# Avoidable structures, II: finite distributive lattices and nicely-structured ordered sets

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Abstract. Let $\langle \mathcal{D}, \leq \rangle$ be the ordered set of isomorphism types of finite distributive lattices, where the ordering is by embeddability. We characterize the order ideals in $\langle \mathcal{D}, \leq \rangle$ that are well-quasi-ordered by embeddability, and thus characterize the members of $\mathcal{D}$ that belong to at least one infinite anti-chain in $\mathcal{D}$.

## 1. Introduction

A quasi-ordering (or pre-ordering) of a set $W$ is a relation $\leq$ that is reflexive over $W$ and transitive. A quasi-ordered set $\langle W, \leq \rangle$ is said to be *well-quasi-ordered* iff it has no infinite strictly decreasing sequence and no infinite anti-chain. We study in this paper the pre-ordering on the class $\mathcal{D}$ of all finite distributive lattices defined by $\mathbf{L} \leq \mathbf{L}'$ iff $\mathbf{L}$ is isomorphic to a sublattice of $\mathbf{L}'$. The order-ideals in this setting are the subclasses $\mathcal{T} \subseteq \mathcal{D}$ satisfying $\mathsf{S}(\mathcal{T}) = \mathcal{T}$ (closed under forming sublattices and their isomorphic images). A universal class of distributive lattices is a class defined by some set of universal first-order sentences, or in other terms, an axiomatizable class closed under subalgebras. Since distributive lattices are locally finite algebras of a finite signature, the map from universal classes to $\mathcal{S}$-closed subclasses of $\mathcal{D}$ defined by $\mathcal{K} \mapsto \mathcal{K} \cap \mathcal{D}$ is an isomorphism from the lattice of universal classes of distributive lattices onto the lattice of order-ideals of the pre-ordered class $\mathcal{D}$.

Throughout this paper, we identify members of $\mathcal{D}$ with their isomorphism types, and identify the class $\mathcal{D}$ with the set of isomorphism types of its members. This allows us, for example to write $\langle \mathcal{D}, \leq \rangle$ to denote the (denumerable) ordered set of isomorphism types of finite distributive lattices, ordered by embeddability. Without this convention, the exposition would become quite awkward. We believe it will cause no harm.

In this setting, a basic observation is that a universal class $\mathcal{K}$ of distributive lattices has only countably many universal subclasses iff $\mathcal{K} \cap \mathcal{D}$ is well-quasi-ordered by embeddability. The principal contribution of this paper is a characterization of all order-ideals in $\langle \mathcal{D}, \leq \rangle$ that are well-quasi-ordered.

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*Key words and phrases*: finite distributive lattice, finite ordered set, universal class, well-quasi-ordered set.

While working on this paper, the second author (Ježek) and the third author (McKenzie) were supported by US NSF grant DMS-0604065. The second author was also supported by the Grant Agency of the Czech Republic, grant #201/05/0002 and by the institutional grant MSM0021620839 financed by MSMT.

An element $a$ in $W$, where $\langle W, \leq\rangle$ is quasi-ordered, will be called *avoidable in* $\mathbf{W}$ iff the set $\{b \in W : a \not\leq b\}$ is not well-quasi-ordered. Since every principal order-ideal in $\langle \mathcal{D}, \leq\rangle$ is finite (up to the isomorphism of lattices), the avoidable members of $\langle \mathcal{D}, \leq\rangle$ are precisely those members that belong to some infinite anti-chain, and the unavoidable members are precisely those that belong to every $\mathcal{S}$-closed class that contains an infinite anti-chain. This paper is an offshoot of the previous paper J. Ježek, W. Dziobiak, R. McKenzie [1], in which we characterized the avoidable finite lattices, ordered sets, and semilattices. Then we turned to finite distributive lattices and found that the characterization problem here seemed to be difficult and deep. A characterization of the avoidable finite distributive lattices will follow from our main result.

Let $\langle W, \leq\rangle$ be a quasi-ordered set. By a *least complete infinite anti-chain* in $\langle W, \leq\rangle$ we will mean an infinite anti-chain $A$ such that $A$ is not a proper subset of any anti-chain of $\langle W, \leq\rangle$ and whenever $B$ is an infinite anti-chain then for every element $b$ of $B$ there exists an element $a$ of $A$ such that $a \leq b$.

Let $\langle W, \leq\rangle$ be a quasi-ordered set such that the order-ideal generated by any element of $W$ is finite up to the equivalence $x \leq y \leq x$. One can easily see that the following are true:

(1) $\langle W, \leq\rangle$ has a least complete infinite anti-chain if and only if the order-filter of all its avoidable elements has infinitely many non-equivalent minimal elements; moreover, in that case the collection of all these non-equivalent minimal avoidable elements is the (essentially unique) least complete infinite anti-chain of $\langle W, \leq\rangle$.

(2) If $U$ is a least complete infinite anti-chain of $\langle W, \leq\rangle$, then an element $a \in W$ is avoidable in $\langle W, \leq\rangle$ if and only if $a \geq u$ for some $u \in U$; it is unavoidable in $\langle W, \leq\rangle$ if and only if $a < u$ for some $u \in U$.

(3) If $U$ is a least complete infinite anti-chain of $\langle W, \leq\rangle$, then a finite anti-chain $V$ of $\langle W, \leq\rangle$ can be extended to an infinite anti-chain if and only if every element of $V$ is above some element of $U$.

Many quasi-ordered sets do not contain a least complete infinite anti-chain. For example, it follows from [1] that this is the case for the quasi-ordered sets of finite ordered sets, finite semilattices, and finite lattices (with respect to embeddability). But $\langle \mathcal{D}, \leq\rangle$ does contain it; we will find it in Corollary 6.6.

For ordered sets $\mathbf{A}$, $\mathbf{B}$ we shall write $\mathbf{A} +_c \mathbf{B}$ for the cardinal sum of $\mathbf{A}$ and $\mathbf{B}$ (which is the disjoint union of ordered subsets isomorphic to $\mathbf{A}$ and to $\mathbf{B}$ with no relations between the two parts). We shall write $\mathbf{A} +_o \mathbf{B}$ for the ordinal sum (in which a copy of $\mathbf{A}$ is placed under a disjoint copy of $\mathbf{B}$), and if $\mathbf{A}$ has a top element and $\mathbf{B}$ has a bottom element, we shall write $\mathbf{A} +'_o \mathbf{B}$ for the glued ordinal sum, in which the top element of $\mathbf{A}$ is identified with the bottom element of $\mathbf{B}$. Thus the operations $+_o$ and $+'_o$ are defined on $\mathcal{D}$.

It is easy to see that $\mathcal{D}$ is not well-quasi-ordered under embeddability. For any positive integer $n$, let $\mathbf{n}$ denote the $n$-element chain. Define $\mathbf{B}_{m,n} = \mathbf{m} \times \mathbf{n}$. Put $\mathbf{D}_0 = \mathbf{1}$, $\mathbf{D}_1 = \mathbf{B}_{2,2}$ and for $n \in \omega$ put

$$\mathbf{D}_n = \mathbf{D}_1 +'_o \mathbf{D}_1 +'_o \cdots +'_o \mathbf{D}_1$$

with $n$ copies of $\mathbf{D}_1$. Define

$$\mathbf{J}_n = \mathbf{B}_{2,3} +_o' \mathbf{D}_n +_o' \mathbf{B}_{2,3}\,.$$

Now $\{\mathbf{J}_n : n \in \omega\}$ is an infinite anti-chain in $\mathcal{D}$. Thus every unavoidable member of $\mathcal{D}$ must be isomorphic to a proper sublattice of some $\mathbf{J}_n$.

We shall prove the converse (Theorem 6.2): every proper sublattice of any $\mathbf{J}_n$ is unavoidable in $\mathcal{D}$. In fact, we prove a much stronger result (Theorem 6.3): an order-ideal $\mathcal{I}$ in $\langle \mathcal{D}, \leq \rangle$ is well-quasi-ordered iff there are only finitely many $n$ for which $\mathbf{J}_n \in \mathcal{I}$.

The following theorem, which is one of the basic results in the theory of well-quasi-order, will play an essential role. Its proof is not difficult.

**Theorem 1.1.** *Let $\mathbf{W} = \langle W, \leq \rangle$ be a quasi-ordered set containing no infinite descending chain. If $\mathbf{W}$ is not well-quasi-ordered, then there exists a sequence $\langle a_n : n < \omega \rangle$ of elements of $W$ such that $a_i \not\leq a_j$ whenever $i < j < \omega$ and such that the set $\{a \in W : a < a_i \text{ for some } i\}$ is well-quasi-ordered by $\leq$.*

A sequence with the property formulated in Theorem 1.1 will be called a *minimal bad sequence* in $\mathbf{W}$. We note that our main result, Theorem 6.3, can be reformulated as: if $\langle \mathbf{L}_n : n < \omega \rangle$ is any minimal bad sequence in $\langle \mathcal{D}, \leq \rangle$, then all but finitely many $\mathbf{L}_n$ are among the terms of the sequence $\langle \mathbf{J}_n : n < \omega \rangle$.

To prove that every order-ideal in $\langle \mathcal{D}, \leq \rangle$ that is not well-quasi-ordered contains infinitely many of the lattices $\mathbf{J}_n$, our strategy will be to develop, for each $N$, a structure theorem for the members of $\mathcal{D}$ that embed no member of the class $\{\mathbf{J}_n : n \geq N\}$. The structure theorem will also allow us to establish analogous results to the mentioned results about $\langle \mathcal{D}, \leq \rangle$ for the remaining three pre-orderings on $\mathcal{D}$ each of which is a modification of $\leq$ by imposing an additional condition requiring preservation of 0 and/or 1.

## 2. $\diamond$-decompositions

For an ordered set $P$ and set $X \subseteq P$, we write $X\!\downarrow$ for the set $\{p \in P : p \leq x \text{ for some } x \in X\}$, and write $X\!\uparrow$ for the set $\{p \in P : p \geq x \text{ for some } x \in X\}$. For $x \in P$, we write $x\!\downarrow$ for $\{x\}\!\downarrow$ and $x\!\uparrow$ for $\{x\}\!\uparrow$. Where $X, Y \subseteq P$, we write $X < Y$ to mean that for all $x \in X$ and $y \in Y$ holds $x < y$. If $Y = \{a\}$ then we write $X < a$.

A *cut-point* in an ordered set $P$, or in a lattice $\mathbf{A}$, is an element $u$ such that $P = u\!\downarrow \cup u\!\uparrow$, or $A = u\!\downarrow \cup u\!\uparrow$, respectively. A finite lattice $\mathbf{A}$ is $+_o'$-decomposable iff it has a cut-point $u$ that is neither the largest nor the least element of $\mathbf{A}$. A finite lattice $\mathbf{A}$ is $+_o$-decomposable iff it has cut-points $u, v$ with $u \prec v$.

The next definitions will be seldom used in this paper until §7, where we focus on the three pre-orderings $\leq_0$, $\leq_1$, $\leq_2$ of $\mathcal{D}$. Suppose that $\mathbf{A}, \mathbf{B} \in \mathcal{D}$. We say that $\mathbf{A}$ is a 0-sublattice of $\mathbf{B}$ provided that $\mathbf{A} \subseteq \mathbf{B}$ and the least element of $\mathbf{B}$ belongs to $\mathbf{A}$. An embedding $f : \mathbf{A} \to \mathbf{B}$ is a 0-embedding iff $f(0_A) = 0_B$. We write $\mathbf{A} \leq_0 \mathbf{B}$ iff there is a 0-embedding of $\mathbf{A}$ into $\mathbf{B}$. The notions of a 1-subalgebra and a 1-embedding are defined dually, and we write $\mathbf{A} \leq_1 \mathbf{B}$ iff there is a 1-embedding of **A** into **B**. By a $0,1$-subalgebra (or $0,1$-embedding) we mean a subalgebra **A** of **B** that is both a 0-subalgebra and a 1-subalgebra (respectively, an embedding $f : \mathbf{A} \to \mathbf{B}$ satisfying $f(0_A) = 0_B$ and $f(1_A) = 1_B$). We shall write $\mathbf{A} \leq_2 \mathbf{B}$ (rather than $\mathbf{A} \leq_{0,1} \mathbf{B}$) to denote that there is a $0,1$-embedding of **A** into **B**.

All lattices discussed in this paper are assumed to be finite and distributive.

**Definition 2.1.** A lattice **A** will be called *diamond-decomposable* iff it has a $0,1$-sublattice isomorphic to $\mathbf{D}_n$ for some $n > 0$. This means that there are elements

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

so that each interval $I[a_i, a_{i+1}]$ has a complemented element $c_i$, $a_i < c_i < a_{i+1}$. Such a chain $a_0, \ldots, a_n$ will be called a *diamond-decomposition of* **A**. Given **A** and $a, b \in A$ with $a < b$, we say that $b$ is *diamond-accessible* from $a$ if the interval lattice $I[a, b]$ has a diamond-decomposition.

We are going to show that a finite distributive lattice of more than one element is $+_o$-indecomposable iff it has a diamond-decomposition.

**Definition 2.2.** The *diamond-height* of a finite $\mathbf{A} \in \mathcal{D}$ is defined to be the largest $n > 0$ such that **A** has a $0,1$-sublattice isomorphic to $\mathbf{D}_n$, and is undefined if **A** has no such $0,1$-sublattice. We denote this quantity by $h_\diamond(\mathbf{A})$. The *diamond-bound* of **A**, or $b_\diamond(\mathbf{A})$, is the largest $n$ such that $\mathbf{D}_n \leq \mathbf{A}$. If $Q$ is a sublattice of **A**, we denote the diamond-bound and the diamond-height of this sublattice (if it exists) by $b_\diamond(Q)$, $h_\diamond(Q)$ respectively.

In dealing with any of the concepts defined above, we may substitute the symbol "$\diamond$" for the word "diamond", as in "$\diamond$-decomposition".

We shall need these elementary facts: if $a, b$ are elements in a distributive lattice **A**, then there are isomorphisms of intervals $I[ab, a] \cong I[b, a+b]$, $I[ab, b] \cong I[a, a+b]$, $I[ab, a+b] \cong I[ab, a] \times I[ab, b]$. Also, $I[ab, a+b] \cong I[ab, a] \times I[a, a+b]$.

**Lemma 2.3.** *Let $\mathcal{A} \in \mathcal{D}$. We have that $h_\diamond(\mathbf{A}) = 1$ iff $\mathbf{A} \cong \mathbf{2} \times \mathbf{L}$ where either $\mathbf{L} = \mathbf{2}$, $\mathbf{L} = \mathbf{D}_1$ or $\mathbf{L} = \mathbf{1} +_o \mathbf{K} +_o \mathbf{1}$ for some* **K**. *In the third case, $b_\diamond(\mathbf{L}) = b_\diamond(\mathbf{K}) < b_\diamond(\mathbf{A})$.*

*Proof.* It is easy to verify that the lattices listed each have diamond height 1.

For the converse, suppose that $h_\diamond(\mathbf{A}) = 1$. We have elements $c, d \in A$ with $0 < c < 1$, $0 < d < 1$, $c + d = 1$, $cd = 0$. Suppose that neither $c$ nor $d$ covers 0. Then we have elements $0 < e < c$ and $0 < f < d$. The elements $c, d, e, f$ generate a $0,1$ sublattice of **A** isomorphic to $\mathbf{3} \times \mathbf{3}$, which has a $0,1$-sublattice isomorphic to $\mathbf{D}_2$. Thus in, fact, one of $c, d$ covers 0. Suppose that $c$ covers 0. Now since $\mathbf{A} \cong I[0, c] \times I[0, d] \cong \mathbf{2} \times I[0, d]$ then if $I[0, d]$ is isomorphic to **2** or to $\mathbf{D}_1$ then we are done. So suppose that $I[0, d]$ is isomorphic with neither of them. Then we claim that 0 is meet-irreducible in $I[0, d]$ and $d$ is join-irreducible in $I[0, d]$. The lemma follows from these claims.

Suppose to the contrary that $0 < x \leq d$, $0 < y \leq d$, $xy = 0$ and $x$ and $y$ are incomparable. We can then choose these elements so that in addition, $x, y$ are covers of 0. Since $I[0, d]$ is not isomorphic with $\mathbf{D}_1$, then it follows that $x + y < d$. Now the elements $\{0, x, y, x + y, x + y + c, d, 1\}$ constitute a $0,1$ sublattice of **A** isomorphic to $\mathbf{D}_2$, contradiction. The proof that $d$ is join-irreducible in $I[0,d]$—equivalently, 1 is join-irreducible in $I[c,1]$—is dual to this.

To see in the third case, that $b_\diamond(\mathbf{A}) > b_\diamond(I[0,d])$, suppose that $D$ is a sublattice of $I[0,d]$ isomorphic to $\mathbf{D}_k$, $k > 0$. Let $e$, $e'$ be the least and largest elements of $D$. We have that $0 < e$ (since we are in the third case). Then $\{c + x : x \in D\}$ is a sublattice of $I[c,1]$ isomorphic to $\mathbf{D}_k$. Also $\{0, c, e, c+e\}$ is a copy of $\mathbf{D}_1$. The union of these two sets is a sublattice isomorphic to $\mathbf{D}_1 +_o' \mathbf{D}_k = \mathbf{D}_{k+1}$. Thus $b_\diamond(\mathbf{A}) > k$. •

**Lemma 2.4.** *Suppose that $a < b$ in $\mathbf{A}$ and that $a$ is minimal among all the elements from which $b$ is $\diamond$-accessible. Then $a$ is a cut-point of the interval $I[0,b]$.*

*Proof.* Suppose that $x \leq b$ and $x$ is incomparable to $a$. We can assume that every element $y < x$ is below $a$. Thus $x \succ ax$ (i.e., $x$ covers $ax$). By diamond-accessibility, we have for some $n \geq 1$, a copy of $\mathbf{D}_n$ stretching from $a$ to $b$. It is constituted by elements $a_0, a, \ldots, a_n$ and pairs $u_i, v_i$, $0 \leq i < n$ so that $a = a_0$, $b = a_n$ and for all $i < n$, $u_i v_i = a_i$ and $u_i + v_i = a_{i+1}$ and $u_i, v_i$ are incomparable. There is a largest $k \geq 0$ with $x, a_k$ incomparable. Here $0 \leq k < n$ and $x < a_{k+1}$. If $x \not\leq u_k$ then $ax \leq u_k x < x$ implies $u_k x = ax$. Likewise for $v_k$. If both $u_k x = ax = v_k x$ then $x = a_{k+1} x = (u_k + v_k)x = ax$, a contradiction. Also it is impossible that both $u_k$ and $v_k$ are above $x$, since $a_k = u_k v_k \not\geq x$. Thus one is above $x$, the other is not. Say $u_k x = ax$ and $x \leq v_k$. Now also, $a_k x = ax$. Note that $a_k, x$ are incomparable so that $\{ax, a_k, x, a_k + x\}$ is a copy of $\mathbf{D}_1$.

Suppose first (Case 1) that $a_k + x = v_k$. Then $u_k + x = a_{k+1}$, while $u_k x = ax$, so we have a $0,1$-sublattice $\{ax, u_k, x, a_{k+1}\}$ of $I[ax, a_{k+1}]$ isomorphic with $\mathbf{D}_1$. Then $ax$ is $\diamond$-accessible from $a_{k+1}$ and $a_{k+1}$ is $\diamond$-accesible from $b$, giving that $ax$ is $\diamond$-accessible from $b$, a contradiction. Finally, suppose that (Case 2) $a_k + x < v_k$. Then we have a $0,1$-sublattice $\{ax, a_k, x, a_k+x, u_k+x, v_k, a_{k+1}\}$ of $I[0,b]$ isomorphic to $\mathbf{D}_2$, which gives the same contradiction. •

**Lemma 2.5.** *Let $\mathbf{A} \in \mathcal{D}$. There is a unique decomposition $\mathbf{A} = \mathbf{C} +_o' \mathbf{D}$ where the least element of $\mathbf{D}$ is the smallest element of $\mathbf{A}$ from which 1 is $\diamond$-accessible. If $\mathbf{D} \neq \mathbf{A}$ then we also have $\mathbf{A} = \mathbf{C}' +_o \mathbf{D}$ for a unique $\mathbf{C}'$.*

*Proof.* Let $a$ be a minimal element from which 1 is $\diamond$-accessible. By Lemma 2.4, $a$ is comparable to all elements of $\mathbf{A}$. Obviously, if $a > 0$ then $a$ is join-irreducible; in this case $a$ has a unique subcover $a' \prec a$. Then $a'$ is also comparable to all elements of $\mathbf{A}$. Thus $A = I[0,a'] +_o I[a,b]$ in this case. It follows that any element from which 1 is $\diamond$-accessible lies in $I[a,1]$. Thus $a$ is the least such element. •

**Corollary 2.6.** *If $|\mathbf{A}| > 1$ then $\mathbf{A}$ is $+_o$-indecomposable iff $h_\diamond(\mathbf{A})$ is defined.*

## 3. Stacks, blocks and towers

In the remainder of the paper, we employ the Birkhoff duality between finite distributive lattices with 0 and 1, $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$, and finite ordered sets $P$

(including the empty set). The dual of $\mathbf{L}$ is the ordered set of join-irreducible non-zero elements of $\mathbf{L}$, denoted by $\mathbf{A}^\partial$. For a finite ordered set $P$, the dual of $P$ is the lattice $P^\partial$ of order-ideals of $P$, including the empty ideal.

Suppose that $Q$ is a convex subset in the finite ordered set $P$. Then $Q$ is the set-theoretic difference of two order-ideals—$Q = I_1 \setminus I_0$, $I_0 \subseteq I_1$, where $I_1 = Q{\downarrow}$ and $I_0 = (Q{\downarrow}) \setminus Q$. In this situation, the map $I \mapsto I \cup I_0$ is an isomorphism of $Q^\partial$ with the interval lattice $I[I_0, I_1]$ in $P^\partial$.

According to Birkhoff duality, we have $\mathbf{L} \leq^{0,1} \mathbf{K}$ where $\mathbf{L}, \mathbf{K} \in \mathcal{D}$ iff there is a monotone surjective map of $\mathbf{K}^\partial$ onto $\mathbf{L}^\partial$. (Here, $\mathbf{L} \leq^{0,1} \mathbf{K}$ denotes that there is an embedding of $\mathbf{L}$ into $\mathbf{K}$ that maps $0_L$ to $0_K$ and $1_L$ to $1_K$.) Translated to our context of arbitrary embeddings, this means that $\mathbf{L} \leq \mathbf{K}$ iff there is a monotone surjective mapping of some convex subset of $\mathbf{K}^\partial$ onto $\mathbf{L}^\partial$.

Suppose that $0 = a_0 < a_1 < \cdots < a_n = 1$ in $\mathbf{A}$. For $0 \leq i < n$ write $Q_i$ for the set $\{p \in \mathbf{A}^\partial : p \leq a_{i+1}, p \not\leq a_i\}$. Then each $Q_i$ is a nonvoid convex subset of $\mathbf{A}^\partial$. The $Q_i$ partition $\mathbf{A}^\partial$, and whenever $i < j$, no element of $Q_i$ is above an element of $Q_j$. For any ordered set $P$, a sequence $P_0, \ldots, P_{n-1}$ of nonvoid convex subsets of $P$ which partitions $P$ and has this property that no element of $P_i$ is above any element of $P_j$ when $i < j$, will be called a *stack for* $P$, or $n$*-stack for* $P$. Thus a stack for $P$ is essentially just a chain from 0 to 1 in $P^\partial$.

When $P_0, \ldots, P_{n-1}$ is an $n$-stack for $P$ then in $P^\partial$ we have the elements $a_i = P_0 \cup \cdots \cup P_{i-1}$ satisfying $0 = a_0 < a_1 < \cdots < a_n = 1$. Moreover, for $0 \leq i < n$, the interval sublattice $L_i = I[a_i, a_{i+1}]$ in $P^\partial$ has a complemented element $b$, $a_i < b < a_{i+1}$, if and only if the ordered set $P_i$ can be partitioned into two nonvoid disjoint order-ideals. In fact, $L_i$ is canonically isomorphic with $P_i^\partial$, or equivalently, $P_i \cong L_i^\partial$.

Where $n$ denotes an integer, we have used $\mathbf{n}$ to denote the $n$-element chain. Dealing with ordered sets, we shall use $n$ (non-bold-faced type) to denote the $n$-element anti-chain.

**Definition 3.1.** We define a *block* to be an ordered set $P$ isomorphic to one of these:

$$J_2 = 2 \text{ and } J_3 = 3 \text{ (the two- and three-element anti-chains);}$$

$$J_2' = \mathbf{1} +_c \mathbf{2}\,; \text{ and}$$

$$J(Q) = 1 +_c (1 +_o Q +_o 1) \text{ where } Q \text{ is any nonvoid finite ordered set.}$$

A block has *type* $i$, $i \in \{2, 3, 2'\}$ iff it is isomorphic, respectively, to $J_2$, $J_3$ or $J_2'$. The *long* blocks are those isomorphic to some $J(Q)$, $Q \neq \emptyset$.

**Lemma 3.2.** *Let* $\mathbf{A} \in \mathcal{D}$ *with* $|\mathbf{A}| > 1$. *Then* $\mathbf{A}$ *is* $+_o$*-indecomposable iff* $\mathbf{A}^\partial$ *has no cut-point; and* $h_\diamond(\mathbf{A}) = 1$ *iff* $\mathbf{A}^\partial$ *is a block.*

*Proof.* This follows easily from Lemma 2.3. •

**Definition 3.3.** Let $P$ be a finite ordered set. We say that $P$ is *primitive* iff $P$ has no cut-point (equivalently, $P^\partial$ is $+_o$-indecomposable). We put $h_\diamond(P) = h_\diamond(P^\partial)$, $b_\diamond(P) = b_\diamond(P^\partial)$, and we define $w(P)$ to be the size of the largest anti-chain in $P$.

**Lemma 3.4.** *Let $P$ be a finite ordered set.*

(1) *If $Q$ is a nonvoid convex subset of $P$ then $Q^\partial \leq P^\partial$, and so $b_\diamond(Q) \leq b_\diamond(P)$.*
(2) *For any finite ordered set $P$, we have $w(P) \leq 2b_\diamond(P) + 1$ and, if $P$ is primitive, $h_\diamond(P) \leq b_\diamond(P)$.*
(3) *If $n$ is a positive integer such that $P^\partial \not\geq \mathbf{J}_n$, then $w(P) < 2n + 6$.*

*Proof.* For (1), notice that if $Q$ is a convex subset of $P$ then there is a monotone map of $P$ onto a convex subset $C$ of $1 +_c Q +_c 1$ identical with either $Q$, $1 +_c Q$, $Q +_c 1$, or $1 +_c Q +_c 1$. In all cases, we have $Q^\partial \leq C^\partial$. Then by duality theory, we get

$$Q^\partial \leq C^\partial \leq P^\partial .$$

For (3), let $C$ be a $6 + 2n$-element anti-chain of $P$. Then $\mathbf{2}^{6+2n} \cong C^\partial \leq P^\partial$ (where the embedding is from (1)), which gives $\mathbf{J}_n \leq P^\partial$. The proof of (2) is analogous. •

**Definition 3.5.** Given an ordered set $P$, a *tower for $P$* is a stack $B_0, \dots, B_{n-1}$ for $P$, in which each $B_i$ is a block.

**Theorem 3.6.** *For any finite ordered set $P$ we have*

(1) *$h_\diamond(P) = 1$ iff $P$ is a block.*
(2) *$P$ admits an $n$-tower for some $n \geq 1$ iff $P$ is primitive. The largest $n$ for which $P$ admits an $n$-tower is $n = h_\diamond(P)$.*
(3) *If $P = J(Q) = 1 +_c (1 +_o Q +_o 1)$ is a long block then $b_\diamond(P) > b_\diamond(Q)$ and $w(P) > w(Q)$.*

*Proof.* Left to the reader. (See Lemma 2.3 and Corollary 2.6.) •

**Definition 3.7.** Suppose that $B$ is a long block or a block of type $J_2'$. The isolated point of $B$ will be called the *orphan* in $B$, and usually will be denoted by $a$. $B \setminus \{a\}$ will be called the *chamber* of $B$. The least and largest elements of $B \setminus \{a\}$ will usually be denoted as $b_0, b_1$, respectively.

**Remark 3.1.** Besides the Birkhoff duality between finite distributive lattices with 0 and 1, and finite ordered sets, we have another duality at work here, and it is referred to in the last sentence of the next lemma. Namely, if $\mathbf{L} = \langle L, \vee, \wedge \rangle$ is a lattice, then so is $\mathbf{L}' = \langle L, \wedge, \vee \rangle$, and if $\langle P, \leq \rangle$ is an ordered set, then so is $\langle P, \geq \rangle$. $\mathbf{L}'$ is often called the dual of $\mathbf{L}$ and $\langle P, \geq \rangle$ is called the dual of $\langle P, \leq \rangle$. It happens that $\langle P, \geq \rangle^\partial$ is canonically isomorphic to $(\langle P, \leq \rangle^\partial)'$. This means, for example, that if $P_0, \dots, P_{n-1}$ is a tower for $\langle P, \leq \rangle$, then $P_{n-1}, \dots, P_0$ (with the order turned upside down inside each block $P_i$) is a tower for $\langle P, \geq \rangle$. Many of the statements proved in this and the following sections thus remain true when we turn the order (and the list of blocks comprising a tower) upside down.

**Lemma 3.8.** (Adjacency Lemma) *Suppose that $B_0, \dots, B_{n-1}$ is a tower for $P$ and that $h_\diamond(P) = n$. Let $B_i$ and $B_{i+1}$ be two adjacent blocks and suppose that $B_{i+1}$ is a long block or of type $2'$. Let $b_0^{i+1}$ be the least non-isolated element of $B_{i+1}$.*

(1) *Suppose that $B_i$ is a long block or of type $2'$. Then $b_1^i < b_0^{i+1}$.*

(2) *Suppose that* $B_i$ *is type* 3*. Then all three elements* $x \in B_i$ *satisfy* $x < b_0^{i+1}$*.*
(3) *Suppose that* $B_i$ *is type* 2 *and* $B_{i+1}$ *is a long block. If one of the elements of* $B_i$ *is not below* $b_0^{i+1}$*, then there is a cut-point* $c$ *of the interval* $I[b_0^{i+1}, b_1^{i+1}]$ *with* $b_0^{i+1} \prec c$*. If neither element of* $B_i$ *is below* $b_0^{i+1}$ *then both elements of* $B_i$ *are below* $c$*.*

*The duals of these statements hold when* $B_i$ *is a long block or a block of type* $2'$*.*

*Proof.* First suppose that $B_i = \{u, v, w\}$ is type 3 and assume, say, that $u \not\leq b_0^{i+1}$. We work in the lattice $P^\partial$. Let $I_0 = \bigcup_{k<i} B_k$, $I_1 = I_0 \cup B_i \cup B_{i+1}$. These are elements of $P^\partial$. The interval $I[I_0, I_1]$ has a $0,1$-embedded $\mathbf{D}_3$ consisting of $I_0$, $I_0 \cup \{v\}$, $I_0 \cup \{w\}$, $I_0 \cup \{v, w\}$, $I_0 \cup \{v, w, b_0^{i+1}\}$, $I_0 \cup \{u, v, w\}$, $I_0 \cup \{u, v, w, b_0^{i+1}\}$, $I_0 \cup \{u, v, w, a^{i+1}, b_0^{i+1}\}$, $(I_0 \cup B_i \cup B_{i+1}) \setminus \{a^{i+1}\}$, $I_1$. Also, the interval $I[\emptyset, I_0]$ has a $0,1$-embedded $\mathbf{D}_i$ and the interval $I[I_1, P]$ has a $0,1$-embedded $\mathbf{D}_{n-i-2}$. Altogether, we get a $0,1$-embedded $\mathbf{D}_{n+1}$ contradicting the fact that $h_\diamond(P^\partial) = n$.

Next, suppose that $B_i$ is a long block and that $b_1^i \not\leq b_0^{i+1}$. We get again a $0,1$-embedded $\mathbf{D}_3$ in $I[I_0, I_1]$, another contradiction. Namely, $I_0$, $(I_0 \cup B_i) \setminus \{a^i, b_1^i\}$, $I_0 \cup \{a^i\}$, $(I_0 \cup B_i) \setminus \{b_1^i\}$, $I_0 \cup B_i$, $(I_0 \cup B_i \cup \{b_0^{i+1}\}) \setminus \{b_1^i\}$, $I_0 \cup B_i \cup \{b_0^{i+1}\}$, $I_0 \cup B_i \cup \{a^{i+1}, b_0^{i+1}\}$, $I_1 \setminus \{a^{i+1}\}$, $I_1$.

Next, suppose that $B_i$ has type 2, say $B_i = \{u, v\}$. Suppose first that, say $v \not\leq b_0^{i+1}$. Assume next, to get a contradiction, that the next cut-point of $I[b_0^{i+1}, b_1^{i+1}]$ above $b_0^{i+1}$ does not cover $b_0^{i+1}$. Then there are two distinct covers, $c$ and $c'$ of $b_0^{i+1}$ below $b_1^{i+1}$. Now we show that the interval between $I_0 = B_0 \cup \cdots \cup B_{i-1}$ and $I_1 = P \cup B_i \cup B_{i+1}$ in $P^\partial$ has a $0,1$-copy of $\mathbf{D}_3$. Since the interval from $\emptyset$ to $I_0$ has a $0,1$ copy of $\mathbf{D}_i$ and the interval from $I_1$ to $P$ has a $0,1$-copy of $\mathbf{D}_{n-i-2}$ then $P^\partial$ has a $0,1$-copy of $\mathbf{D}_{n+1}$ which contradicts the assumption that $h_\diamond(P) = n$. The copy of $\mathbf{D}_3$ consists of $I_0 \cup \{v\}$ and $I_0 \cup \{u, b_0^{i+1}\}$ which intersect to $I_0$ and whose union is $I_0 \cup B_i \cup \{b_0^{i+1}\}$; $I_0 \cup B_i \cup \{b_0^{i+1}, c\}$ and $I_0 \cup B_i \cup \{b_0^{i+1}, c'\}$ which intersect to $I_0 \cup B_i \cup \{b_0^{i+1}\}$ and join to to $I_0 \cup B_i \cup \{b_0^{i+1}, c, c'\}$; and $I_0 \cup B_i \cup \{b_0^{i+1}, c, c', a^{i+1}\}$ and $I_0 \cup B_i \cup I[b_0^{i+1} b_1^{i+1}]$ which intersect to $I_0 \cup B_i \cup \{b_0^{i+1}, c, c'\}$ and join to $I_1$. Here $a^{i+1}$ is the isolated point of $B_{i+1}$. Hence we do have $b^{i+1} \prec c$ where

$$I[b_0^{i+1}, b_1^{i+1}] = I[b_0^{i+1}, c] \cup I[c, b_1^{i+1}]\,.$$

Now suppose that also, $u \not\leq b_0^{i+1}$ and that not both $u$ and $v$ are below $c$. Thus we may assume that $u \not\leq c$ and $v \not\leq b_0^{i+1}$ (or exchange $u, v$). Since $B_{i+1}$ is a long block, then $b_1^{i+1} > c$. Now we have again a $0,1$-copy of $\mathbf{D}_3$ in the interval from $P$ to $Q$, giving the same contradiction. Its construction is left to the reader. •

**Definition 3.9.** A *knot* is a tower with all blocks of types $2, 3, 2'$.

Let $T$ be a finite ordered set and $B_0, \ldots, B_{n-1}$ be an $n$-knot for $T$ where $n = h_\diamond(T) \geq 1$. Let $D$ be an order-ideal in $T$. Given a block $B$ which is a convex subset of $T$, we say that $B$ is *green with respect to* $D$ iff either $B$ is a $J_3$ and intersects $D$, or $B$ is a $J_2$ and contained in $D$, or $B$ is a $J_2'$ and the least member of its chamber belongs to $D$.

**Lemma 3.10.** (Knot Lemma) *Let $B_0, \ldots, B_{n-1}$ be an $n$-knot for $T$ where $n = h_\diamond(T) \geq 1$, and let $D$ be an order-ideal in $T$. Suppose that $B_{n-1}$ is green with respect to $D$. Then $T$ admits a stack of one of the following types: (1) an $n$-knot $C_0, \ldots, C_{n-1}$ in which $C_0$ is green; (2) a stack $E, C_1, \ldots, C_{n-1}$ where $C_1, \ldots, C_{n-1}$ is a knot and $E$ is a two-element chain contained in $D$; (3) a stack $E, C_0, \ldots, C_{n-1}$ where $E = \{p\}$, $p$ is a minimal element of $T$ and $p \in D$, and where $C_0, \ldots, C_{n-1}$ is a knot.*

*Proof.* If $n = 1$ or if $B_0$ is green, there is nothing to prove. Suppose that $n > 1$ and $B_0$ is not green. Let $i_0$ be the least $i$, $i < n$, such that $B_i$ is green. We have $i_0 > 0$.

Among the blocks $B_0, \ldots, B_{i_0-1}$, no block $B_i$ contains more than one element of $D$, blocks of type $J_3$ are disjoint from $D$, and in blocks of type $J_{2'}$, only the isolated point can belong to $D$. Suppose that there are $k$ elements of $D$ altogether in the blocks, $B_1, \ldots, B_{i_0-1}$.

First, suppose that $k = 0$. If $B_{i_0}$ is a $J_2$, then all elements of $B_{i_0}$ are minimal elements of $T$. So, we can move $B_{i_0}$ down to the bottom to form a stack of type (1). If $B_{i_0}$ is a $J_2'$ or a $J_3$, then the least non-isolated point of the $J_2'$ or at least one point of the $J_3$, respectively, is minimal in $T$. We can move this element down to the bottom to form a stack of type (3).

Henceforth, we assume that $k > 0$. Let $i_1 > \cdots > i_k$, with $i_1 < i_0$, be the indices of the blocks in which these elements of $D$ occur; and let $B_{i_j} \cap D = \{b_j\}$ for $1 \leq j \leq k$.

<u>Case 1:</u> $B_{i_0}$ is a $J_3$ or $J_{2'}$. Choose for $b_0$ any element of $D \cap B_{i_0}$ in the first case, and in the second, choose for $b_0$ the lesser of the two non-isolated elements. Thus $b_0 \in B_{i_0} \cap D$.

We define a stack $F, R_1, \ldots, R_{i_0}$ for the ordered set $Q = B_1 \cup B_2 \cup \cdots \cup B_{i_0}$. Put $R_i = B_i$ for $1 \leq i \leq i_0 - 1$, $i \notin \{i_1, \ldots, i_k\}$. Put $R_{i_0} = B_{i_0} \setminus \{b_0\}$. For $1 \leq j \leq k$ put $R_{i_j} = (B_{i_j} \setminus \{b_j\}) \cup \{b_{j-1}\}$. Finally, take $F = \{b_k\}$. It is easy to see that this is a stack for $Q$ and, moreover, $F \subseteq D$ and $R_1, \ldots, R_{i_0}$ is a knot for $Q \setminus F$.

Now there are cases depending on the character of $B_0$. If $b_k$ is above no member of $B_0$ then $F, B_0, R_1, \ldots, R_{i_0}, B_{i_0+1}, \ldots, B_{n-1}$ is a stack of type (3). Assume that $b \in B_0$ and $b < b_k$. Then $b \in D$ and since $B_0$ is not green, then $B_0$ is a $J_2$ or $J_2'$ and $b$ is an isolated point in $B_0$. In this case, we take $E = \{b\}$ and $R_0 = (B_0 \setminus \{b\}) \cup \{b_k\}$. Now $E, R_0, R_1, \ldots, R_{i_0}, B_{i_0+1}, \ldots, B_{n-1}$ is a stack for $T$ of type (3).

<u>Case 2:</u> $B_{i_0}$ is a $J_2$. Write $b_{-1}, b_0$ for the two elements of $B_{i_0}$. We again define a stack for $Q$. We put $F = \{b_{k-1}, b_k\}$. For $2 \leq i \leq i_0$ we define $R_i$ by cases. If $B_{i-1} \cap D = \emptyset$, we put $R_i = B_{i-1}$. Otherwise, we have, say $i - 1 = i_j$, $j \geq 1$, and we put $R_i = R_{i_j+1} = (B_{i_j} \setminus \{b_j\}) \cup \{b_{j-2}\}$. This gives us a stack $F, R_2, R_3, \ldots, R_{i_0}$ for $Q$ where $R_2, \ldots, R_{i_0}$ are blocks.

The construction now depends on the characters of $F$ and $B_0$ and the relations between elements of $B_0$ and the elements $b_k$, and $b_{k-1}$ of $F$. If no member of $F$ is above a member of $B_0$ then $F, B_0, R_2, \ldots, R_{i_0}, B_{i_0+1}, \ldots, B_{n-1}$ is a stack for $T$ of type (1) or (2) (depending on whether $b_k < b_{k-1}$ or $b_k$ and $b_{k-1}$ are incomparable). Suppose that $b \in B_0$ and $b < b_j$, $j \in \{k, k-1\}$. Then $b \in D$ and so $B_0$ is a $J_2$ or $J_2'$, and $b$ is an isolated point of $B_0$ since $B_0$ is not green.

We now take $E = \{b, b_k\}$, $C_1 = \{b_{k-1}\} \cup (B_0 \setminus \{b\}$, $C_i = R_i$ for $2 \leq i \leq i_0$, and $C_i = B_i$ for $i_0 + 1 \leq i < n$. Since $b_{k-1} \not\leq b_k$, then $E, C_1, \ldots, C_{n-1}$ is a stack for $T$ of type (1) if $b \not\leq b_k$, and of type (2) if $b < b_k$.

We have now handled all cases. •

**Definition 3.11.** Let $T$ be a primitive finite ordered set. An $n$-tower for $T$ is called *fitting* iff $h_\diamond(T) = n$. An $n$-tower $B_0, \ldots, B_{n-1}$ for $T$ is called *aligned* iff it is fitting and whenever $i < j$ and $B_i$ and $B_j$ are long blocks then $b_1^i < b_0^j$.

In Fig. 1 we show an example of an ordered set with two different arrangements into a 4-tower; the first tower is not aligned, while the second is.

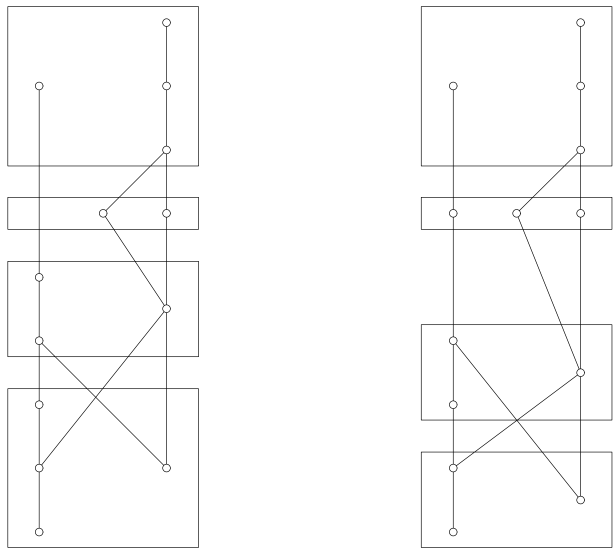


Fig. 1

**Theorem 3.12.** (Alignment Theorem) *Every primitive finite ordered set admits an aligned tower.*

*Proof.* Let $B_0, B_1, \ldots, B_{n-1}$ be a fitting tower expression of $P$. Our proof is by induction on several quantities. First on $n = h_\diamond(P)$. Next on two parameters of the particular tower expression. First, the number $\nu$ of long blocks $B_i$. Second, the number $b$ of "bad gaps" where we have long blocks $B_i$ and $B_{i+k+1}$, and $B_{i+1}, \ldots, B_{i+k}$ is a knot (or $k = 0$), and where $b_1^i \not\leq b_0^{i+k+1}$.

Let the given tower expression for $P$ have value sequence $(n, \nu, b)$. The induction assumption is that every finite primitive ordered set $P'$ which has a fitting tower with value sequence $(n', \nu', b')$ that is lexicographically less than $(n, \nu, b)$, does admit an aligned tower.

Clearly, we can assume that $B_{n-1}$ is a long block (else the induction assumption immediately yields the desired conclusion) and that $b > 0$ (else there is nothing to be proved). Let $B_i$ be the long block that begins the highest gap. Say $B_{i+1}, \dots, B_{i+k}$ is a knot (or $k = 0$) and $B_{i+k+1}$ is the next long block. We are going to find a different fitting tower for $P$ with the same $n$ and either smaller $\nu$ or same $\nu$ and smaller $b$.

Step 1: We make adjustments to $B_{i+k}$ and $B_{i+k+1}$.

Suppose that $B_{i+k}$ is a $J_2$, say $B_{i+k} = \{u, v\}$ is a $J_2$. If neither of $u, v$ is below $b_0^{i+k+1}$ then by Lemma 3.8, we have a cut-point $c \succ b_0^{i+k+1}$ in $B_{i+k+1} \setminus \{a^{i+k+1}\}$ and $u < c$, $v < c$. Let $B'_{i+k+1} = B_{i+k+1} \setminus \{b_0^{i+k+1}\}$ and $B'_{i+k} = B_{i+k} \cup \{b_0^{i+k+1}\}$. We have now a tower

$$B_0, \dots, B_{i+k-1}, B'_{i+k}, B'_{i+k+1}, B_{i+k+2}, \dots, B_{n-1}$$

in which, if $|B_{i+k+1}| > 4$ then $B'_{i+k+1}$ is a long block and $B'_{i+k}$ is a $J_3$. In this case, for the new tower, $n$ and $\nu$ are the same and either $b$ is decreased by 1 or $b$ is unchanged. On the other hand, if $|B_{i+k+1}| = 4$ then $B_{i+k+1}$ is a $J'_2$ and the new tower has the same $n$ (of course) but smaller $\nu$. Thus we achieve that the highest block below the long block at the top of the highest gap is a $J_3$, or we are done, in this case.

Now consider the case when, say, $v < b_0^{i+k+1}$ and $u \not< b_0^{i+k+1}$. We remove $b_0^{i+k+1}$ from $B_{i+k+1}$ and place it in $B_{i+k}$. By the Adjacency Lemma, this gives a new tower in which $B'_{i+k}$ is a $J'_2$ and $B'_{i+k+1}$ is a long block or of type $J'_2$. In the second case, the parameter $\nu$ is reduced. In the first, we have arranged that the block just below the top long block is a $J'_2$.

Step 2: Setting $D = b_0^{i+k+1}\!\downarrow$, we define "green blocks" with respect to $D$ as in Lemma 3.10, and observe that with Lemma 3.8 and the above manipulations, we can assume that $B_{i+k}$ is green.

Step 3: Using Lemma 3.10, we find a stack for $Q = B_{i+1} \cup \dots \cup B_{i+k}$ of type (1), (2), or (3).

Case 1: The new stack is a $k$-knot $C_{i+1}, \dots, C_{i+k}$ with $C_{i+1}$ green.

Subcase 1.1: $C_{i+1}$ is a $J_3$. Then by (the dual of) Lemma 3.8, all members of $C_{i+1}$ are above $b_1^i$. Since one of them, $x$, belongs to $D$ then $b_1^i < x < b_0^{i+k+1}$. This contradicts the badness of our gap.

Subcase 1.2: $C_{i+1}$ is a $J'_2$. Then by (the dual of) Lemma 3.8, the least element $x$ among the two comparable members of $C_{i+1}$ is above $b_1^i$. But $x \in D$, so $b_1^i < x < b_0^{i+k+1}$—a contradiction again.

Subcase 1.3: $C_{i+1}$ is a $J_2$. Then $C_{i+1}$ is all green. By (the dual of) Lemma 3.8, one of the next two subcases pertains.

Subcase 1.3.1: One element of $C_{i+1}$ is above $b_1^i$. Then of course we get the same contradiction.

Subcase 1.3.2: $C_{i+1} = \{u, v\}$ where $\{u, v, b_1^i\}$ is a set of three incomparable elements, all of which are above $c$, $c \prec b_1^i$ and $c$ is a cut-point of $I[b_0^i, b_1^i]$. Now we replace $B_i$ by $B'_i = B_i \setminus \{b_1^i\}$ and replace $C_{i+1}$ by $C'_{i+1} = C_{i+1} \cup \{b_1^i\}$. Now we have a new tower for $P$, namely

$$B_0, \dots, B_{i-1}, B'_i, C'_{i+1}, \dots, C_{i+k}, B_{i+k+1}, \dots, B_{n-1}\,.$$

If $B'_i$ is a $J'_2$ then we have changed $(n,\nu,b)$ to $(n,\nu-1,\ldots)$ and the desired result follows by induction. Otherwise $B'_i$ is a long block and $b'^i_1 = c < b_0^{i+k+1}$, so we have changed $(n,\nu,b)$ to $(n,\nu,b-1)$, and again we are done.

Case 2: The new stack is $E, C_{i+2},\ldots,C_{i+k}$ where $E=\{u,v\}$ is a green two-element chain, $u<v$, and $C_{i+2},\ldots,C_{i+k}$ is a knot. Neither of $u,v$ is comparable with $b_1^i$, since they can only be above it, and both are below $b_0^{i+1+1}$. Now we put $R = E\cup\{b_1^i\}$, $S = B_i\setminus\{b_1^i\}$. The argument that there is $c \prec b_1^i$, a cut-point of $I[b_0^i,b_1^i]$, is strictly analogous to one we have seen before. Now $S$ is either a $J'_2$ or a long block. In either case, we have a tower

$$B_0,\ldots,B_{i-1},S,R,C_{i+2},\ldots,C_{i+k},B_{i+k+1},\ldots,B_{n-1}$$

for $P$ and in the first case, $(n,\nu,b)$ has been replaced by $(n,\nu-1,\ldots)$, and in the second case, we know by Lemma 3.8 that, since $R$ is a $J'_2$, $c<u<b_0^{i+k+1}$. So in the second case, we have replaced $(n,\nu,b)$ by $(n,\nu,b-1)$.

Case 3: The new stack is $E,C_{i+1},\ldots,C_{i+k}$ where $C_j$ are blocks and $E=\{p\}$, $p<b_1^{i+k+1}$. We have, of course, that $p$ must be incomparable with $b_1^i$. Now we take $B'_i = B_i\setminus\{b_1^i\}$, $E'=\{p,b_1^i\}$ and we have a tower

$$B_0,\ldots,B_{i-1},B'_i,E',C_{i+1},\ldots,C_{i+k},B_{i+k+1},\ldots,B_{n-1}$$

for $P$. The argument that $b_1^i$ has in $I[b_0^i,b_1^i]$ a unique subcover $c$—and so that $B'_i$ is a block—is the same one we have seen before. We have a contradiction, of course, to the assumption that $h_\diamond(P)=n$.

We have treated all cases, and justified the theorem. •

## 4. Avoiding $\{\mathbf{J}_k : k\ge N\}$, I: critical blocks and critical intervals

We define $\mathcal{A}_N$ as the class of finite distributive lattices $\mathbf{L}$ such that for all $k\ge N$, $\mathbf{L}\not\geq \mathbf{J}_k$. We define $\mathcal{P}_N$ to be the class of finite ordered sets $P$ such that $P^\partial\in\mathcal{A}_N$.

**Definition 4.1.** A *$k$-ladder* is a poset $L_k$ isomorphic to $\mathbf{D}_k^\partial$, that is, the ordinal sum of $k$ copies of the two-element anti-chain.

**Lemma 4.2.** *Let $P\in\mathcal{P}_N$. Let $Q\subseteq P$ be a convex subset of $P$ which in its induced order is a $k+N$-ladder for some $k\ge 2$. There do not exist two distinct elements $x,y\in P\setminus Q$ such that $x$ is incomparable to both minimal elements of $Q$ and $y$ is incomparable to both maximal elements of $Q$.*

*Proof.* For $0\le i<k+N$ let $a_i,b_i$ be the two incomparable elements of $Q$ at height $i$. Suppose that there exists a pair $x,y$ of distinct elements of $P$ such that $x$ is incomparable with both $a_0$ and $b_0$ and $y$ is incomparable with both $a=a_{k+N-1}$ and $b=b_{k+N-1}$. First we are going to prove that there exists a pair $x',y'$ with the same properties and, moreover, such that $y'\not\leq x'$ and $Q\cup\{x',y'\}$ is a convex subset of $P$. Denote by $M_1$ the set of the maximal elements $z$ with the property that $z$ is incomparable with both $a_0$ and $b_0$. Denote by $M_2$ the set of the minimal elements $z$ with the property that $z$ is incomparable with both $a$ and $b$. It can be easily checked that for any $m\in M_1\cup M_2$, the set $Q\cup\{m\}$ is a convex subset of $P$.

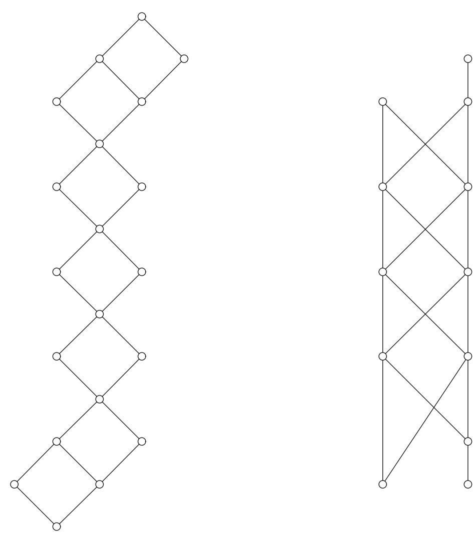
Consider first the case when $M_1 = M_2 = \{p\}$ for an element $p$. Then $x \leq p \leq y$. If $x < p$, put $y' = p$ and let $x'$ be a subcover of $p$ above $x$. Otherwise, if $x = p < y$, put $x' = p$ and let $y'$ be a cover of $p$ below $y$.

It remains to consider the case when there are two distinct elements $p \in M_1$ and $q \in M_2$. If $p, q$ are incomparable, put $x' = p$ and $y' = q$. If $p < q$, put $x' = p$ and $y' = q$. Finally, if $q < p$, put $x' = q$ and let $y'$ be a cover of $q$ below $p$.

So, in the rest of the proof we can assume that $x, y$ is a pair of distinct elements such that $x||a_0$, $x||b_0$, $y||a$, $y||b$, $y \not\leq x$ and the set $C = Q \cup \{x, y\}$ is convex. Put $E_i = \{a_0, b_0, \ldots, a_i, b_i\}$ and $E = E_{k+N-2}$. The following are order-ideals in $C$:

$$\emptyset, \quad \{a_0\}, \quad \{b_0\}, \quad \{a_0, b_0\} \quad \{a_0, x\}, \quad E_0 \cup \{x\},$$
$$E_i \cup \{a_{i+1}, x\}, \quad E_i \cup \{b_{i+1}, x\}, \quad E_{i+1} \cup \{x\}, \quad (0 \leq i < k + N - 2)$$
$$E \cup \{a, x\}, \quad E \cup \{b, x\}, \quad E \cup \{a, x, y\}, \quad E \cup \{a, b, x\}, \quad E \cup \{a, b, x, y\}.$$

It is easy to verify that these are all order-ideals in $C$, and form a sublattice of $C^\partial$ isomorphic with $\mathbf{J}_{N+k-2}$. Since $C^\partial \leq P^\partial$, this contradicts that $P \in \mathcal{P}_N$. •



$\mathbf{J}_3$ $\qquad\qquad$ $\mathbf{J}_3^\partial$

Fig. 2

**Corollary 4.3.** *Let $P \in \mathcal{P}_N$. Let $Q \subseteq P$ be a convex subset of $P$ which in its induced order is a $k + N$-ladder for some $k \geq 2$. Assume that $x_0 \in P \setminus Q$ and $x_0$ is incomparable to all elements of $Q$. Let $Q'$ be the convex subset of $P$ formed by all elements of $Q$ except those elements that are either maximal or minimal in $Q$ (so that $Q'$ is a $k - 2 + N$-ladder). Then every element $x \in P \setminus Q'$ different from $x_0$ is either strictly above all elements of $Q'$ or strictly below all elements of $Q'$.*

**Remark 4.1.** Recall that we defined a *primitive ordered set* to be an ordered set without cut-points. Thus a long block is a primitive ordered set that decomposes internally as a cardinal sum, $Q = \{a\} +_c I[b,c]$, of a one-element ordered set and a bounded ordered set of at least three elements. In such a long block $Q$, the isolated point $a$ is called the *orphan in* $Q$, and the interval $I[b_0, b_1]$ that makes up the rest of $Q$ is called the *chamber in* $Q$.

**Definition 4.4.** Let $P$ be an ordered set. $P$ has a unique internal ordinal sum decomposition into singletons $\{c\}$ ($c$ ranging through the cut-points of $P$) and primitive ordered sets. The primitive ordered subsets of $P$ occuring in this decomposition will be called the *primitive vertical components of* $P$. They are, of course, convex subsets of $P$, and each one of them is either identical to $\{x : x < c\}$ where $c$ is the least cutpoint of $P$, or of the form $I(c, c')$ where $c < c'$ are successive cut-points, or is identical to $\{x : c < x\}$ where $c$ is the largest cutpoint of $P$, or finally, is identical with $P$ itself if $P$ has no cut-points.

We define critical blocks of depth $n$ and critical intervals of depth $n$ in $P$, for any positive integer $n$.

Each primitive vertical component of $P$ can be expressed in at least one way as a tower. By a *critical block of depth* 1 *in* $P$, we mean a long block that occurs as a block in some tower expression for some primitive vertical component of $P$. By a *critical interval of depth* 1 *in* $P$ we mean the chamber of some critical block of depth 1 in $P$.

Now suppose that $a_0, \dots, a_{n-1}$ is an $n$-element anti-chain in $P$ and

$$b_0 < b_1 < \cdots < b_{n-1} < c_{n-1} < c_{n-2} < \cdots < c_1 < c_0$$

is a chain in $P$ such that the following hold.

- $a_i$ is incomparable to all elements of $I[b_i, c_i]$ for $0 \le i < n$ and $b_i < a_{i+1} < c_i$ for $0 \le i < n-1$.
- Where $Q_i = \{a_i\} \cup I[b_i, c_i]$, $Q_0$ is a critical block of depth 1 in $P$ and for all $i < n-1$, $Q_{i+1}$ is a critical block of depth 1 in $I[b_i, c_i]$.

In this situation, we call $Q_{n-1}$ a *critical block of depth* $n$ *in* $P$, and its chamber, $I[b_{n-1}, c_{n-1}]$, a *critical interval of depth* $n$ *in* $P$.

The critical intervals and critical blocks of depth $n$ in $P$ are precisely those intervals and blocks for which the auxiliary elements exist as above.

Notice that in the above situation the elements $a_0, \dots, a_{n-1}$ form an $n$-element anti-chain of elements incomparable to all elements of $I[b_{n-1}, c_{n-1}]$. Thus

**Lemma 4.5.** *If* $I[b,c]$ *is a critical interval of depth* $n$ *in* $P$ *then* $w(P) \ge n + w(I[b,c])$.

**Definition 4.6.** Let $P$ be a poset. An interval $I[b_0, b_1] \subseteq P$ is *undivided (in P)* iff $b_0 < b_1$ and for all $x \in P \setminus I[b_0, b_1]$, and all $b_0 < y < b_1$, if $x < y$ then $x < b_0$, and if $y < x$ then $b_1 < x$. Otherwise, the interval is said to be *divided (in P)*.

**Lemma 4.7.** *Suppose that* $Q = \{a\} +_c I[b_0, b_1]$ *is a long block, and* $c_0 < c_1$ *are successive cut-points of* $I[b_0, b_1]$. *Then the interval* $I[c_0, c_1]$ *is undivided in* $Q$.

*Proof.* Trivial, from the definitions. •

**Proposition 4.8.** *Suppose that $P \in \mathcal{P}_N$ and that $Q = \{a\} +_c I[b_0, b_1]$ is a long block that is a critical block of depth $n \geq 1$ in $P$. Suppose that $c_0 < c_1$ are successive cut-points of $I[b_0, b_1]$ and that $I(c_0, c_1)$ is a $k+N$-ladder for some $k \geq 2$. Then $n = 1$ and the interval $I[c_0, c_1]$ is undivided in $P$.*

*Proof.* To prove the proposition for the case $n = 1$, suppose that $Q$ is a critical block of depth 1 in $P$. There is a primitive vertical component $P'$ of $P$ that has a tower $Q_0, \ldots, Q_{k-1}$ with, say, $Q = Q_{i_0}$. Obviously, every $x \in P \setminus P'$ satisfies either $x < P'$ and $x < c_0$, or $x > P'$ and $x > c_1$. So we work inside the convex set

$$P' = Q_0 \cup \cdots \cup Q_{k-1}$$

to show that $I[c_0, c_1]$ is undivided in $P'$. By the previous lemma, the interval $I[c_0, c_1]$ is undivided in $Q = Q_{i_0}$.

We have the element $a$ that is incomparable to all elements in the interval $I[c_0, c_1]$, especially, to all elements of the $k+N$-ladder $I(c_0, c_1)$. Thus by Lemma 4.2, every element of any $Q_i$, $i < i_0$, must be below at least one of the two minimal elements of $I(c_0, c_1)$, and every element of any $Q_i$, $i > i_0$, must be above at least one of the two maximal elements of $I(c_0, c_1)$. Suppose that there is $x \in P'$ such that $x < u_0$ and $x$ is incomparable to $u_1$ where $u_0, u_1$ are the two minimal elements of $I(c_0, c_1)$. We can assume that $x$ is maximal with this property. We have that $x \in Q_i$ for some $i < i_0$. It is easy to see that $I(c_0, c_1) \cup \{a, x\}$ is convex in $P$. Now where

$$I(c_0, c_1) = L_0 \cup L_1 \cup \cdots L_{N+k-1}$$

and $L_i$ are the two-element levels of $I(c_0, c_1)$, we have a tower $S, L_1, \ldots, L_{N+k-2}, T$ for the convex set $I(c_0, c_1) \cup \{a, x\}$, with $S = \{x, u_0, u_1\} = \{x\} \cup L_0$ and $T = \{a\} \cup L_{N+k-1}$. The block $S$ in this tower is of type $J_2'$ while the block $T$ is of type $J_3$. This tower gives a copy of $\mathbf{J}_{N+k-2}$ in $P^\partial$, and that is a contradiction.

Thus $Q_i < I(c_0, c_1]$ for $i < i_0$. Dually, we have $I[c_0, c_1) < Q_i$ for $i > i_0$. It remains to show that every $x$ that belongs to some $Q_i$, $i < i_0$, satisfies $x < c_0$; and dually, every $x$ that belongs to some $Q_i$, $i > i_0$, satisfies $x > c_1$.

The proof that this is so for $i < i_0$ is by induction on the quantity $i_0 - i$. First, suppose that $x \in Q_{i_0-1}$ and $x \not< c_0$. We can choose $x$ to be maximal in $Q_{i_0-1}$. We do have that $x < I(c_0, c_1]$. Let $y$ be a maximal element of $Q_{i_0-1}$ that is incomparable to $x$. (Such a $y$ must exist!) Clearly, if $y < c_0$, then $y$ is covered by $c_0$. Since $I[c_0, c_1]$ is undivided in $Q_{i_0}$, to $I[c_0, c_1]$, then the set $\{x, y, a\} \cup I[c_0, c_1)$ is convex in $P'$. We have a tower $S = \{x, y, c_0\}, L_0, \ldots, L_{N+k-2}, T = \{a\} \cup L_{N+k-1}$ for this ordered set. Here $S$ is a block of type $J_2'$ or $J_3$ and $T$ is a block of type $J_3$. The tower produces a copy of $\mathbf{J}_{N+k-2}$ in $P^\partial$, a contradiction.

So we have the result for $i = i_0 - 1$. Now suppose that it is true for $i \in \{i_0 - 1, i_0 - 2, \ldots, j+1\}$ where $0 \leq j \leq i_0 - 2$. Let $x \in Q_j$ and suppose that $x \not< c_0$. Since $Q_p < c_0$ for $j < p \leq i_0$, then $x$ is incomparable to all elements of $Q_{j+1} \cup \cdots \cup Q_{i_0-1}$. We can assume that $x$ is maximal in $Q_j$ and pick another maximal element $y \in Q_j$. If $y \not< c_0$ take $z = y$, else take $z$ to be maximal in the interval $I[y, c_0)$. Here, $\{x, z, a\} \cup I[c_0, c_1)$ is convex and has the tower, as above, that leads to a contradiction.

We have now shown that the interval $I[c_0, c_1]$ is not divided by any member of $P'$ that lies in a block $Q_j$ with $j \leq i_0$. The fact that the interval is not divided by any member of $P'$ lying in a block $Q_j$ with $j > i_0$ follows from what has been proved above and Remark 3.1.

Thus we have handled the case $n = 1$ of this Proposition. It is impossible that $n > 1$. For if $I[b_0, b_1]$ is a critical interval of depth $\geq 2$ then there are two incomparable elements $a, a'$ in $P$ that are both incomparable to the entire interval $I[c_0, c_1]$. By Lemma 4.2, this cannot occur. •

**Lemma 4.9.** *Let $P \in \mathcal{P}_N$ be a primitive ordered set. Let $Q_0, \dots, Q_{m-1}$ be any tower for $P$ where $m = h_\diamond(P)$. Then either $P$ is an $m$-ladder (and every $Q_i$ is a $J_2$-block and $Q_i < Q_j$ when $i < j$) or else there are $0 \leq i_0 \leq i_1 < m$ such that $i_1 - i_0 \leq N + 3$ and $Q_0 \cup \dots \cup Q_{i_0-1}$ is an $i_0$-ladder and $Q_{i_1} \cup \dots \cup Q_{m-1}$ is an $m - i_1$-ladder.*

*Proof.* For the purpose of this proof, a block $Q_i$ that is not of type $J_2$, or a pair of consecutive blocks $Q_i, Q_{i+1}$ such that $Q_i \not< Q_{i+1}$ will be called a "tangle" in the tower. If there is no tangle in this tower then, obviously, $P$ is an $m$-ladder. So suppose there is a tangle. Let $i_0$ be the least $i$ such that either $Q_i$ or $Q_i, Q_{i+1}$ is a tangle. Let $i'$ be the greatest $i$ such that either $Q_i$ or $Q_{i-1}, Q_i$ is a tangle. Then $i_0 \leq i_1$. Put $i_1 = i' + 1$ (and note that $i_1 = m$ is possible). Obviously, $Q_0, \dots, Q_{i_0-1}$ is an $i_0$-ladder (empty if $i_0 = 0$), and $Q_{i_1}, \dots, Q_{m-1}$ is an $m - i_1$-ladder (empty if $i_1 = m$).

We need to show the bound on $i_1 - i_0$. Write $I_0 = Q_0 \cup \cdots Q_{i_0-1}$ and $I_3 = Q_0 \cup \dots \cup Q_{i'}$. If the lowest tangle is simply $Q_{i_0}$, put $I_1 = I_0 \cup Q_{i_0}$, and otherwise put $I_1 = I_0 \cup Q_{i_0} \cup Q_{i_0+1}$. If the highest tangle is simply $Q_{i'}$ put $I_2 = Q_0 \cup \cdots \cup Q_{i'-1}$, and otherwise put $I_2 = Q_0 \cup \dots \cup Q_{i'-2}$. To demonstrate the claimed bound on $i_1 - i_0$, it suffices, given that $P \in \mathcal{P}_N$, to show that the interval $I[I_0, I_1]$ in $P^\partial$ has a copy of the lattice $\mathbf{B}_{2,3}$ with its top coinciding with $I_1$, while the interval $I[I_2, I_3]$ has a copy of $\mathbf{B}_{2,3}$ with its bottom coinciding with $I_2$. (This is because the interval $I[I_1, I_2]$ has a $0,1$-copy of $\mathbf{D}_k$ provided by a middle part of our tower, with $k \geq i_1 - i_0 - 4$.) The verification is left to the reader. •

The next remark will be needed in §5.

**Remark 4.2.** If in Lemma 4.9, the lattice $P^\partial$ satisfies $P^\partial \not\geq_0 \mathbf{D}_k +'_o \mathbf{B}_{2,3}$ for every $k \geq N$, and $P^\partial$ is not a ladder, then $i_0 \leq N - 1$. This is because the interval $I[I_0, I_1]$ in $P^\partial$ contains not only a 1-embedded $\mathbf{B}_{2,3}$ but also a 0-embedded $\mathbf{B}_{2,3}$. Furthermore, and more trivially, if $\mathbf{D}_k \not\leq P^\partial$ for every $k \geq N$ then we can take $i_0 = 0$ and $i_1 = m$ in the Lemma, because in fact we have $m \leq N - 1$.

**Lemma 4.10.** *Suppose that $P \in \mathcal{P}_N$ is a long block, say $P = \{a\} \cup I[b_0, b_1]$ where $a$ is the orphan in $P$ and $|I[b_0, b_1]| \geq 3$. Let $c_0 < c_1$ be two successive cut-points in*

*$I[b_0, b_1]$ such that $I(c_0, c_1)$ is nonvoid. Then either $I(c_0, c_1)$ is a $k$-ladder for some $k > 0$ or else, $h_\diamond(I(c_0, c_1)) \leq 2N + 2$.*

*Proof.* The interval $I(c_0, c_1)$ is a primitive ordered set. We apply Lemma 4.9 to it. Suppose that the interval is not a ladder. Then where $m = h_\diamond(I(c_0, c_1))$, we have the tower $Q_0, \ldots, Q_{m-1}$ and it contains a tangle $Q_{i_0}$ or $Q_{i_0}, Q_{i_0+1}$ (see the proof of Lemma 4.9). We claim that $i_0 \leq N$ and $m - i_0 - 2 \leq N$ (which will prove that $h_\diamond(I(c_0, c_1)) \leq 2N + 2$.

Thus assume that $N < i_0$. (The proof that $Q_{i_0+1}, \ldots, Q_{m-1}$ is a tower of length at most $N$ is dual (in the simple sense) to the proof we give, and will be omitted.) Using that the orphan $a$ is incomparable to all elements of $I(c_0, c_1)$, we can put $a$ together with $Q_0$, and find a block $Q'_0$ of type $J_3$ that is an upset in $\{a\} \cup Q_0$. If the tangle is $Q_{i_0}, Q_{i_0+1}$, we can find a down-set $Q'_{i_0}$ in $Q_{i_0} \cup Q_{i_0+1}$ that is a block of type $J_3$ or $J'_2$. If the tangle is simply $Q_{i_0}$ then we take $Q'_{i_0} = Q_{i_0}$. In either event, we get a tower $Q'_0, Q_1, \ldots, Q_{i_0-1}, Q'_{i_0}$ for the ordered set

$$Q = Q'_0 \cup Q_1 \cup \cdots \cup Q_{i_0-1} \cup Q'_{i_0}$$

and $Q$ is a convex subset of $P$. This yields a copy of $\mathbf{J}_{i_0-1}$ in $Q^\partial$, and hence a copy of this lattice in $P^\partial$, contradicting the fact that $P \in \mathcal{P}_N$. •

## 5. Avoiding $\{\mathbf{J}_k : k \geq N\}$, II: nicely structured ordered sets

**Definition 5.1.** By a *basic skeleton* we mean an ordered set $S$ and a system $\overline{S} = (X; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ where

- $S = X \cup \{b_i : i < n\} \cup \{c_i : i < n\}$;
- $X \cap (\{b_i : i < n\} \cup \{c_i : i < n\}) = \emptyset$, and

$$b_0 \prec c_0 \leq b_1 \prec c_1 \leq \cdots \leq b_{n-1} \prec c_{n-1}$$

(i.e., $b_i$ is covered by $c_i$ for $0 \leq i < n$).

In this situation, we also say that $\overline{S}$ is a *basic skeleton for $S$*.

By a *skeleton* we mean an ordered set $S$ and a system

$$\overline{S} = (X, Y_0, Y_1; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$$

where

- $X$, $Y_0$, $Y_1$ are disjoint sets and $X \cup Y_0 \cup Y_1 \subseteq S$;
- $(X; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ is a basic skeleton for $S \setminus (Y_0 \cup Y_1)$;
- $Y_0$ is an order-ideal of $S$ and $Y_1$ is an order filter of $S$; moreover, either $Y_0 < b_0$ and $c_{n-1} < Y_1$, or $n = 0$ and $Y_0 < Y_1$;
- each of $Y_0$, $Y_1$ is empty, or a ladder;
- there is at most one $x \in S \setminus Y_0$ such that $Y_0 < x$ fails;
- there is at most one $x \in S \setminus Y_1$ such that $x < Y_1$ fails.

In this situation, we also say that $\overline{S}$ is a *skeleton for* $S$.

Where $M$ is a positive integer, a skeleton or basic skeleton for $S$, as above, is an $(M)$-*skeleton for* $S$, or a *basic* $(M)$-*skeleton for* $S$ iff $|X \cup \{b_i : i < n\} \cup \{c_i : i < n\}| \leq M$.

**Definition 5.2.** Suppose that $S$ is an ordered set,

$$S = X \cup Y_0 \cup Y_1 \cup \{b_i : i < n\} \cup \{c_i : i < n\}\,,$$

and $\overline{S} = (X, Y_0, Y_1; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ is a skeleton for $S$. We define the *character* of the skeleton $\overline{S}$. Suppose that $Y_i$ is an $\ell_i$-ladder ($i \in \{0, 1\}$). (Note that $\ell_i = 0$ signifies that $Y_i = \emptyset$.) Put $\bar{c}_0 = \langle \ell_0 \rangle$ if $Y_0 < x$ for all $x \in S \setminus Y_0$. In the contrary case, put $\bar{c}_0 = \langle \ell_0, d_0, \varepsilon_0, x_0 \rangle$ where $d_0$, $\varepsilon_0$, $x_0$ are defined as follows: $x_0$ is the unique element $x \in S \setminus Y_0$ such that $Y_0 < x$ fails. We have $0 \leq d_0 < \ell_0$ and $x_0$ fails to be above both elements of depth $d_0$ in $Y_0$, while it is above all elements of $Y_0$ of depth greater than $d_0$. $\varepsilon_0$ is the number of elements at depth $d_0$ in $Y_0$ that are below $x_0$ (either 0 or 1). We define $\bar{c}_1$ dually (so to speak). That is, we put $\bar{c}_1 = \langle \ell_1 \rangle$ if $x < Y_1$ holds for all $x \in S \setminus Y_1$. In the contrary case, we put $\bar{c}_1 = \langle \ell_1, d_1, \varepsilon_1, x_1 \rangle$ where $d_1$, $\varepsilon_1$, $x_1$ are defined as follows: $x_1$ is the unique element $x \in S \setminus Y_1$ such that $x < Y_1$ fails. We have $0 \leq d_1 < \ell_1$ and $x_1$ fails to be below both elements of height $d_1$ in $Y_1$, while it is below all elements of $Y_1$ of height greater than $d_1$. $\varepsilon_1$ is the number of elements at height $d_1$ in $Y_1$ that are above $x_1$ (either 0 or 1).

The *character of* $\overline{S}$ is the pair $(\bar{c}_0, \bar{c}_1)$.

**Definition 5.3.** Suppose that $\overline{S} = (X, Y_0, Y_1; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ is a skeleton for an ordered set $S$, and $\overline{S'} = (X, Y_0', Y_1'; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ is a skeleton for an ordered set $S'$ having the same underlying basic skeleton $(X; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$. Let $(\bar{c}_0, \bar{c}_1)$, $(\bar{c}_0', \bar{c}_1')$ be the characters of $\overline{S}$ and $\overline{S}'$, respectively. We write $(\bar{c}_0, \bar{c}_1) \leq (\bar{c}_0', \bar{c}_1')$ to mean that for $i = 0$ and $i = 1$:

- $\bar{c}_i = \langle \ell_i \rangle$ iff $\bar{c}_i' = \langle \ell_i' \rangle$ and if $\bar{c}_i = \langle \ell_i \rangle$ then $\ell_i \leq \ell_i'$;
- if $\bar{c}_i = \langle \ell_i, d_i, \varepsilon_i, x_i \rangle$ and $\bar{c}_i' = \langle \ell_i', d_i', \varepsilon_i', x_i' \rangle$ then $x_i = x_i'$, $\varepsilon_i = \varepsilon_i'$, $d_i \leq d_i'$ and $\ell_i \leq \ell_i'$.

**Definition 5.4.** Let $P$ be a finite ordered set, and $M$ be a positive integer. By a *basic* $(M)$-*system for* $P$ we mean an ordered subset $S \subseteq P$ together with a basic $(M)$-skeleton for $S$, $\overline{S} = (X; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$, such that $P = S \cup \bigcup_{0 \leq i < n} I(b_i, c_i)_P$ and this union is disjoint. (Recall that the interior intervals $I(b_i, c_i)$ are empty in $S$, since $b_i \prec c_i$ in $S$, but in $P$ they need not be empty.)

By an $(M)$-*system for* $P$ we mean an ordered subset $S \subseteq P$ together with an $(M)$-skeleton for $S$,

$$\overline{S} = (X, Y_0, Y_1; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})\,,$$

such that $S \setminus (Y_0 \cup Y_1)$ and its basic $(M)$-skeleton $(X; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ is a basic $(M)$-system for $P \setminus (Y_0 \cup Y_1)$. We define the *width of* $\overline{S}$, written $w(\overline{S})$, to be the maximum, over all the intervals $I[b_i, c_i]$ $(0 \leq i < n)$ that are divided in $P$, of the numbers $w(I[b_i, c_i]_P)$.

**Definition 5.5.** An $(M)$-system $\overline{S} = (X, Y_0, Y_1; b_0, \dots, b_{n-1}; c_0, \dots, c_{n-1})$ or basic $(M)$-system $\overline{S} = (X; b_0, \dots, b_{n-1}; c_0, \dots, c_{n-1})$ for $P$ is said to be *nice* if none of the intervals $I[b_i, c_i]$, $0 \le i < n$, is divided in $P$. We say that $P$ is *nicely* $(M)$*-structured* iff $P$ admits a nice $(M)$-system.

Suppose that $P$ admits the nice $(M)$-system

$$\overline{S} = (X, Y_0, Y_1; b_0, \dots, b_{n-1}; c_0, \dots, c_{n-1}),$$

where

$$S = X \cup Y_0 \cup Y_1 \cup \{b_0, \dots, b_{n-1}\} \cup \{c_0, \dots, c_{n-1}\}$$

is an ordered subset of $P$. Then $P$ arises through the following construction. We have the $(M)$-skeleton $\overline{S}$ for $S$, and the pairwise disjoint ordered sets (ordered subsets of $P$) $P_i = I(b_i, c_i)_P$, $0 \le i < n$, which are all disjoint from $S$. The ordered set

$$\overline{S}(P_0, \dots, P_{n-1})$$

(which is actually identical to $P$) is then defined as follows: The universe of $\overline{S}(P_0, \dots, P_{n-1})$ is $S \cup \bigcup_i P_i$. The order is defined to be the transitive closure of the union of the order on $S$ and the orders on each $P_i$ and the pairs $b_i < x$ and $x < c_i$ for all $x \in P_i$ and all $0 \le i < n$. In this ordered set $\overline{S}(P_0, \dots, P_{n-1})$ $(= P)$, we have that each $P_i = I(b_i, c_i)_P$, and each interval $I[b_i, c_i]$ is undivided in $P$.

The utility of these ideas is manifested in this theorem.

**Theorem 5.6.** *Suppose that* $\overline{S} = (X, Y_0, Y_1; b_0, \dots, b_{n-1}; c_0, \dots, c_{n-1})$ *and* $\overline{S}' = (X, Y_0', Y_1'; b_0, \dots, b_{n-1}; c_0, \dots, c_{n-1})$ *are two* $(M)$*-skeletons with the same underlying basic* $(M)$*-skeleton. Suppose that* $P_i$ *(*$0 \le i < n$*) are pairwise disjoint ordered sets disjoint from the universe of* $\overline{S}$ *and* $P_i'$ *(*$0 \le i < n$*) are pairwise disjoint ordered sets disjoint from the universe of* $\overline{S}'$*. If the characters* $(\bar{c}_0, \bar{c}_1)$, $(\bar{c}_0', \bar{c}_1')$ *of* $\overline{S}$ *and* $\overline{S}'$ *satisfy* $(\bar{c}_0, \bar{c}_1) \le (\bar{c}_0', \bar{c}_1')$ *and for all* $0 \le i < n$ *we have that* $P_i^\partial \le P_i'^\partial$*, then* $\overline{S}(P_0, \dots, P_{n-1})^\partial \le \overline{S}'(P_0', \dots, P_{n-1}')^\partial$.

*Proof.* Under all the given assumptions, we need to find a monotone mapping of some convex subset of $P' = \overline{S}'(P_0', \dots, P_{n-1}')$ onto $P = \overline{S}(P_0, \dots, P_{n-1})$.

We do have for each $i$, a monotone mapping of a convex subset $Q_i \subseteq P_i'$ onto $P_i$ (by Birkhoff duality). We can extend this (in several possible ways) to a monotone mapping $f_i$ of $I[b_i, c_i]_{P'}$ onto $I[b_i, c_i]_P$. Setting $f = \mathrm{id}_X \cup \bigcup_i f_i$ we have that $f$ is a monotone map of $P' \setminus (Y_0' \cup Y_1')$ onto $P \setminus (Y_0 \cup Y_1)$. This is true because of the way in which the order on $P' \setminus (Y_0' \cup Y_1')$ (respectively, on $P \setminus (Y_0 \cup Y_1)$) is the minimum extension of the order on $X \cup \{b_0, \dots, b_{n-1}\} \cup \{c_0, \dots, c_{n-1}\}$ joined to the disjoint union of the orders on the individual $I[b_i, c_i]_{P'}$ (respectively, on the individual $I[b_i, c_i]_P$).

It remains to extend the monotone mapping $f$ to map a convex subset of $P'$ onto $P$. We do this by adding to the domain of $f$ the top $\ell_0$ levels of $Y_0'$ and the bottom $\ell_1$ levels of $Y_1'$ and mapping, for each $0 \le j < \ell_0$, the two elements of depth $j$ in $Y_0'$ to the two elements of depth $j$ in $Y_0$, making sure, if $\bar{c}_0 = \langle \ell_0, d_0, 1, x_0 \rangle$ and $\bar{c}_0' = \langle \ell_0', d_0, 1, x_0 \rangle$, to map the element of depth $d_0$ in $Y_0'$ below $x_0' = x_0$ to the element of depth $d_0$ in $Y_0$ below $x_0$; and likewise mapping, for each $0 \leq j < \ell_1$, the two elements of height $j$ in $Y_1'$ to the two elements of height $j$ in $Y_1$, making sure, if $\bar{c}_1 = \langle \ell_1, d_1, 1, x_1 \rangle$ and $\bar{c}_1' = \langle \ell_1', d_1, 1, x_1 \rangle$, to map the element of height $d_1$ in $Y_1'$ above $x_1' = x_1$ to the element of height $d_1$ in $Y_1$ above $x_1$.

The detailed verification that this works is left to the reader. •

**Remark 5.1.** An inspection of the proof of Theorem 5.6 reveals the following refined version of the conclusion of the theorem:

(1) If $Y_0 = \emptyset = Y_1$, then $\overline{S}(P_0, \ldots, P_{n-1})^\partial$ is isomorphic to a $0,1$-sublattice of $\overline{S}'(P_0', \ldots, P_{n-1}')^\partial$.
(2) If $Y_0 = \emptyset$, then $\overline{S}(P_0, \ldots, P_{n-1})^\partial$ is isomorphic to a 0-sublattice of $\overline{S}'(P_0', \ldots, P_{n-1}')^\partial$.
(3) If $Y_1 = \emptyset$, then $\overline{S}(P_0, \ldots, P_{n-1})^\partial$ is isomorphic to a 1-sublattice of $\overline{S}'(P_0', \ldots, P_{n-1}')^\partial$.

These statements will be used in the proof of Theorem 7.1.

The next theorem justifies all the definitions above.

**Theorem 5.7.** *Let $N \geq 3$. Suppose that $P$ is a primitive ordered set belonging to $\mathcal{P}_N$. Then $P$ is nicely $(M)$-structured, for $M = 5 \cdot (18N)^{6N+4}$. In fact, $P$ admits a nice $(M)$-system $\overline{S} = (X, Y_0, Y_1; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$ such that*

(i) *if $h_\diamond(P) < N$ then $Y_0 = \emptyset = Y_1$; and*
(ii) *if $\mathbf{D}_k +_o' \mathbf{B}_{2,3} \not\leq_0 P^\partial$ for every $k \geq N$ then either $Y_0 = \emptyset$, or $P = Y_0$ is a ladder and hence $X \cup Y_1 = \emptyset$ and $n = 0$.*

This theorem is established with the next proposition, which will be proved by induction.

**Proposition 5.8.** *Let $P$ be a primitive ordered set in $\mathcal{P}_N$, $N \geq 3$. For each $k$, $1 \leq k < w(P)$, $P$ admits an $((18N)^k)$-system $\overline{S}_k$ such that all intervals of $\overline{S}_k$ that are divided in $P$ are critical intervals of depth $k$ in $P$ (and thus $w(\overline{S}_k) \leq w(P) - k$). Moreover, the system $\overline{S}_k$ can be chosen so that the assertions expressed in statements (i) and (ii) of Theorem 5.7 are true of it.*

The base step in the inductive argument is the case $k = 1$, established in the next lemma.

**Lemma 5.9.** *Let $P$ be a primitive ordered set in $\mathcal{P}_N$, $N \geq 3$. Then $P$ admits an $(M)$-system $\overline{S}_1$, $M \leq 3(3N + 7) \leq 18N$, such that all intervals in $\overline{S}_1$ are critical intervals of depth 1 in $P$ and the assertions expressed in statements (i) and (ii) of Theorem 5.7 are true of $\overline{S}_1$.*

*Proof.* According to Lemma 4.9, $P$ admits a fitting tower

$$Z_0^0, \ldots, Z_{r-1}^0, M_0, \ldots, M_{s-1}, Z_0^1, Z_1^1, \ldots, Z_{t-1}^1$$

where $r + s + t = h_\diamond(P)$, $Z_0^0, \ldots, Z_{r-1}^0$ and $Z_0^1, \ldots, Z_{t-1}^1$ are ladders and $s \leq N + 3$. (Any of $r, s, t$ may be 0.) Moreover, by Remark 4.2, we can assume that $r = 0 = s$ if $h_\diamond(P) < N$; and if $\mathbf{D}_k +_o' \mathbf{B}_{2,3} \not\leq_0 P^\partial$ for every $k \geq N$ then we have $r \leq N - 1$.

Assume for the moment that $r > N+2$. Let $Y^0 = Z^0_0 \cup \cdots Z^0_{r-N-3}$. We claim that there is at most one $x \in P \setminus Y^0$ such that $Y^0 < \{x\}$ fails. To see the truth of this claim, note that if $Y^0 < \{x\}$ fails and $x \notin Y^0$ then $x$ belongs to a block $M_0$ or higher, so that $x$ cannot be equal or less than any member of the $N+2$-ladder $W^0 = Z^0_{r-N-2} \cup \cdots Z^0_{r-1}$. In fact, $x$ must be incomparable to all members of $W^0$. Since $W^0$ is a convex set in $P$, then according to Lemma 4.2, there is at most one such $x$ in $P$.

Analogously, if $t > N+2$ then where $Y^1 = Z^1_{N+2} \cup \cdots \cup Z^1_{t-1}$, there is at most one $x \in P \setminus Y^1$ such that $x < Y^1$ fails.

We now adjoin to the middle segment of blocks $M_i$ the highest $N+2$ of the blocks $Z^0_j$ (or all of these blocks if $r \leq N+2$). Likewise we adjoin to the middle segment the lowest $N+2$ of the blocks $Z^1_j$ (or all of these blocks if $t \leq N+2$). Thus we rewrite our tower as

$$Y^0_0, \ldots, Y^0_{r'-1}, M'_0, \ldots, M'_{s'-1}, Y^1_0, \ldots, Y^1_{t'-1}$$

with $s' \leq 3N+7$. Here, if $r' > 0$ then $r' = r-N-2$, and if $t' > 0$ then $t' = t-N-2$. Where $Y^0$ is the union of the blocks of the lower ladder, and $Y^1$ the union of the blocks of the upper ladder, our observations in the two paragraphs above imply that there is at most one $x \in P \setminus Y^0$ such that $Y^0 < \{x\}$ fails, and at most one $x \in P \setminus Y^1$ such that $\{x\} < Y^1$ fails.

Now according to Theorem 3.12, the primitive ordered set $M' = M'_0 \cup \cdots \cup M'_{s'-1}$ with $h_\diamond(M') = s'$ admits an aligned tower. We need to ensure that the elements of the chambers of the long blocks in the aligned tower are all entirely above $Y^0$ and below $Y^1$. This is important, and easily accomplished, but requires some space to demonstrate. We do it as follows.

All the long blocks in $M' = M'_0, \ldots, M'_{s'-1}$ lie in the middle section $M_0, \ldots, M_{s-1}$. Applying Theorem 3.12 to this middle section, we are able to replace $M_0, \ldots, M_{s-1}$ by an aligned tower $N_0, \ldots, N_{s-1}$. Suppose that $i_0$ is the least index $i$ such that $N_i$ is a long block and that $I[b^{i_0}_0, b^{i_0}_1]$ is the chamber of $N_i$. We want it to be the case that the union of the chambers of all the long blocks in $N_0, \ldots, N_{s-1}$ is entirely above $Y^0$. Since at most one element of $P \setminus Y^0$ is not above $Y^0$, the only element of the union of the chambers that can possibly fail to be above $Y^0$ is $b^{i_0}_0$. Suppose that $Y^0 \not< \{b^{i_0}_0\}$. Then $Y^0 \neq \emptyset$ and so the aligned tower replacing $M'_0, \ldots, M'_{s'-1}$ is

$$Z^0_{r-N-2}, \ldots, Z^0_{r-1}, N_0, \ldots, N_{s-1}, \ldots .$$

Since all elements outside of $Y^0$ but $b^{i_0}_0$ are above $Y^0$, then $b^{i_0}$ is incomparable to all elements of the preceeding block—$N_{i_0-1}$ if $i_0 > 0$, or the $J_2$ block $Z^0_{r-1}$ if $i_0 = 0$. Call this preceeding block $N$. We know from the Adjacency Lemma (3.8) that, since the tower $Z^0_{r-N-2}, \ldots,$ is fitting the only way it can happen that $b^{i_0}_0$ is incomparable to all elements of $N$ is if $N$ is a $J_2$-block, $N = \{u, v\}$, and $b^{i_0}_0$ has a unique cover $c$ in the chamber of $N_{i_0}$ and $N' = \{u, v, b^{i_0}_0\} < \{c\}$. Now $\{c\} > Y^0$ since $c \neq b^{i_0}_0$. Thus after replacing $N$ by $N'$ (a $J_3$-block) and $N_{i_0}$ by $Q = N_{i_0} \setminus \{b^{i_0}_0\}$, we achieve an aligned tower

$$Z^0_{r-N-2}, \ldots, Z^0_{r-2}, \ldots, N', Q, N_{i_0+1}, \ldots, N_{s-1}, \ldots ,$$

whose chambers in the long blocks are all totally above $Y^0$.

Now if these chambers are not all below $Y^1$, then $Y^1 \neq \emptyset$ and so the above tower actually ends with $Z_0^1, \ldots, Z_{N+1}^1$. Now the same argument, applied to the right end, yields the small adjustment needed to get the aligned tower $M_0'', \ldots, M_{s'-1}''$ to replace $M_0', \ldots, M_{s'-1}'$ in which all the chambers of the long blocks are above $Y^0$ and below $Y^1$.

We can now define the required $(M)$-system. Namely, let $i_0 < i_1 < \cdots < i_{u-1}$ be the list of those $i$, $0 \leq i < s'$, such that $M_i''$ is a long block. Let $a^j$ be the orphan and $I[b_0^{i_j}, b_1^{i_j}]$ be the chamber, in $M_{i_j}''$. Take $n = u$ (which may be 0) and let $b_j = b_0^{i_j}$ and $c_j = b_1^{i_j}$ for $0 \leq j < u$. Take $X$ to be the set consisting of the union of all the blocks $M_i''$ that are not long blocks together with all the elements $a^j$.

It is easy to see that $\overline{S}_1 = (X, Y^0, Y^1; b_0, \ldots, b_{u-1}; c_0, \ldots, c_{u-1})$ is an $(M)$-system for $P$, where $M \leq 3(3N+7) \leq 18N$. The intervals of this system are all, by definition, critical intervals of depth 1 in $P$. Since by Lemma 3.4, $w(P) \leq 2N+5$, these intervals each have width at most $2N+4$.

Furthermore, we have obviously ensured that the assertions made in statements (i) and (ii) of Theorem 5.7 are true of $\overline{S}_1$. •

**Lemma 5.10.** *Let $P$ be a primitive ordered set in $\mathcal{P}_N$, let $k \geq 1$, and let $\overline{S}$ be an $(M)$-system for $P$ in which all the divided intervals are critical intervals of depth $k$ in $P$. Then $P$ admits a $(5M)$-system $\overline{S}'$ such that for every divided interval $I[b_i', c_i']$ of $\overline{S}'$, there is a divided interval of $\overline{S}$ which contains it and in which $b_i'$ and $c_i'$ are two successive cut-points and $I(b_i', c_i')$ is a primitive vertical component. Moreover, if statements (i) and (ii) of Theorem 5.7 are true of $\overline{S}$ then they are true of $\overline{S}'$ as well.*

*Proof.* Write $\overline{S} = (X, Y_0, Y_1; b_0, \ldots, b_{n-1}; c_0, \ldots, c_{n-1})$. Let $i_0 < i_1 < \cdots < i_{u-1}$ where $I[b_{i_j}, c_{i_j}]$, $0 \leq j < u$, are the divided intervals in $\overline{S}$. For each $j$, let

$$b_{i_j} = c_0^j < \cdots < c_{\ell_j - 1}^j = c_{i_j}$$

where $c_r^j$ are the cut-points in the interval $I[b_{i_j}, c_{i_j}]$.

For each $x \in X$, there is at most one $j$ such that and $x < c_{i_j}$ and $x \not\leq b_{i_j}$. For this $j = j_x^0$, if it exists, there is a unique $r$, $0 \leq r < \ell_j - 1$, such that $x < c_{r+1}^j$ and $x \not\leq c_r^j$. Put $d_x^0 = c_r^{j_x^0}$ and $d_x^1 = c_{r+1}^{j_x^0}$.

For each $x \in X$, there is at most one $j$ such that $b_{i_j} < x$ and $c_{i_j} \not\leq x$. For this $j = j_x^1$, if it exists, there is a unique $r$, $0 \leq r < \ell_j - 1$, such that $c_r^j < x$ and $c_{r+1}^j \not\leq x$. Put $e_x^0 = c_r^{j_x^1}$ and $e_x^1 = c_{r+1}^{j_x^1}$.

The set $Z = \{d_x^0, d_x^1, e_x^0, e_x^1\}_{x \in X}$ satisfies $|Z| \leq 4M$. For each $z \in Z$, $z$ is a cutpoint in a unique interval $I[b_{i_j}, c_{i_j}]$ and either $z$ has a predecessor $z' < z$ in the sequence of cut-points of the interval and $z' \in Z$, or $z$ has a succesor $z < z'$ in the sequence of cut-points of the interval and $z' \in Z$.

For $0 \leq j < u$, write

$$b_{i_j} = f_0^j < f_1^j < \cdots < f_{m_j - 1}^j = c_{i_j}$$

for the ordered sequence of all members of $Z \cup \{b_{i_j}, c_{i_j}\}$ in the interval $I[b_{i_j}, c_{i_j}]$.

We define the system $\overline{S}'$ to be the same as $\overline{S}$ except that the intervals will be all the undivided intervals $I[b_i, c_i]$ of $\overline{S}$, together with all the intervals $I[f_r^j, f_{r+1}^j]$ where $0 \leq j < u$ and $0 \leq r < m_j - 1$.

This is a system for $P$ (easy to show). The divided intervals in $\overline{S}'$ are all among the intervals $I[d_x^0, d_x^1]$ and $I[e_x^0, d_x^1]$ and each such interval has primitive interior (since the endpoints of the interval are successive cut-points of a larger interval and the interval is divided).

We added the set $Z$ of points to the universe of the underlying basic skeleton in order to generate these intervals. Thus $\overline{S}'$ is a $(5M)$-system for $P$. •

**Lemma 5.11.** *Let $P$ be a primitive ordered set in $\mathcal{P}_N$, $N \geq 3$. Let $k \geq 1$, and let $\overline{S}$ be an $(M)$-system for $P$ in which all the divided intervals are critical intervals of depth $k$ in $P$. Then $P$ admits an $(18N \cdot M)$-system $\overline{S}''$ such that every divided interval of $\overline{S}'$ is a critical interval in $P$ of depth $k+1$. Moreover, if statements (i) and (ii) of Theorem 5.7 are true of $\overline{S}$, they are true of $\overline{S}''$ as well.*

*Proof.* First we produce the $(5M)$-system $\overline{S}'$ for $P$, from Lemma 5.10.

Let $I[c, c']$ be any one of the divided intervals of $\overline{S}'$; say $c, c'$ are successive cut-points of the critical interval $I[b_{i_j}, c_{i_j}]$ of depth $k$ in $P$. (See the proof of Lemma 5.10). By Lemma 4.10, $I(c, c')$ is either a ladder $L_m$, or else $h_\diamond(I(c, c')) \leq 2N + 2$. By Proposition 4.8, if $I(c, c')$ is a ladder $L_m$, then $m \leq N + 2$. Thus in both cases, $h_\diamond(I(c, c')) \leq 2N + 2$.

Now we apply Theorem 3.12 to the primitive ordered set $I(c, c')$ to obtain an aligned tower $Q_0, \ldots, Q_{m-1}$ for this set. Here $m \leq 2N + 2$.

Thus for every divided interval in $\overline{S}'$ we have an aligned tower of at most $2N+2$ blocks.

To construct $\overline{S}''$, we take $X''$ to be the union of $X' = X$ and all the elements in all these intervals $I(c, c')$, excluding the members of the chambers of the long blocks in these various towers.

For the intervals in $\overline{S}''$ we take the undivided intervals of $\overline{S}$ together with the chambers of the long blocks in the chosen aligned towers for the divided intervals of $\overline{S}$.

The lower and upper ladders of $\overline{S}'$, $Y_0$ and $Y_1$, are unchanged.

The only possible divided intervals in $\overline{S''}$ are some of the chambers of these long blocks in the towers for divided intervals of $\overline{S}'$. These intervals are, by definition, critical intervals in $P$ of depth $k + 1$.

Finally, how large is the universe $S''$ of the underlying basic skeleton of $\overline{S}''$. We have added points to $X$ and points to define the new intervals. In each of the divided intervals of $\overline{S}'$, we have added at most $3 \cdot (2N + 2)$ points altogether. The number of those divided intervals is at most $2|X| \leq 2M$. (See the proof of Lemma 5.10). Thus $\overline{S}''$ is an $(M'')$-system where

$$M'' \leq 5M + 2|X| \cdot 3(2N + 2) \leq 5M + 12M(N + 1) = M(17 + 12N) \leq$$

$$M(6N + 12N) = 18N \cdot M\,.$$

•

**Proof of Proposition 5.8 and Theorem 5.7.** The proposition follows immediately by induction, from Lemmas 5.9 and 5.11. Thus, by this proposition, where $k = w(P)$, we get a $((18N)^{k-1})$-system $\overline{S}_{k-1}$ for $P$ where each divided interval (if any) is a critical interval of depth $k-1$. By Lemma 4.5, each of these divided intervals $I[b,c]$ has width at most 1; i.e., the interval is a chain. Thus the procedure followed in the proof of Lemma 5.10 now produces a $(5 \cdot (18N)^{k-1})$-system $\overline{S}$ for $P$ that is nice (has no divided intervals). By Lemma 3.4 (3), we have $k \leq 6N + 5$. This proves Theorem 5.7.

## 6. Avoiding $\{\mathbf{J}_k : k \geq N\}$, III: conclusion

**Theorem 6.1.** *For each $N \geq 1$ the class $\mathcal{A}_N$ is well-quasi-orderd by embeddability.*

*Proof.* We can assume that $N \geq 3$. Assume that this theorem is false. Choose, by Theorem 1.1, a minimal bad sequence $\langle \mathbf{L}_n : n < \omega \rangle$ in $\mathcal{A}_N$. By removing finitely many terms from the sequence, we can assume that every $\mathbf{L}_n$ is $+_o$-indecomposable. Let $P_n = \mathbf{L}_n^\partial$ for $n < \omega$. The $P_n$ are primitive. By Theorem 5.7, $P_n$ is nicely $(M)$-structured by a system $\overline{S}_n$, $M = 5(18N)^{6N+4}$. By successively cutting down to subsequences, we can assume that:

- There is a basic skeleton $\overline{S}^b = (X; b_0, \ldots, b_{m-1}, c_0, \ldots, c_{m-1})$, such that every $\overline{S}_n$ is an expansion $(X, Y_0, Y_1; b_0, \ldots, b_{m-1}; c_0, \ldots, c_{m-1})$ of $\overline{S}^b$.
- We have $P_n = \overline{S}_n(Q_0^n, \ldots, Q_{m-1}^n)$ where $Q_j^n$ are convex subsets of $P_n$ with ${Q_j^n}^\partial < P_n^\partial \cong \mathbf{L}_n$.
- The characters $(\bar{c}_0^n, \bar{c}_1^n)$ of $\overline{S}_n$ satisfy $(\bar{c}_0^i, \bar{c}_1^i) \leq (\bar{c}_0^j, \bar{c}_1^j)$ whenever $i < j < \omega$.

Since ${Q_r^n}^\partial < \mathbf{L}_n$, then the collection $\{{Q_r^n}^\partial : 0 \leq r < m, 0 \leq n < \omega\}$ is well-quasi-ordered. Thus by successive further cutting down to infinite subsequences, we can assume that

- When $i < j$ we have ${Q_r^i}^\partial \leq {Q_r^j}^\partial$ for all $0 \leq r < m$.

But now, by Theorem 5.6, $P_0^\partial \leq P_1^\partial$, equivalently, $\mathbf{L}_0 \leq \mathbf{L}_1$. This is a contradiction, and it concludes our proof of the theorem. •

**Theorem 6.2.** *The unavoidable members of $\langle \mathcal{D}, \leq \rangle$ are precisely the lattices isomorphic to a proper sublattice of some $\mathbf{J}_n$.*

*Proof.* Since $\{\mathbf{J}_n : n < \omega\}$ is an anti-chain, any unavoidable finite distributive lattice must be isomorphic to a proper sublattice of some $\mathbf{J}_n$. But we immediately see that if $\mathbf{L}$ is isomorphic to a proper sublattice of $\mathbf{J}_n$, then $\mathbf{L} < \mathbf{J}_{n+m}$ for all $m \geq 0$. Thus in this case, $\{\mathbf{K} \in \mathcal{D} : \mathbf{L} \not\leq \mathbf{K}\}$ is contained in $\mathcal{A}_{n-1}$ and, by Theorem 6.1, is well-quasi-ordered. This means that $\mathbf{L}$ is unavoidable. •

**Theorem 6.3.** *Let $\mathcal{A}$ be any order-ideal in $\langle \mathcal{D}, \leq \rangle$. Then $\langle \mathcal{A}, \leq \rangle$ is well-quasi-ordered iff $\mathcal{A} \subseteq \mathcal{A}_N$ for some $N$. If $\langle \mathbf{L}_n : n < \omega \rangle$ is any minimal bad sequence in $\langle \mathcal{D}, \leq \rangle$, then all but finitely many $\mathbf{L}_n$ are among the terms of the sequence $\langle \mathbf{J}_n : n < \omega \rangle$.*

*Proof.* This is an immediate consequence of Theorem 6.1 and the fact that $\{\mathbf{J}_n : n < \omega\}$ is an anti-chain. •

**Corollary 6.4.** *A universal class $\mathcal{K}$ of distributive lattices has uncountably many universal subclasses if and only if it contains infinitely many members of the sequence $\langle \mathbf{J}_n : n < \omega \rangle$.*

*Proof.* Every subset of an infinite set of pairwise incomparable finite distributive lattices in $\mathcal{K}$ generates a different order-ideal of $\langle \mathcal{K} \cap \mathcal{D}, \leq \rangle$. If $\mathcal{K}$ contains only finitely many members of $\langle \mathbf{J}_n : n < \omega \rangle$, then $\langle \mathcal{K} \cap \mathcal{D}, \leq \rangle$ is well-quasi-ordered; by Theorem 6.3, every order-ideal $I$ in $\langle \mathcal{K} \cap \mathcal{D}, \leq \rangle$ is uniquely determined by the set of minimal elements of $\langle (\mathcal{K} \cap \mathcal{D}) - I, \leq \rangle$, and the set of these minimal elements is finite (up to isomorphism); thus there are only countably many order-ideals in $\langle \mathcal{K} \cap \mathcal{D}, \leq \rangle$. •

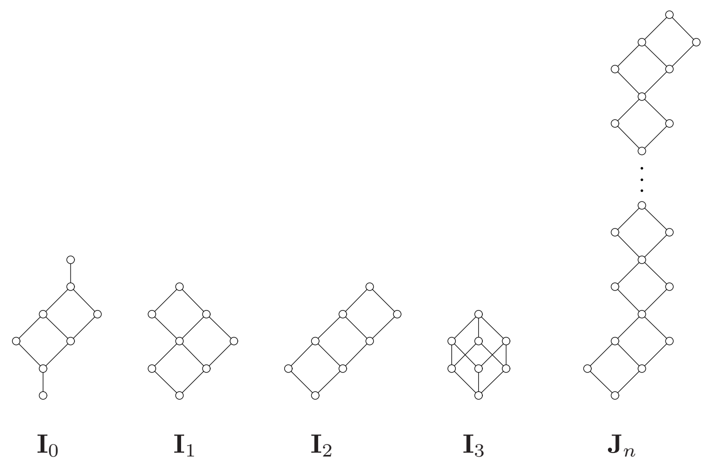



Fig. 3

Consider the following four lattices pictured together with $\mathbf{J}_n$ in Fig. 3: $\mathbf{I}_0 = \mathbf{1} +_o \mathbf{B}_{2,3} +_o \mathbf{1}$, $\mathbf{I}_1$ is the lattice $\mathbf{B}_{3,3}$ with one of the two doubly irreducible elements removed, $\mathbf{I}_2 = \mathbf{B}_{2,4}$ and $\mathbf{I}_3 = \mathbf{2}^3$. Clearly, every proper sublattice of any of these four lattices is also a proper sublattice of some $\mathbf{J}_n$. Thus, by Theorem 6.2, it is also true that unavoidable members of $\langle \mathcal{D}, \leq \rangle$ are precisely the lattices isomorphic to proper sublattices of the lattices from the extended sequence $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, $\mathbf{J}_n$ $(n \geq 0)$. The next theorem says that the extended sequence is a borderline between unavoidable and avoidable members of $\langle \mathcal{D}, \leq \rangle$.

**Theorem 6.5.** *A finite distributive lattice is avoidable in* $\langle \mathcal{D}, \leq \rangle$ *if and only if it has a sublattice isomorphic to a member of the sequence* $\mathbf{I}_0$*,* $\mathbf{I}_1$*,* $\mathbf{I}_2$*,* $\mathbf{I}_3$*,* $\mathbf{J}_n$ $(n \geq 0)$*.*

*Proof.* Let us first prove the converse implication. If $\mathbf{L}$ is unavoidable then, by Theorem 6.2 and the fact that no one of the four lattices $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ is embeddable into any of the lattices $\mathbf{J}_n$ $(n \geq 0)$, it follows that no member of the extended sequence can be isomorphic to a sublattice of $\mathbf{L}$.

In order to prove the direct implication, assume that $\mathbf{L} \in \mathcal{D}$ and $\mathbf{L}$ does not contain a sublattice isomorphic to any of the lattices in the extended sequence $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, $\mathbf{J}_n$ $(n \geq 0)$.

Claim: If $\mathbf{L}$ is $+_o$-indecomposable, then $\mathbf{L}$ is isomorphic to either $\mathbf{B}_{2,3} +_o' \mathbf{D}_k$ or $\mathbf{D}_k +_o' \mathbf{B}_{2,3}$ or $\mathbf{D}_{k+1}$ for some $k \geq 0$.

Since $\mathbf{L}$ is $+_o$-indecomposable, $\mathbf{L}^\partial$ is a primitive ordered set. By Theorem 3.6(2), $\mathbf{L}^\partial$ admits an $n$-tower, where $n = h_\diamond(\mathbf{L}^\partial)$. Let $B_0, \dots, B_{n-1}$ be a tower for $\mathbf{L}^\delta$. Recall from Section 3 that a lattice $\mathbf{K}$ is isomorphic to a sublattice of $\mathbf{L}$ if and only if there is a monotone surjective mapping of some convex subset of $\mathbf{L}^\partial$ onto $\mathbf{K}^\partial$. This, by $\mathbf{I}_2 \not\leq \mathbf{L}$ and $\mathbf{I}_3 \not\leq \mathbf{L}$, implies that every block of the tower $B_0, \dots, B_{n-1}$ has type 2 or 2′. Moreover, as $\mathbf{I}_0 \not\leq \mathbf{L}$, it follows that if $B_i$ has type 2′, then either $i = 0$ or $i = n-1$. Also, since $\mathbf{I}_1 \not\leq \mathbf{L}$, it follows that every two consecutive blocks of the tower form an ordinal sum in the induced order of $\mathbf{L}^\partial$. Since $\mathbf{J}_n \leq \mathbf{L}$ for no $n$, refining the previous observation about blocks of type 2′ we conclude that the tower $B_0, \dots, B_{n-1}$ has at most one block of type 2′. Thus $\mathbf{L}$ must be isomorphic to either $\mathbf{B}_{2,3} +_o' \mathbf{D}_k$ or $\mathbf{D}_k +_o' \mathbf{B}_{2,3}$ or $\mathbf{D}_{k+1}$ for some $k \geq 0$. The claim has been proved.

If $\mathbf{L}$ is $+_o$-indecomposable then, by the above Claim, $\mathbf{L}$ is isomorphic to a proper sublattice of some $\mathbf{J}_n$ and so, by Theorem 6.2, $\mathbf{L}$ is unavoidable.

Let now $\mathbf{L}$ be $+_o$-decomposable. Then $\mathbf{L}$ is of the form $\mathbf{A}_1 +_o \cdots +_o \mathbf{A}_m$, where $m > 1$ and each $\mathbf{A}_i$ is either a one-element lattice or is $+_o$-indecomposable. If $\mathbf{A}_i$ is $+_o$-indecomposable then, by the above Claim and the assumption that $\mathbf{I}_0 \not\leq \mathbf{L}$, it follows that:

- If $1 < i < m$, then $\mathbf{A}_i \cong \mathbf{D}_k$ for some $k \geq 0$
- If $i = 1$, then $\mathbf{A}_i \cong \mathbf{B}_{2,3} +_o' \mathbf{D}_k$ or $\cong \mathbf{D}_{k+1}$ for some $k \geq 0$
- If $i = m$, then $\mathbf{A}_i \cong \mathbf{D}_k +_o' \mathbf{B}_{2,3}$ or $\cong \mathbf{D}_{k+1}$ for some $k \geq 0$

As $\mathbf{J}_n \not\leq \mathbf{L}$ for all $n$, any combination of the three possibilities implies that $\mathbf{L}$ is a proper sublattice of some $\mathbf{J}_n$. Thus, by Theorem 6.2, $\mathbf{L}$ is unavoidable. •

**Corollary 6.6.** *The set consisting of the lattices* $\mathbf{I}_0$*,* $\mathbf{I}_1$*,* $\mathbf{I}_2$*,* $\mathbf{I}_3$ *and* $\mathbf{J}_n$ $(n \geq 0)$ *is a least complete infinite anti-chain in* $\langle \mathcal{D}, \leq \rangle$*.*

*Proof.* It follows from Theorems 6.2 and 6.5. •

## 7. Companions of $\leq$ on $\mathcal{D}$

Each finite distributive lattice has both the smallest and greatest element. These elements are denoted by 0 and 1, respectively. Thus there are three natural companions of the pre-ordering $\leq$ on the class $\mathcal{D}$. They are $\leq_2$, $\leq_1$, $\leq_0$ defined at the beginning of Section 2.

Following notations from section 1, we define for $n \geq 0$

$$\begin{aligned} \mathbf{J}_n^0 &= \mathbf{1} +_o \mathbf{B}_{2,3} +'_o \mathbf{D}_n \\ \mathbf{J}_n^1 &= \mathbf{D}_n +'_o \mathbf{B}_{2,3} +_o \mathbf{1} \\ \mathbf{J}_n^2 &= \mathbf{1} +_o \mathbf{J}_n +_o \mathbf{1}. \end{aligned}$$

We also define for $N > 0$

$$\mathcal{A}_N^2 = \{\mathbf{L} \in \mathcal{D} : \ \forall k \geq N, \ \mathbf{D}_k \not\leq_2 \mathbf{L}, \ \mathbf{J}_k^0 \not\leq_2 \mathbf{L}, \ \mathbf{J}_k^1 \not\leq_2 \mathbf{L}, \text{ and } \mathbf{J}_k^2 \not\leq_2 \mathbf{L}\}$$

$$\mathcal{A}_N^1 = \{\mathbf{L} \in \mathcal{D} : \ \forall k \geq N, \ \mathbf{B}_{2,3} +'_o \mathbf{D}_k \not\leq_1 \mathbf{L} \text{ and } \ \mathbf{J}_k +_o \mathbf{1} \not\leq_1 \mathbf{L}\}$$

$$\mathcal{A}_N^0 = \{\mathbf{L} \in \mathcal{D} : \ \forall k \geq N, \ \mathbf{D}_k +'_o \mathbf{B}_{2,3} \not\leq_0 \mathbf{L} \text{ and } \mathbf{1} +_o \mathbf{J}_k \not\leq_0 \mathbf{L}\}$$

For $P$ a finite ordered set and for $i \in \{0, 1, 2\}$ and $N > 0$, we define $P \in \mathcal{P}_N^i$ to mean that $P^\partial \in \mathcal{A}_N^i$.

**Theorem 7.1.** *For $i = 0, 1, 2$ and each order-ideal $\mathcal{A}$ of $\langle \mathcal{D}, \leq_i \rangle$, $\langle \mathcal{A}, \leq_i \rangle$ is well-quasi-ordered iff $\mathcal{A} \subseteq \mathcal{A}_N^i$ for some $N > 0$.*

*Proof.* Case 1: $i = 2$. The direct implication follows from the observation that the set consisting of $\mathbf{D}_n$ $(n \geq 1)$, $\mathbf{J}_n^0$ $(n \geq 0)$, $\mathbf{J}_n^1$ $(n \geq 0)$, and $\mathbf{J}_n^2$ $(n \geq 0)$ is an infinite anti-chain with respect to $\leq_2$.

For the converse implication, we have to show that $\langle \mathcal{A}_N^2, \leq_2 \rangle$ is well-quasi-ordered for each $N > 0$. Suppose that this is false, for a certain $N > 0$. Clearly, we can assume that $N \geq 3$. Choose, by Theorem 1.1, a minimal bad sequence $\langle \mathbf{L}_n : n < \omega \rangle$ in $\langle \mathcal{A}_N^2, \leq_2 \rangle$.

Claim 1: $\mathbf{L}_n \in \mathcal{A}_N$ for all $n < \omega$.

The claim follows from the observation that, for all $n$, $\mathbf{D}_{n+2} \leq_2 \mathbf{J}_n$, $\mathbf{J}_{n+1}^0 \leq_2 \mathbf{1} +_o \mathbf{J}_n$, $\mathbf{J}_{n+1}^1 \leq_2 \mathbf{J}_n +_o \mathbf{1}$, and $\mathbf{J}_n^2 = \mathbf{1} +_o \mathbf{J}_n +_o \mathbf{1}$.

Now let $P_n = \mathbf{L}_n^\partial$.

Claim 2: There are only a finite number of $n$ for which $P_n$ has a cut-point other than a top or bottom element.

For suppose that this fails. By cutting down to an infinite subsequence, we can suppose that for every $n$, $P_n$ has the cut-point $c_n$ that is neither least nor greatest element in $P_n$. This means that $\mathbf{L}_n = \mathbf{A}_n +_o \mathbf{B}_n$ where each of $\mathbf{A}_n$, $\mathbf{B}_n$ has at least two elements. Put $\mathbf{A}'_n = \mathbf{A}_n +_o \mathbf{1}$, $\mathbf{B}'_n = \mathbf{1} +_o \mathbf{B}_n$. It follows that $\mathbf{A}'_n <_2 \mathbf{L}_n$ and $\mathbf{B}'_n <_2 \mathbf{L}_n$. (The reader can easily supply a $0,1$-embedding of $\mathbf{A}'_n$ into $\mathbf{L}_n$, and it is a proper embedding.) By the minimality of the bad sequence $\langle \mathbf{L}_n : n \in \omega \rangle$, there is an infinite subsequence $\langle \mathbf{L}_{i_n} : n \in \omega \rangle$ such that $\mathbf{A}'_{i_n} \leq_2 \mathbf{A}'_{i_m}$ whenever $n < m$.

By the same token, this sequence has an infinite subsequence, which we denote as $\langle \mathbf{L}_{j_n} : n \in \omega \rangle$, such that $\mathbf{B}'_{j_n} \leq_2 \mathbf{B}'_{j_m}$ whenever $n < m$, and also $\mathbf{A}'_{j_n} \leq_2 \mathbf{A}'_{j_m}$ whenever $n < m$. So now, we have $\mathbf{A}'_{j_0} \leq_2 \mathbf{A}'_{j_1}$ and $\mathbf{B}'_{j_0} \leq_2 \mathbf{B}'_{j_1}$. This implies that $\mathbf{A}_{j_0} \leq_0 \mathbf{A}_{j_1}$ and $\mathbf{B}_{j_0} \leq_1 \mathbf{B}_{j_1}$. But these relations clearly imply that

$$\mathbf{L}_{j_0} = \mathbf{A}_{j_0} +_o \mathbf{B}_{j_0} \leq_2 \mathbf{A}_{j_1} +_o \mathbf{B}_{j_1} = \mathbf{L}_{j_1} .$$

This contradiction establishes the claim.

Claim 3: There are only a finite number of $n$ for which $P_n$ has a bottom element and a top element.

Suppose that this fails. Thus (by cutting down the original sequence) suppose that every $P_n$ has top and bottom elements. Thus $\mathbf{L}_n = \mathbf{1} +_o \mathbf{A}_n +_o \mathbf{1}$ for some non-empty lattice $\mathbf{A}_n$, for each $n$. By Claim 1, $\mathbf{L}_n$ and thus also $\mathbf{A}_n$ belong to $\mathcal{A}_N$. Since $\langle \mathcal{A}_N, \leq \rangle$ is well-quasi-ordered (Theorem 6.3), there is $i < j$ for which $\mathbf{A}_i \leq \mathbf{A}_j$. But this clearly gives that $\mathbf{L}_i \leq_2 \mathbf{L}_j$.

The three claims above demonstrate that we can assume (by cutting down the sequence) that there are primitive ordered sets $Q_n$ such that either $P_n = Q_n$ for all $n$, or $P_n = Q_n +_o 1$ for all $n$, or $P_n = 1 +_o Q_n$ for all $n$. Since the second and third cases are dual, we shall treat only the first and second case.

Case 1.1: $P_n = Q_n$ is primitive for all $n$.

Now $P_n \in \mathcal{P}^2_N \subseteq \mathcal{P}_N$ and, in particular, $\mathbf{D}_k \not\leq_2 P^\partial_n$ for every $k \geq N$, implying that $h_\diamond(P_n) < N$. Thus by Theorem 5.7 (especially statement (i) of the theorem), $P_n$ is nicely $(M)$-structured, $M = 5 \cdot (18N)^{6N+4}$ and in fact admits a nice basic $(M)$-system $\overline{S}_n = (X; b_0, \dots, b_{k_n-1}; c_0, \dots, c_{k_n-1})$. The ordered subset of $P_n$ of which this is the basic skeleton, namely $S_n = X \cup \{b_0, \dots, b_{k_n-1}\} \cup \{c_0, \dots, c_{k_n-1}\}$, has at most $M$ elements.

Now by cutting down our sequence, and by replacing some $P_n$ by isomorphic ordered sets, we can assume that the ordered sets $S_n$ are the same for all $n$, and likewise the basic skeletons $\overline{S}_n$. Write

$$\overline{S} = (X; b_0, \dots, b_{k-1}; c_0, \dots, c_{k-1})$$

for this skeleton. Then $P_n = \overline{S}(P^0_n, \dots, P^{k-1}_n)$ where the $P^i_n$ are pairwise disjoint convex subsets of $P_n$. (See the paragraph following Definition 5.5.) For $0 \leq r < k$, $I_{P_n}[b_r, c_r] = P^r_n \cup \{b_r, c_r\}$ is a bounded convex subset of $P_n$, and there is a monotone map of $P_n$ onto this interval. Since $P_n$ is primitive, the interval is certainly a proper subset of $P_n$. It follows that $I_{P_n}[b_r, c_r]^\partial <_2 \mathbf{L}_n$.

Now again using the minimality of our sequence and cutting it down $k$ times, we find that we can assume that for all $i < j$ and for all $0 \leq r < k$ we have that there is a monotone map of $I_{P_j}[b_r, c_r]$ onto $I_{P_i}[b_r, c_r]$. Now returning to the proof of Theorem 5.6 and visiting Remark 5.1, we see that this gives a monotone map of $P_j$ onto $P_i$, consequently a $0,1$-embedding of $\mathbf{L}_i$ into $\mathbf{L}_j$. This is a contradiction.

Case 1.2: $P_n = Q_n +_o 1$ for all $n$ where $Q_n$ is primitive. To prove this, we need the following claim, which is obvious; recall that $\langle \mathbf{L}_n : n < \omega \rangle$ is a minimal bad sequence in $\langle \mathcal{A}^2_N, \leq_2 \rangle$.

<u>Claim 4</u>: There are only finitely many $n$ for which $Q_n$ is a ladder.

Continuing with Case 1.2, we next observe that since $P_n \in \mathcal{P}^2_N$ and so $P_n^\partial \not\geq_2 \mathbf{J}^1_k$ for all $k \geq N$, it follows that $Q_n^\partial \not\geq_0 \mathbf{D}_k +'_o \mathbf{B}_{2,3}$ for every $k \geq N$. Thus by Claim 4 and Theorem 5.7 (especially statement (ii) of the theorem), $Q_n$ is nicely $(M)$-structured, $M = 5 \cdot (18N)^{6N+4}$, in fact admits a nice $(M)$-system

$$\overline{S}_n = (X, \emptyset, Y_1; b_0, \ldots, b_{k_n-1}; c_0, \ldots, c_{k_n-1}) .$$

As we did above for Case 1.1, we now work through the steps of our argument for Theorem 6.1.

By cutting down (and replacing some $P_n$ by isomorphic ordered sets), we can assume that for all $n$, the basic skeletons

$$\overline{S}'_n = (X; b_0, \ldots, b_{k_n-1}; c_0, \ldots, c_{k_n-1})$$

are the same sets, i.e.,

$$\overline{S}'_n = \overline{S}' = (X; b_0, \ldots, b_{k-1}; c_0, \ldots, c_{k-1}) ;$$

and have the same induced order from $Q_n$. Thus we can write

$$P_n = \overline{S}_n(Q^0_n, \ldots, Q^{k-1}_n) +_o 1 .$$

It is true, also in Case 1.2, that for $0 \leq r < k$, $I_{P_n}[b_r, c_r] = Q^r_n \cup \{b_r, c_r\}$ is a bounded convex subset of $P_n$, implying that there is a monotone map of $P_n$ onto this interval. Thus $I_{P_n}[b_r, c_r] = I_{Q_n}[b_r, c_r]^\partial <_2 \mathbf{L}_n$.

Now the proof goes just as in Case 1.1, except in the very last step, where the application of Remark 5.1 following Theorem 5.6 gives that where $\mathbf{A}_n = Q^\partial_n$ (and $\mathbf{L}_n \cong \mathbf{A}_n +_o \mathbf{1}$) there is $i < j$ with $\mathbf{A}_i \leq_0 \mathbf{A}_j$. Clearly, this implies that $\mathbf{L}_i \leq_2 \mathbf{L}_j$. This contradiction finishes our proof of this theorem in Case 1.

<u>Case 2</u>: $i = 1$. The direct implication follows from the fact that the set consisting of $\mathbf{B}_{2,3} +'_o \mathbf{D}_n$ ($n \geq 1$) and $\mathbf{J}_n +_o \mathbf{1}$ ($n \geq 0$) is an infinite anti-chain with respect to $\leq_1$. The converse follows from the case $i = 2$ just proved and the observations that for any $\mathbf{A}, \mathbf{B} \in \mathcal{D}$, $\mathbf{A} \leq_1 \mathbf{B}$ iff $\mathbf{1} +_o \mathbf{A} \leq_2 \mathbf{1} +_o \mathbf{B}$, and $\mathbf{A} \in \mathcal{A}^1_N$ iff $\mathbf{1} +_o \mathbf{A} \in \mathcal{A}^2_N$.

Finally, the result for the pre-order $\leq_0$ is the dual of the result for Case 2, so we omit the proof of it. •

Recall that $\mathbf{I}_0$ is the lattice $\mathbf{1} +_o \mathbf{B}_{2,3} +_o \mathbf{1}$ and $\mathbf{I}_3$ is the eight-element Boolean lattice.

**Theorem 7.2.** *The following are true for a finite distributive lattice* $\mathbf{L}$*:*

1. $\mathbf{L}$ *is avoidable in* $\langle \mathcal{D}, \leq_2 \rangle$ *if and only if it has a* $0,1$*-sublattice isomorphic to either* $\mathbf{1}$ *or* $\mathbf{D}_1$ *or* $\mathbf{D}_1 +_o \mathbf{1}$ *or* $\mathbf{1} +_o \mathbf{D}_1$ *or* $\mathbf{I}_0$*.*
2. $\mathbf{L}$ *is avoidable in* $\langle \mathcal{D}, \leq_1 \rangle$ *if and only if it has a* $1$*-sublattice isomorphic to either* $\mathbf{D}_1$ *or* $\mathbf{I}_0$ *or* $\mathbf{I}_3 +_o \mathbf{1}$*.*
3. $\mathbf{L}$ *is avoidable in* $\langle \mathcal{D}, \leq_0 \rangle$ *if and only if it has a* $0$*-sublattice isomorphic to either* $\mathbf{D}_1$ *or* $\mathbf{I}_0$ *or* $\mathbf{1} +_o \mathbf{I}_3$*.*

*Proof.* (1): Each of the five lattices is avoidable in $\langle \mathcal{D}, \leq_2 \rangle$, since the first four of them constitute a $0,1$-anti-chain together with the lattices $\mathbf{J}_n^2$ ($n \geq 0$), and the last constitutes a $0,1$-anti-chain together with $\mathbf{D}_n$ ($n \geq 1$). Thus every finite distributive lattice containing a $0,1$-sublattice isomorphic to one of the five lattices is avoidable in $\langle \mathcal{D}, \leq_2 \rangle$. Let $\mathbf{L}$ be a finite distributive lattice containing no $0,1$-sublattice isomorphic to one of the five lattices. Since $\mathbf{L}$ $0,1$-avoids the first four lattices, $\mathbf{L}$ is of the form $\mathbf{1} +_o \mathbf{K} +_o \mathbf{1}$ for a finite lattice $\mathbf{K}$ (or $\mathbf{L}$ is the two-element chain). Since it $0,1$-avoids $\mathbf{I}_0$, the lattice $\mathbf{K}$ avoids $\mathbf{B}_{2,3}$ and hence $\mathbf{L}$ is of the form $\mathbf{C}_0 +'_o \dots +'_o \mathbf{C}_{k-1}$ where $k \geq 1$, $\mathbf{C}_0$ and $\mathbf{C}_{k-1}$ are two-element lattices and every $\mathbf{C}_i$ is either $\mathbf{D}_1$ or the two-element lattice. Clearly, all such lattices $\mathbf{L}$ are $0,1$-embeddable into all sufficiently large lattices $\mathbf{D}_n$, into all sufficiently large lattices $\mathbf{J}_n^0$, into all sufficiently large lattices $\mathbf{J}_n^1$ and into all sufficiently large lattices $\mathbf{J}_n^2$. Thus it follows from Theorem 7.1(1) that $\mathbf{L}$ is unavoidable in $\langle \mathcal{D}, \leq_2 \rangle$.

(2): The three lattices are avoidable in $\langle \mathcal{D}, \leq_2 \rangle$, since the first of them constitutes a 1-anti-chain (i.e., anti-chain with respect to $\leq_1$) together with the lattices $\mathbf{B}_{2,3} +'_o \mathbf{D}_n$ and the last two constitute a 1-anti-chain together with the lattices $\mathbf{J}_n +_o \mathbf{1}$. Let $\mathbf{L}$ be a finite distributive lattice containing no 1-sublattice isomorphic to one of the three lattices. Since $\mathbf{D}_1 \not\leq_1 \mathbf{L}$, $\mathbf{L}$ is of the form $\mathbf{K} +_o \mathbf{1}$ for a finite lattice $\mathbf{K}$ (or $\mathbf{L}$ is trivial). Since $\mathbf{K}$ avoids both $\mathbf{1} +_o \mathbf{B}_{2,3}$ and $\mathbf{I}_3$, $\mathbf{L}$ is of the form $\mathbf{C}_0 +'_o \dots +'_o \mathbf{C}_{k-1}$ where $k \geq 1$, $\mathbf{C}_{k-1}$ is the two-element lattice, $\mathbf{C}_0$ is a sublattice of $\mathbf{B}_{2,3}$ and every $\mathbf{C}_i$ with $i > 0$ is either $\mathbf{D}_1$ or the two-element lattice. (To see that $\mathbf{C}_0$ is a sublattice of $\mathbf{B}_{2,3}$, use the facts that $\mathbf{1} +_o \mathbf{B}_{2,3} \leq \mathbf{B}_{2,4}$ and $\mathbf{I}_3 \not\leq \mathbf{K}$.) Clearly, all such lattices $\mathbf{L}$ are 1-embeddable into all sufficiently large lattices $\mathbf{B}_{2,3} +'_o \mathbf{D}_n$ and into all sufficiently large lattices $\mathbf{J}_n +_o \mathbf{1}$. Thus it follows from Theorem 7.1(2) that $\mathbf{L}$ is unavoidable in $\langle \mathcal{D}, \leq_1 \rangle$.

(3): This statement is dual to (2). •

It follows that the quasi-ordered sets $\langle \mathcal{D}, \leq_2 \rangle$, $\langle \mathcal{D}, \leq_1 \rangle$ and $\langle \mathcal{D}, \leq_0 \rangle$ contain no least complete infinite anti-chain. The quasi-ordered set $\langle \mathcal{D}, \leq \rangle$ is thus rather exceptional.

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