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SEMILATTICE-VALUED MEASURES

JIŘÍ TŮMA

ABSTRACT. We review some of recent and classical results in algebraic representations of lattices and put them into a framework of semilattice-valued measures. This enables to define categorial versions of many classical lattice representation problems.

INTRODUCTION

In this paper we present three variations on algebraic representations of lattices. The common theme is the concept of semilattice-valued measures that has been gaining importance recently. A special case, the semilattice-valued distances, appeared in various papers on algebraic representations of lattices but only recently F. Wehrung observed (see [8]) that there exists a natural concept of morphisms for distances. The concept of morphisms of distances allows to formulate categorial versions of many lattice representation problems. In particular, it allowed to formulate a sequence of problems of increasing difficulty about existence of simultaneous representations of diagrams of (distributive) semilattices satisfying some additional conditions. The first two problems in the sequence had a positive solution (Theorem 7.1 in [8], Theorem 9.2 in [12]). But the complexity of constructions needed for these positive solutions was increasing and it finally led F. Wehrung to the negative solution of CLP - the discovery of a distributive semilattice that is not isomorphic to the compact congruence semilattice of any lattice, see [14].

The negative solution of CLP illustrates nicely the fact that attempts to find simultaneous representations of diagrams of semilattices could finally lead to the discovery of an obstacle that prevents the existence of representations of a particular type for a single semilattice. That is why it is of interest to look at proofs of classical results on algebraic representations of lattices and to

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see if the methods used to prove the results allow to be extended to categorial versions of these results.

The three variations are on representations of lattices as sublattices of partition lattices, as congruence lattices of algebras of a given type, and as intervals in subgroup lattices of groups. In all three cases, a representation of a given type for an algebraic lattice L can be stated as the existence of a special mapping α into the $\langle\vee,0\rangle$-semilattice R of compact element of L. In all three cases, the mapping (here called a semilattice-valued measure) α into R induces a pair of mappings $\alpha^\dagger : K \to L$ and $\alpha^* : L \to K$ satisfying

$$\alpha^\dagger(x) \le y \quad \text{if and only if} \quad x \le \alpha^*(y) \tag{0.1}$$

for every $x \in K$ and $y \in L$. Such a pair of mappings is called an *adjoint pair*. In the first variation, the lattice K is a partition lattice, in the second variation it is the congruence lattice of an algebra of the given type, and in the third variation it is the subgroup lattice of a group.

The concept of adjoint pairs is very useful since it allows to translate properties of α^* into the corresponding properties of $\alpha^\dagger$ and then to the properties of the measure α itself. If we replace the lattice L by its dual L^d, then both mappings $\alpha^\dagger : K \to L^d$ and $\alpha^* : L^d \to K$ reverse the order. Also $\operatorname{Im}\alpha^\dagger$ is a complete meet-subsemilattice of L^d and $\operatorname{Im}\alpha^*$ is a complete meet-subsemilattice of K. Thus the mappings $\alpha^\dagger : K \to L^d$ and $\alpha^* : L^d \to K$ define a Galois connection between K and L^d. This observation explains why it is possible to translate properties of $\alpha^\dagger$ in terms of properties of α^* and vice versa. A general theorem about the correspondence between the properties of α^* and $\alpha^\dagger$ in our context is Theorem 1.7 of [7]. This theorem is specialized to Propositions 1.3, 2.3 and 3.2 in each of the three variations. In each variation we also present some of classical and recent results on algebraic represenatations in the language of measures and formulate categorial versions of theses results as problems.

1. Semilattice-valued distances

If R is a $\langle\vee,0\rangle$-semilattice, then by $\mathrm{J}(R)$ we denote the ideal lattice of R. In some cases it is useful to identify every $r \in R$ with the principal ideal $(r\rangle$ of R generated by r. If this identification is made, then R becomes the subsemilattice of compact elements of $\mathrm{J}(R)$.

The following concept was used in various papers on algebraic representations of lattices, but it was only in [8], where the natural concept of morphisms for distances was introduced.

Definition 1.1. Let R be a $\langle\vee,0\rangle$-semilattice and let A be a set. A mapping $\alpha\colon A \times A \to R$ is an *R-valued distance* on A, shortly a *distance* on A, if the following statements hold:

(D1) $\alpha(a,a) = 0_R$, for all $a \in A$,
(D2) $\alpha(a,b) = \alpha(b,a)$, for all $a, b \in A$,
(D3) $\alpha(a,c) \leq \alpha(a,b) \vee \alpha(b,c)$, for all $a, b, c \in A$.

We will refer to the condition (D3) as the *triangle inequality.*

If both A and R are finite, then we say that a distance $\alpha\colon A \times A \to R$ is a *finite distance.*

If $\alpha : A \times A \to R$ is a distance, then by assigning to every ideal $I \in \mathrm{J}(R)$ the set

$$\alpha^*(I) = \{(a,b) \in A \times A \mid \alpha(a,b) \in I\}$$

we obtain a mapping from $\mathrm{J}(R)$ to the partition lattice $\Pi(A)$ on the set A. It is straightforward to verify that the mapping $\alpha^* : \mathrm{J}(R) \to \Pi(A)$ preserves arbitrary meets (thus also the top element), hence by [10] there exists a unique complete $\langle \vee, 0\rangle$-preserving mapping $\alpha^\dagger : \Pi(A) \to \mathrm{J}(R)$ such that

$$\alpha^\dagger(\pi) \leq I \quad \text{if and only if} \quad \pi \leq \alpha^*(I) \tag{1.1}$$

for every $\pi \in \Pi(A)$ and $I \in \mathrm{J}(R)$. The pair of mappings $\langle \alpha^\dagger, \alpha^* \rangle$ will be called the *adjoint pair* defined by α.

It is also easy to verify that the mapping α^* preserves joins of up-directed subsets of $\mathrm{J}(R)$. By [13], this is equivalent to the fact that the mapping $\alpha^\dagger : \Pi(A) \to \mathrm{J}(R)$ maps compact elements of $\Pi(A)$ to compact elements of $\mathrm{J}(R)$, shortly that $\alpha^\dagger$ is *compactness preserving.*

For every $a, b \in A$ we denote by $\pi_{a,b}$ the partition of A consisting only of singletons except possibly (if $a \neq b$) the block $\{a, b\}$. Then $\alpha^\dagger(\pi_{a,b})$ is a compact element of $\mathrm{J}(R)$, i.e. a principal ideal of R. Thus $\alpha^\dagger(\pi_{a,b}) = (r\rangle$ for some element $r \in R$.

Lemma 1.2. *Let $\alpha : A \times A \to R$ be an R-valued distance on a set A and let $\langle \alpha^\dagger, \alpha^* \rangle$ be the adjoint pair defined by α. Then*

$$\alpha^\dagger(\pi_{a,b}) = (\alpha(a,b)\rangle$$

for every $a, b \in A$.

Using the identification of elements of R with the principal ideals in $\mathrm{J}(R)$ they generate we can write directly

$$\alpha^\dagger(\pi_{a,b}) = \alpha(a,b)$$

for every $a, b \in A$.

Proof. Let $\alpha^\dagger(\pi_{a,b}) = (r\rangle$ for $r \in R$. For every $s \in R$ we have $\alpha(a,b) \leq s$ if and only if $(a,b) \in \alpha^*((s\rangle)$ and this is the case if and only if $\pi_{a,b} \leq \alpha^*((s\rangle)$. From the formula (1.1) defining $\alpha^\dagger$ we obtain that $\pi_{a,b} \leq \alpha^*((s\rangle)$ if and only if $\alpha^\dagger(\pi_{a,b}) \leq (s\rangle$. Since $\alpha^\dagger(\pi_{a,b}) = (r\rangle$, the last inequality holds if and only if

$r \leq s$. Thus $\alpha(a,b) \leq s$ if and only if $r \leq s$. Since this is true for any $s \in R$, we get $\alpha(a,b) = r$. This proves $\alpha^\dagger(\pi_{a,b}) = (\alpha(a,b)\rangle$. □

Conversely, if we have an adjoint pair of mappings $\alpha^\dagger : \Pi(A) \to \mathrm{J}(R)$ and $\alpha^* : \mathrm{J}(R) \to \Pi(A)$ satisfying (1.1) and such that $\alpha^\dagger$ is compactness-preserving, then we can define a mapping $\alpha : A \times A \to R$ by

$$\alpha(a,b) = r \quad \text{if and only if} \quad \alpha^\dagger(\pi_{a,b}) = (r\rangle. \tag{1.2}$$

It is straightforward to verify that α is in fact an R-valued distance on A and that $\langle \alpha^\dagger, \alpha^* \rangle$ is the adjoint pair defined by α. Thus constructing R-valued distances on A is equivalent either to constructing complete $\langle 1, \wedge \rangle$-homomorphisms from $\mathrm{J}(R)$ to $\Pi(A)$ or to constructing complete $\langle \vee, 0 \rangle$-homomorphisms from $\Pi(A)$ to $\mathrm{J}(R)$.

This connection between a distance α and the adjoint pair $\langle \alpha^\dagger, \alpha^* \rangle$ defined by α allows us to construct complete $\langle 1, \wedge \rangle$-preserving homomorphisms $\alpha^* : \mathrm{J}(R) \to \Pi(A)$ using R-valued distances on A and to prove their properties using the properties of the adjoint $\langle \vee, 0 \rangle$-homomorphisms $\alpha^\dagger : \Pi(A) \to \mathrm{J}(R)$. Since the set of partitions $\pi_{a,b}$ of A, $a, b \in A$, is a join-generating subset of $\Pi(A)$ and the vaules $\alpha^\dagger(\pi_{a,b})$ are determined by the distances $\alpha(a,b)$ by Lemma 1.2, we can determine various properties of $\alpha^\dagger$ by the properties of the distance α. As an example we state the following proposition characterizing the R-valued distances on A such that the mappings α^* are embeddings of $\mathrm{J}(R)$ into the partition lattice $\Pi(A)$. The proposition is a special case of Theorem 1.7 of [7].

Proposition 1.3. *Let $\alpha : A \times A \to R$ be an R-valued distance on A and let $\langle \alpha^\dagger, \alpha^* \rangle$ be the adjoint pair defined by α. Then*

(i) *the mapping α^* is injective if and only if the set $\{\alpha(a,b) \mid (a,b) \in A \times A\}$ join-generates R. In particular, the mapping α^* is injective if the mapping α is onto.*
(ii) *The mapping α^* preserves joins if and only if for every $a, b \in A$ and every $r, s \in R$, if $\alpha(a,b) \leq r \vee s$, then $(a,b) \in \alpha^*(r) \vee \alpha^*(s)$.*

The following natural concept of morphisms for distances was introduced in [8].

Definition 1.4. If $\alpha\colon A \times A \to R$ is a distance on A and $\beta\colon B \times B \to S$ is a distance on B, then by a *morphism* from α to β we mean a pair $(\boldsymbol{f}, f)$, where $\boldsymbol{f} : R \to S$ is a $\langle \vee, 0 \rangle$-homomorphism and $f : A \to B$ is a mapping such that for every $a, b \in A$

$$\boldsymbol{f}(\alpha(a,b)) = \beta(f(a), f(b)). \tag{1.3}$$

We also write $(\boldsymbol{f}, f) : \alpha \to \beta$.

So the following diagram must commute.

$$
\begin{array}{ccc}
R & \xrightarrow{\boldsymbol{f}} & S \\
\alpha \big\downarrow & & \big\downarrow \beta \\
A \times A & \xrightarrow[f \times f]{} & B \times B
\end{array} \tag{1.4}
$$

It is straihgtforward to check that if $\gamma : C \times C \to T$ is another distance and $(\boldsymbol{f}, f) : \alpha \to \beta$, $(\boldsymbol{g}, g) : \beta \to \gamma$ are morphisms of distances, then the *product* $(\boldsymbol{gf}, gf)$ is a morphism from α to γ. With this definition of product (composition) of morphisms the class of all distances together with morphisms defined in Definition 1.4 form a category. The identity morphism on α is the pair $(\boldsymbol{i}, i)$, where $\boldsymbol{i}$ is the identity morphism on R and i is the identity mapping on A. We will denote this category by $\mathbf{D}$ and refer to it as the *category of distances*. The full subcategory of finite distances will be denoted by $\mathbf{D}_{\text{fin}}$. The *forgetful functor* $\mathcal{F}$ on the category of distances $\mathbf{D}$ assigns to every distance $\alpha : A \times A \to R$ the semilattice R and to every morphism $(\boldsymbol{f}, f) : \alpha \to \beta$ from α to $\beta : B \times B \to S$ the $\langle \vee, 0 \rangle$-homomorphism $\boldsymbol{f} : R \to S$. Thus the forgetful functor maps the category of distances $\mathbf{D}$ to the category $\mathbf{S}$ of $\langle \vee, 0 \rangle$-semilattices with $\langle \vee, 0 \rangle$-homomorphisms. The full subcategory of $\mathbf{S}$ of finite $\langle \vee, 0 \rangle$-semilattices will be denoted by $\mathbf{S}_{\text{fin}}$.

We also state a simple lemma translating extensions of distances to the language of complete $\langle 1, \wedge \rangle$-homomorphisms to partition lattices.

Lemma 1.5. *Let $A \subseteq B$ be sets, R be a $\langle \vee, 0 \rangle$-semilattice, and $\alpha : A \times A \to R$ and $\beta : B \times B \to R$ be distances. Then β extends α if and only if for every $r \in R$*

$$\beta^*((r\rangle) \cap (A \times A) = \alpha^*((r\rangle). \tag{1.5}$$

Proof. Let β extend α and $(a, b) \in \beta^*((r\rangle) \cap (A \times A)$. Then $\beta(a, b) \leq r$, thus also $\alpha(a, b) \leq r$ and $(a, b) \in \alpha^*((r\rangle)$. It follows that $\beta^*((r\rangle) \cap (A \times A) \subseteq \alpha^*((r\rangle)$. The opposite inclusion is obvious, hence (1.5) holds.

Conversely, let (1.5) hold for every $r \in R$. Take $a, b \in A$. Then for any $r \in R$, $\alpha(a, b) \leq r$ if and only if $(a, b) \in \alpha^*((r\rangle)$ if and only if $(a, b) \in \beta^*((r\rangle) \cap (A \times A)$ and this holds if and only if $\beta(a, b) \leq r$. Thus $\alpha(a, b) \leq r$ if and only if $\beta(a, b) \leq r$. Since this is true for every $r \in R$, we get $\alpha(a, b) = \beta(a, b)$. $\square$

The following concept of a simultaneous representation, or a lifting, was probably used in the area of algebraic representations of lattices for the first time in [4].

Definition 1.6. Let $\mathbf{P}$ and $\mathbf{Q}$ be categories and let $\mathcal{F} : \mathbf{P} \to \mathbf{Q}$ be a functor. We say that a functor $\mathcal{A} : \mathbf{Q} \to \mathbf{P}$ is a *simultaneous representation*, or a *lifting*, of the category $\mathbf{Q}$ in the category $\mathbf{P}$ with respect to the functor $\mathcal{F}$ if the composition functor $\mathcal{F} \circ \mathcal{A} : \mathbf{Q} \to \mathbf{Q}$ is naturally equivalent to the identity functor on $\mathbf{Q}$.

A classical result of P. Whitman [15] says that every lattice can be embedded into an infinite partition lattice. Since every lattice L is a sublattice of a lattice L_0 with a least element 0_L and this in turn is a sublattice of the ideal lattice $\mathrm{J}(L_0)$, Whitman's theorem can be restated in the language of distances as follows.

Theorem 1.7. *For every $\langle \vee, 0 \rangle$-semilattice R there exists an R-valued distance $\alpha : A \times A \to R$ satisfying the two conditions of Proposition 1.3.*

In [1], B. Jónsson presented a different proof of Whitman's theorem. His proof is a kind of a free, or a canonical, construction extending a given distance $\alpha : A \times A \to R$ satisfying only the condition (i) of Proposition 1.3 to a distance $\beta : B \times B \to R$ that satisfies both (i) and (ii) of Proposition 1.3. Since the extension is free we conjecture that the following is true.

Conjecture 1.8. There exists a functor $\mathcal{F} : \mathbf{S} \to \mathbf{D}$ such that the composition of $\mathcal{F}$ with the forgetful functor from $\mathbf{D}$ to $\mathbf{S}$ is equal to the identity functor on $\mathbf{S}$ and for every object R of $\mathbf{S}$, the distance $\mathcal{F}(R)$ satisfies the conditions (i) and (ii) of Proposition 1.3.

A special case of the conjecture for the full subcategory of $\mathbf{S}$ consisting of distributive semilattices was proved in [8].

Unlike Jónsson's proof of Whitman's theorem, the proof that every finite lattice can be embedded into a finite partition lattice given in [5] is not free or canonical in any sense. Thus there is no reasonable conjecture about a possible answer to the following problem.

Problem 1. Is there a lifting of the category $\mathbf{S}_{\mathrm{fin}}$ into the category $\mathbf{D}_{\mathrm{fin}}$ of finite distances with respect to the forgetful functor from $\mathbf{D}_{\mathrm{fin}}$ to $\mathbf{S}_{\mathrm{fin}}$ such that the distance $\mathcal{F}(R)$ satisfies the conditions (i) and (ii) of Proposition 1.3 for every object R of $\mathbf{S}_{\mathrm{fin}}$?

2. Congruence distances

In this section we will consider semilattice-valued distances defined on algebras.

Definition 2.1. Let $\mathbf{A} = (A, F)$ be a finitary algebra and let $\alpha : A \times A \to R$ be a distance on A. We say that α is a *congruence distance* on $\mathbf{A}$ if

(D4) $\alpha(f(a_1,\dots,a_n), f(b_1,\dots,b_n)) \le \bigvee_{k=1}^{n} \alpha(a_k, b_k)$ for every operation $f \in F$ (of arity n), and every $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A$.

Becuase of the condition (D4), the equivalence relation $\alpha^*(I)$ is in fact a congruence of the algebra $\mathbf{A}$, thus we can restrict the range of α^* to the congruence lattice $\mathrm{Con}(\mathbf{A})$ of $\mathbf{A}$. Since $\alpha^* : \mathrm{J}(R) \to \mathrm{Con}(\mathbf{A})$ preserves arbitrary meets and the top element, there exists again a unique mapping $\alpha^\dagger : \mathrm{Con}(\mathbf{A}) \to \mathrm{J}(R)$ such that

$$\alpha^\dagger(\pi) \le I \quad \text{if and only if} \quad \pi \le \alpha^*(I) \tag{2.1}$$

for every $\pi \in \mathrm{Con}(\mathbf{A})$ and $I \in \mathrm{J}(R)$. If $\alpha : A \times A \to R$ is a congruence distance on an algebra $\mathbf{A}$, then by the *adjoint pair defined by* α we mean the pair of adjoint mappings $\langle \alpha^\dagger, \alpha^* \rangle$.

Since $\alpha^* : \mathrm{J}(R) \to \mathrm{Con}(\mathbf{A})$ also preserves joins of directed subsets of $\mathrm{J}(R)$, the mapping $\alpha^\dagger$ preserves not only arbitrary joins and the least element, but is also compactness-preserving. It follows that for any compact congruence π of $\mathbf{A}$, the ideal $\alpha^\dagger(\pi)$ of R is principal, hence $\alpha^\dagger(\pi) = (r\rangle$ for some $r \in R$.

If $a, b \in A$, then we denote by $\Theta(a, b)$ the principal congruence defined by the pair (a, b). The following lemma is an analogue of Lemma 1.2 for congruence distances.

Lemma 2.2. *Let* $\alpha : A \times A \to R$ *be a congruence distance on an algebra* $\mathbf{A} = (A, F)$ *and let* $\langle \alpha^\dagger, \alpha^* \rangle$ *be the adjoint pair defined by* α. *Then*

$$\alpha^\dagger(\Theta(a, b)) = (\alpha(a, b)\rangle$$

for every $a, b \in A$.

Using the identification of elements of R *with the principal ideals in* $\mathrm{J}(R)$ *they generate we can write directly*

$$\alpha^\dagger(\Theta(a, b)) = \alpha(a, b)$$

for every $a, b \in A$.

Proof. We can follow the proof of Lemma 1.2 with only small changes. Let $\alpha^\dagger(\Theta(a, b)) = (r\rangle$ for $r \in R$. For every $s \in R$ we have $\alpha(a, b) \le s$ if and only if $(a, b) \in \alpha^*((s\rangle)$ and this is the case if and only if $\Theta(a, b) \le \alpha^*((s\rangle)$. From the formula (2.1) defining $\alpha^\dagger$ we obtain that $\Theta(a, b) \le \alpha^*((s\rangle)$ if and only if $\alpha^\dagger(\Theta(a, b)) \le (s\rangle$. Since $\alpha^\dagger(\Theta(a, b)) = (r\rangle$, the last inequality holds if and only if $r \le s$. Thus $\alpha(a, b) \le s$ if and only if $r \le s$. Since this is true for any $s \in R$, we get $\alpha(a, b) = r$. This proves $\alpha^\dagger(\Theta(a, b)) = (\alpha(a, b)\rangle$. □

As in the case of semilattice-valued distances, if we have adjoint mappings $\alpha^\dagger : \mathrm{Con}(\mathbf{A}) \to \mathrm{J}(R)$ and $\alpha^* : \mathrm{J}(R) \to \mathrm{Con}(\mathbf{A})$ satisfying (2.1) and such that

$\alpha^\dagger$ is compactness-preserving, then we can define a mapping $\alpha : A \times A \to R$ by

$$\alpha(a,b) = r \quad \text{if and only if} \quad \alpha^\dagger(\Theta(a,b)) = (r\rangle. \tag{2.2}$$

It is straightforward to verify that α is in fact an R-valued congruence distance on the algebra $\mathbf{A} = (A,F)$ and $\langle \alpha^\dagger, \alpha^* \rangle$ is the adjoint pair defined by α. Thus constructing R-valued congruence distances on $\mathbf{A}$ is equivalent either to constructing complete $\langle 1, \wedge \rangle$-preserving mappings from $\mathrm{J}(R)$ to $\mathrm{Con}(\mathbf{A})$ or to constructing complete $\langle \vee, 0 \rangle$-homomorphisms from $\mathrm{Con}(\mathbf{A})$ to $\mathrm{J}(R)$.

The connection between a congruence distance α on an algebra $\mathbf{A}$ and the adjoint pair $\langle \alpha^\dagger, \alpha^* \rangle$ can be used again to describe properties of α^* in terms of properties of the congruence distance α. Since the set of principal congruences $\Theta(a,b)$ of $\mathbf{A}$ is a join-generating subset of $\mathrm{Con}\,\mathbf{A}$ and the values $\alpha^\dagger(\Theta(a,b))$ are determined by the distances $\alpha(a,b)$ by Lemma 2.2, we can determine as in Section 1 various properties of $\alpha^\dagger$ by the properties of α. This is summarized in the following proposition that is again a special case of Theorem 1.7 of [7].

Proposition 2.3. *Let $\alpha : A \times A \to R$ be an R-valued congruence distance on an algebra $\mathbf{A} = (A,F)$ and let $\langle \alpha^\dagger, \alpha^* \rangle$ be the adjoint pair defined by α. Then*

(i) *the mapping α^* is injective if and only if the set $\{\alpha(a,b) \mid (a,b) \in A \times A\}$ join-generates R. In particular, the mapping α^* is injective if the mapping α is onto.*
(ii) *The mapping α^* preserves joins if and only if for every $a,b \in A$ and every $r,s \in R$, if $\alpha(a,b) \leq r \vee s$, then $(a,b) \in \alpha^*(r) \vee \alpha^*(s)$.*
(iii) *The mapping α^* is a lattice homomorphism from $\mathrm{J}(R)$ onto the interval $[\alpha^*(0_R), A \times A]$ in $\mathrm{Con}\,\mathbf{A}$ if and only if it preserves joins and for every $a,b,c,d \in A$, if $\alpha(c,d) \leq \alpha(a,b)$, then $(c,d) \in \alpha^*(0_R) \vee \Theta(a,b)$.*

Thus to represent the ideal lattice $\mathrm{J}(R)$ of a $\langle \vee, 0 \rangle$-semilattice R as the congruence lattice of an algebra in a class of algebras $\mathcal{V}$ closed under quotients requires to find an algebra $\mathbf{A} = (A,F)$ of $\mathcal{V}$ and a congruence distance $\alpha : A \times A \to R$ satisfying the conditions (i), (ii) and (iii) of Proposition 2.3. Then $\mathrm{J}(R)$ is isomorphic to the congruence lattice of the quotient of the algebra $\mathbf{A}$ by the congruence $\alpha^*(0_R)$.

The assignement of the compact congruence semilattice $\mathrm{Con_c}\,\mathbf{A}$ to any algebra $\mathbf{A}$ can be extended to a functor from any category $\mathbf{U}$ of algebras of the same similarity type with morphisms if we define for any morphism $f : A \to B$ the mapping $\mathrm{Con_c}(f) : \mathrm{Con_c}\,\mathbf{A} \to \mathrm{Con_c}\,\mathbf{B}$ as the mapping assigning to any compact congruence π of $\mathbf{A}$ the smallest congruence of $\mathbf{B}$ containing all the pairs $(f(u), f(v))$ for $(u,v) \in \pi$. Thus

$$\mathrm{Con_c}(f)(\pi) = \bigvee_{(u,v) \in \pi} \Theta_{\mathbf{B}}(f(u), f(v)). \tag{2.3}$$

The congruence of $\mathbf{B}$ on the right-hand side of (2.3) is compact provided π is a compact congruence of $\mathbf{A}$.

P. Pudlák initiated in [4] the study of liftings of the subcategory of $\mathbf{S}$ consisting of distributive $\langle\vee,0\rangle$-semilattices in the category of lattices with respect to the functor $\mathrm{Con_c}$. For a survey of results in this topic till 2002 see [11].

The study of liftings of subcategories of $\mathbf{S}$ in other categories of algebras with respect to $\mathrm{Con_c}$ has been very limited. W.A. Lampe states in [3] the following problem. This is in fact the problem of lifting the only possible $\langle\vee,0\rangle$-homomorphism from a one-element semilattice into another $\langle\vee,0\rangle$-semilattice.

Problem 2. Is every semilattice isomorphic to the compact congruence semilattice of an algebra with a one-element subalgebra?

W.A. Lampe also studied in [2] the problem of lifting a single $\langle\vee,0,1\rangle$-homomorphism $\boldsymbol{f} : R \to S$ between two $\langle\vee,0,1\rangle$-semilattices R and S in the category of groupoids and proved that if $\boldsymbol{f}$ is 0-separating, then it can be lifted in the category of groupoids with respect to the functor $\mathrm{Con_c}$.

In view of F. Wehrung's example of a distributive semilattice that is not isomorphic to the compact congruence semilattice of any semilattice presented in [14], the following problem gains more interest.

Problem 3. Is every distributive semilattice isomorphic to the compact congruence semilattice of a groupoid?

3. Group measures

The concept of group measures (under the name of group valuations) was introduce by V. Repnitskiĭ in [6].

Definition 3.1. Let G be a group and R a $\langle\vee,0\rangle$-semilattice. A mapping $\alpha : G \to R$ is called an *R-valued group measure*, or simply a *group measure*, if the following three conditions are satisfied.

(G1) $\alpha(1_G) = 0_R$, for all $g \in G$,
(G2) $\alpha(g^{-1}) = \alpha(g)$, for all $g \in G$,
(G3) $\alpha(gh) \leq \alpha(g) \vee \alpha(h)$, for all $g, h \in G$.

A group measure $\alpha : G \to R$ is *finite* if both G and R are finite.

In this case we assign to every ideal $I \in \mathrm{J}(R)$ the set

$$\alpha^*(I) = \{g \in G \mid \alpha(g) \in I\}.$$

Obviously, the set $\alpha^*(I)$ is a subgroup of G, hence α^* is a mapping from $\mathrm{J}(R)$ to $\operatorname{Sub} G$, where $\operatorname{Sub} G$ denotes the lattice of subgroups of the group G. It can be also verified easily that the mapping α^* preserves arbitrary meets and joins of up-directed subsets of $\mathrm{J}(R)$. Thus the adjoint mapping $\alpha^\dagger : \operatorname{Sub} G \to \mathrm{J}(R)$ is uniquely determined by

$$\alpha^\dagger(H) \leq I \quad \text{if and only if} \quad H \leq \alpha^*(I) \tag{3.1}$$

for every $H \in \operatorname{Sub}(G)$ and $I \in \mathrm{J}(R)$, and it preserves arbitrary joins and the compact elements. Thus $\alpha^\dagger(\langle g \rangle)$ is compact for every cyclic subgroup $\langle g \rangle$ of G. Similarly as in Lemma 1.2 or Lemma 2.2 we can prove that

$$\alpha^\dagger(\langle g \rangle) = (\alpha(a, b)\rangle$$

for every $g \in G$, or if we identify every $r \in R$ with the principal ideal $(r\rangle$,

$$\alpha^\dagger(\langle g \rangle) = \alpha(g).$$

Once again, we call the pair of mappings $\langle \alpha^\dagger, \alpha^* \rangle$ the *adjoint pair* defined by the group distance α.

Conversely, if a pair of adjoint mappings $\alpha^\dagger : \operatorname{Sub} G \to \mathrm{J}(R)$ and $\alpha^* : \mathrm{J}(R) \to \operatorname{Sub} G$ such that $\alpha^\dagger$ is compactness-preserving is given, then we can define an R-valued group measure $\alpha : G \to R$ by

$$\alpha(g) = r \quad \text{if and only if} \quad \alpha^\dagger(\langle g \rangle) = (r\rangle$$

for any $g \in G$. Then $\langle \alpha^\dagger, \alpha^* \rangle$ is the adjoint pair defined by α.

Since the set of all cyclic subgroups of G is a join-generating subset of $\operatorname{Sub} G$, Theorem 1.7 of [7] specializes in the case of group measures to the following proposition.

Proposition 3.2. *Let $\alpha : G \times G \to R$ be an R-valued group measure on a group G and let $\langle \alpha^\dagger, \alpha^* \rangle$ be the adjoint pair defined by α. Then*

(i) *the mapping α^* is injective if and only if the set $\{\alpha(g) \mid g \in G\}$ join-generates R. In particular, the mapping α^* is injective if the mapping α is onto.*

(ii) *The mapping α^* preserves joins if and only if for every $g \in G$ and every $r, s \in R$, if $\alpha(g) \leq r \vee s$, then $g \in \alpha^*(r) \vee \alpha^*(s)$.*

(iii) *The mapping α^* is a lattice homomorphism from $\mathrm{J}(R)$ onto the interval $[\alpha^*(0_R), G]$ in $\operatorname{Sub} G$ if and only if it preserves joins and for every $g, h \in G$, if $\alpha(h) \leq \alpha(g)$, then $h \in \alpha^*(0_R) \vee \langle g \rangle$.*

Thus to represent a given algebraic lattice L as an interval in the subgroup lattice of a group one has to find a group G and a group measure $\alpha : G \to R$, where R is the semilattice of compact elements of L, satisfying the conditions (i), (ii) and (iii) of Proposition 3.2. This is how V. Repnitskiĭ presents in [6] his proof that every algebraic lattice is isomorphic to an interval in the subroup lattice of an infinite group, the result originally proved by the author of this paper in [9]. When this theorem was proved, there were some discussions if there is a suitable categorial setting for the result. This setting is provided by the following definition of morphisms for group measures.

Definition 3.3. If $\alpha\colon G \to R$ is a group measure on a group G and $\beta\colon H \to S$ is a group measure on H, then by a *morphism* from α to β we mean a pair $(\boldsymbol{f}, f)$, where $\boldsymbol{f} : R \to S$ is a $\langle\vee, 0\rangle$-homomorphism and $f : G \to H$ is a group homomorphism such that

$$\boldsymbol{f}(\alpha(g)) = \beta(f(g)) \tag{3.2}$$

for every $g \in G$. We also write $(\boldsymbol{f}, f) : \alpha \to \beta$.

The class of all semilattice-valued group measures together with morphisms of group measures is a category, we will denoted it by $\mathbf{M}$. The *forgetful functor* from $\mathbf{M}$ to $\mathbf{S}$ assigns to every group measure $\alpha\colon G \to R$ the $\langle\vee, 0\rangle$-semilattice R and to every morphism $(\boldsymbol{f}, f) : \alpha \to \beta$ the $\langle\vee, 0\rangle$-homomorphism $\boldsymbol{f} : R \to S$.

The main result of [9] can be formulated in the language of group measures as follows.

Theorem 3.4. *For everry $\langle\vee, 0\rangle$-semilattice R there exists a group G and a group measure $\alpha : G \to R$ satisfying the conditions (i), (ii) and (iii) of Proposition 3.2.*

The proof of Theorem 3.4 presented in [6] seems to be suitable for solving the following problem.

Problem 4. Is there a lifting $\mathcal{A} : \mathbf{S} \to \mathbf{M}$ such that its composition with the forgetful funtor $\mathcal{F}$ on $\mathbf{M}$ is equal to the identity functor on $\mathbf{S}$ and moreover, the group measure $\mathcal{A}(R)$ satisfies the conditions (i), (ii) and (iii) of Proposition 3.2 for every object R of $\mathbf{S}$? If not, find "large" subcategories of $\mathbf{S}$ that can be lifted in $\mathbf{M}$ with respect to the forgetful functor on $\mathbf{M}$.

A combination of methods of [6] with the methods of [5] led in [7] to the following theorem.

Theorem 3.5. *For every at most countable $\langle\vee, 0\rangle$-semilattice R there exists a* locally finite *group G and a group distance $\alpha : G \to R$ satisfying the conditions (i), (ii) and (iii) of Proposition 3.2.*

A lifting of sufficiently general diagrams in $\mathbf{S}_{\mathrm{f}in}$ in the category of finite group measures satisfying conditions (i) and (ii) and a weaker form of (iii) of Proposition 3.2 with respect to the forgetful functor could lead to a proof of the following conjecture.

Conjecture 3.6. Every algebraic lattice is isomorphic to an interval in the subgroup lattice of a locally finite group.

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Charles University in Prague, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 00 Praha 8, Czech Republic

E-mail address: `tuma@karlin.mff.cuni.cz`