let $\mathbf{A}$ be a nontrivial finite semilattice avoiding $\mathbf{N}_0$. Then $T_{\mathbf{A}} - \{0\}$ is a dual forest containing all atoms of $\mathbf{A}$. For every $x \in A$ the set $\{y \in T_{\mathbf{A}} : y \leq x\}$ contains a largest element. Denote this largest element by $\tau(x)$. Then τ is a meet-preserving mapping of $\mathbf{A}$ onto $T_{\mathbf{A}}$.*

For every element $t \in T_{\mathbf{A}}$ denote by $\mathbf{A}_t$ the subsemilattice $\tau^{-1}\{t\}$ of $\mathbf{A}$. Then $A_0 = \{0\}$, $t < s$ implies that every element of A_t is less than s, $t \neq s$ implies that $A_t \cap A_s$ is empty, and if $x \in A_t$ and $y \in A_s$ where t, s are incomparable then $x \wedge y = 0$.

Proof. Since any two incomparable elements of $T_{\mathbf{A}}$ meet to 0, for every element $a \in T_{\mathbf{A}} - \{0\}$ the set $\{x \in A : x \geq a\}$ is a chain. This means that $T_{\mathbf{A}} - \{0\}$ is a dual forest. Clearly, every atom belongs to $T_{\mathbf{A}}$.

Let $x \in A$ and a, b be two incomparable elements from $T_{\mathbf{A}}$ below x. The join c of a and b exists and is also below x. In order to prove that there is a largest element in $T_{\mathbf{A}}$ below x, it is sufficient to prove that $c \in T_{\mathbf{A}}$. Suppose, on the contrary, that there exists an element $d \in A$ incomparable with c, such that $c \wedge d > 0$. If d is incomparable with a then $a \wedge d = 0$ (since $a \in T_{\mathbf{A}}$) and thus the five elements $a, c, d, 0, c \wedge d$ constitute a subsemilattice isomorphic with $\mathbf{N}_0$, a contradiction. Similarly, d cannot be incomparable with b. Since c is incomparable with d, we have $d \not\leq a$ and $d \not\leq b$. Thus $d \geq a$ and $d \geq b$, so that $d \geq c$, a contradiction.

It is easy to see that $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$ for all $x, y \in A$.

The set A_0 contains no element other than 0, because every non-zero element is above an atom. If $t < s$ where $t, s \in T_{\mathbf{A}}$ and $x \in A_t$, then either $t = 0$ in which case $x = 0 < s$, or else s is comparable with x and thus $x < s$. It is clear that if t, s are distinct then $A_t \cap A_s = \emptyset$. Let t, s be two incomparable elements of $T_{\mathbf{A}}$ and suppose that there exist elements $x \in A_t$ and $y \in A_s$ with $x \wedge y > 0$. Since $t \wedge s = 0$, we have $\{x, y\} \neq \{t, s\}$ and we can assume without loss of generality that $x > t$. Since t is incomparable with y, we have $t \wedge y = 0$. But then the elements $x, t, 0, x \wedge y, y$ constitute a subsemilattice isomorphic with $\mathbf{N}_0$, a contradiction. □

3.5. Theorem. *Let $\mathbf{A}, \mathbf{B}$ be two semilattices avoiding $\mathbf{N}_0$. Let $\lambda : T_{\mathbf{A}} \to T_{\mathbf{B}}$ be an order embedding with $\lambda(0) = 0$, and for every $t \in T_{\mathbf{A}}$ let φ_t be a semilattice embedding of $\mathbf{A}_t$ into $\mathbf{B}_{\lambda(t)}$. Then $\varphi = \bigcup\{\varphi_t : t \in T_{\mathbf{A}}\}$ is a semilattice embedding of $\mathbf{A}$ into $\mathbf{B}$.*

Proof. It follows easily from 3.4. □

3.6. Theorem. *A finite semilattice is unavoidable if and only if it is isomorphic to a subsemilattice of $\mathbf{N}_0$.*

Proof. By 3.2, it remains to prove that the class of semilattices avoiding $\mathbf{N}_0$ is well-quasi-ordered by embeddability. Suppose that it is not. By 1.1 there exists a minimal bad sequence $\mathbf{B}_n$ ($n \geq 0$) for this class. Denote by U the set of the semilattices that are a proper subsemilattice of $\mathbf{B}_n$ for some n. So, U is well-quasi-ordered. By 3.4, every $\mathbf{B}_n$ can be considered as a dual forest $T_{\mathbf{B}_n}$ labeled by its proper subsemilattices, with an extra least element added. The labels are elements of the well-quasiordered set U. It follows from (the dual of) 1.2 that $T_{\mathbf{B}_n}$ is embeddable into $T_{\mathbf{B}_m}$ (as a labeled forest) for some $n < m$. By 3.5, this implies $\mathbf{B}_n \leq \mathbf{B}_m$, a contradiction. □

4. Lattices

4.1. Theorem. *There are, up to isomorphism, only seven unavoidable lattices: the four chains of height at most three, the four-element Boolean lattice, the four-element Boolean lattice with a top element added and the four-element Boolean lattice with a bottom element added.*

Proof. The amalgamated ordinal sum of two finite lattices $\mathbf{A}, \mathbf{B}$ is the almost disjoint union of the two lattices, with only the top element of $\mathbf{A}$ identified with the bottom element of $\mathbf{B}$, such that $\mathbf{A}$ and $\mathbf{B}$ are sublattices and every element of A is below every element of B. Denote by $\mathbf{R}_n$ ($n \geq 1$) the amalgamated ordinal sum of one copy of the pentagon, n copies of the four-element Boolean lattice, and one additional copy of the pentagon. Denote by $\mathbf{S}_n$ ($n \geq 1$) the amalgamated ordinal sum of one copy of the lattice $\mathbf{M}_3$ (the five-element modular but non-distributive lattice), n copies of the four-element Boolean lattice, and one additional copy of $\mathbf{M}_3$. For example, $\mathbf{R}_2$ and $\mathbf{S}_2$ are pictured in Fig. 4.

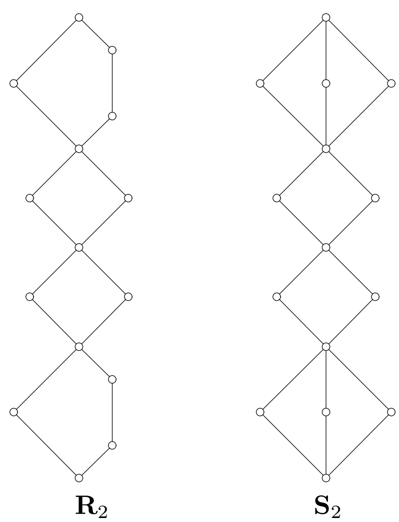


Fig. 4

The lattices $\mathbf{S}_n$ are not pairwise incomparable as semilattices, but they are pairwise incomparable as lattices. Suppose, on the contary, that there is a lattice embedding f of $\mathbf{S}_n$ into $\mathbf{S}_m$, where $n < m$. Clearly, the bottom copy of $\mathbf{M}_3$ in $\mathbf{S}_n$ must be mapped onto the bottom copy of $\mathbf{M}_3$ in $\mathbf{S}_m$, and similarly the top copy must be mapped onto the top copy. Denote by $a_0, a_1, \ldots, a_{n+2}$ the chain of the elements of $\mathbf{S}_n$ that are comparable with each element of $\mathbf{S}_n$, and denote by $b_0, b_1, \ldots, b_{m+2}$ the similar chain in $\mathbf{S}_m$. Thus $f(a_1) = b_1$. The two covers of a_1 are the only pair of incomparable elements that meet to a, so they must be mapped onto the two covers of b_1. But then their join must be mapped onto the join of the images, i.e., $f(a_2) = b_2$. Similarly $f(a_3) = b_3, \ldots, f(a_{n+1}) = b_{n+1}$. But a_{n+1} is the meet of three incomparable elements while b_{n+1} is not, a contradiction.

Quite similarly, also the lattices $\mathbf{R}_n$ are pairwise incomparable. Clearly, the crowns with both the top and the bottom elements added are also pairwise incomparable lattices. Thus every unavoidable lattice is of height at most three, is a sublattice of some $\mathbf{R}_n$ and is a sublattice of some $\mathbf{S}_n$. The only such lattices are the (up to isomorphism) seven proper sublattices of the direct product D of the two-element chain with the three-element chain, i.e., the above mentioned seven lattices.

It remains to prove that the seven lattices are unavoidable. The four-element chain is unavoidable because the only lattices avoiding it are the lattices of height at most two, and these are comparable with each other. The four-element Boolean lattice is unavoidable because the only lattices avoiding it are the chains, and chains are again comparable with each other.

Lattices avoiding the four-element Boolean lattice with top element added are precisely the lattices that become trees after removing the top element. Thus the unavoidability of this five-element lattice follows from 1.2.

The remaining lattice is dual to this last one, so it is also unavoidable. □

5. $0,1$-lattices

5.1. Theorem. *There are, up to isomorphism, precisely three finite unavoidable* $0,1$-*lattices: the three chains with at most four but at least two elements.*

Proof. Similarly as in the case of lattices, every unavoidable $0,1$-lattice must be one of the seven unavoidable lattices. The one-element $0,1$-lattice is avoidable since it is not embeddable into any nontrivial $0,1$-lattice. The four-element Boolean $0,1$-lattice is avoidable because it is avoided by the $0,1$-lattices $\mathbf{R}_n$. The four-element Boolean lattice with the top element added is, as a $0,1$-lattice, avoidable because it is avoided by the $0,1$-lattices that are the ordinal sum of the one-element lattice with the lattice $\mathbf{R}_n$, and these $0,1$-lattices are also pairwise incomparable. The four-element Boolean lattice with the bottom element added is avoidable by a similar reason. □

6. Four open problems

Naturally related to the above results are the following four problems.

Let K be any of the four classes: ordered sets, semilattices, lattices, and $0,1$-lattices. Decide which finite sets of finite structures from K are avoidable in K, i.e., which finite sets of finite structures from K can be extended to an infinite set of pairwise incomparable finite structures from K.

These questions (also for more general classes) are particularly interesting in the case when the class is a locally finite universal class. A class K of structures is called *locally finite* if every finitely generated structure from K is finite; it is called *universal* if it is axiomatizable by a set of universal sentences (i.e., universal closures of quantifier-free formulas). A locally finite class is universal if and only if it is closed under substructures and contains every structure $\mathbf{A}$ such that every finite substructure of $\mathbf{A}$ belongs to K. Notice that the four classes investigated in this paper are all universal, and two of them are locally finite: the class of ordered sets and the class of semilattices.

Let K be a locally finite universal class of structures of a finite signature. Denote by F the set of finite members of K and consider F as a quasi-ordered set with respect to the quasi-ordering $\leq$, where $\mathbf{A} \leq \mathbf{B}$ if and only if $\mathbf{A}$ is embeddable into $\mathbf{B}$. It is easy to see that the mapping $U \mapsto F \cap U$ is an isomorphism of the lattice of universal subclasses of K onto the lattice of order ideals of F. (Strictly speaking, F is not a set and it is illegitimate to form the lattice of proper subclasses of K; this inconvenience could be corrected easily but at the cost of technical difficulties.)

For an antichain X in F denote by $F \ominus X$ the set of the structures from F that avoid all members of X. This is an order ideal of F. Since descending chains in F are all finite, every order ideal of F can be expressed as $F \ominus X$ for some antichain X. It is natural to ask what can be said, in terms of the size of X and the order properties of $F \ominus X$, about the universal subclass U of K corresponding to the order ideal $F \ominus X$. The following five observations are answers to such questions.

(1) U has $2^{\aleph_0}$ universal subclasses if and only if $F \ominus X$ is not well-quasi-ordered, if and only if $F \ominus X$ contains an infinite antichain.

(2) U has at most $\aleph_0$ universal subclasses if and only if $F \ominus X$ is well-quasi-ordered.

(3) U is finitely axiomatizable relative to K if and only if X is finite.

(4) Every universal subclass of U, including U, is finitely axiomatizable relative to K if and only if X is finite and $F \ominus X$ is well-quasi-ordered.

In order to formulate the last observation, we need to state some definitions. A formula (not necessarily a sentence) is said to be valid in K if it is satisfied in every structure from K under any interpretation of its free variables. A *clause* is a disjunction of atomic formulas and/or their negations. By a K-*maximal clause* we mean a clause C such that C is not valid in K but whenever f is an atomic formula containing no variables other than those in C then either $C \vee f$ or $C \vee \neg f$ is valid in K.

(5) Let K be a locally finite quasivariety of a finite signature. U is axiomatizable relative to K by a single K-maximal clause while its every proper universal subclass is finitely axiomatizable relative to K, if and only if $|X| = 1$ and $F \ominus X$ is well-quasi-ordered.

The proofs of (1) through (4) are simple. Here is a hint for the proof of (5).

Let U be axiomatized relative to K by a K-maximal clause C. Denote by Y the set of variables occurring in C and denote by Γ the set of the atomic formulas f such that $\neg f$ occurs in C. Denote by A the finitely presented structure in K with defining relations Γ on the set Y. One can check that a structure from K belongs to U if and only if it avoids A.

Conversely, let $X = \{A\}$ and let $F \ominus X$ be well-quasi-ordered. Let us fix a bijection β of a set Y of variables onto the set A. There is a finite set T of terms over Y such that every term over Y is K-equivalent to precisely one term from T. Denote by B the set of atomic formulas with the term-components all in T. Denote by C the clause $\neg f_1 \vee \cdots \vee \neg f_n \vee g_1 \vee \cdots \vee g_m$ where f_i are all the atomic formulas from B that are satisfied in A under the interpretation β and g_j are all the remaining atomic formulas from B. One can check that C is a K-maximal clause and U is axiomatized relative to K by (the universal closure of) C.

One can now also observe that a universal subclass of K is finitely axiomatizable relative to K if and only if it is axiomatized relative to K by a finite set of K-maximal clauses.

Theorems 2.3 and 3.6 characterize those finite ordered sets or finite semilattices for which the corresponding universal classes of ordered sets or semilattices, respectively, possess the property described in (5). The first four observations are a motivation for our above formulated open problems in the cases of ordered sets and semilattices.

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