TRANSITIVE CLOSURES OF BINARY RELATIONS III.

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Abstract. Transitive closures of the covering relation in lattices are investigated.

Vyšetřují se tranzitivní uzávěry pokrývací relace ve svazech.

This extremely short expository note collects a few more or less notoriously known results on the covering relation β in lattices. Special attention is paid to the property that any two β-sequences connecting two given elements are of the same length. We refer to [1] and [2] as for terminology, notation, further references, etc.

1. The covering relation in lattices

Throughout the note, let $L = L(+,\cdot)$ be a lattice (i.e., both $L(+)$ and $L(\cdot)$ are semilattices and $a(a+b) = a = a + (ab)$ for all $a, b \in L$). Define a relation α on L by $(a, b) \in \alpha$ if and only if $a + b = b$.

1.1. **Proposition.**

- (i) *The relation α is a stable (reflexive) ordering of the lattice and $(a, b) \in \alpha$ if and only if $ab = a$.*
- (ii) *$(a, a + b) \in \alpha$, $(b, a + b) \in \alpha$, $(ab, a) \in \alpha$ and $(ab, b) \in \alpha$ for all $a, b \in L$. (In fact, $a + b = \sup_\alpha(a, b)$ and $ab = \inf_\alpha(a, b)$.)*
- (iii) *An element $a \in L$ is maximal in $L(\alpha)$ (i.e., a is right α-isolated) if and only if $a = 1_L$ is an absorbing element of $L(+)$ if and only if a is a neutral element of $L(\cdot)$. (Then a is the (unique) greatest element of $L(\alpha)$.)*
- (iv) *An element $a \in L$ is minimal in $L(\alpha)$ (i.e., a is left α-isolated) if and only if $a = 0_L$ is a neutral element of $L(+)$ if and only if a is an absorbing element of $L(\cdot)$. (Then a is the (unique) smallest element of $L(\alpha)$.)*

Proof. It is obvious. □

1.2. **Lemma.**

- (i) *Every weakly pseudoirreducible finite α-sequence is pseudoirreducible.*
- (ii) *Every weakly pseudoirreducible right (left, resp.) directed infinite α-sequence is pseudoirreducible.*

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(iii) *If there exists no pseudoirreducible right (left, resp.) directed infinite* α*-sequence then* $1_L \in L$ *(*$0_L \in L$*, resp.).*

Proof. It is obvious. □

1.3. **Lemma.** *Let* $(a, b) \in \alpha$ *and* $I = \mathrm{Int}_\alpha(a, b) = \{c \in L | (a, c) \in \alpha$ *and* $(c, b) \in \alpha\}$*. Then:*

(i) I *is a sublattice of* L *and* $\{a, b\} \subseteq I$*.*
(ii) $a = 0_I$ *and* $b = 1_I$*.*
(iii) $\alpha_I = \alpha_L \restriction I$*.*

Proof. It is obvious. □

In the sequel, put $\beta = \sqrt{\alpha}$ and $\gamma = \mathbf{r}\,\mathbf{t}(\beta)$, so that β is the covering relation of L and γ is its reflexive and transitive closure. Notice that $\mathbf{i}(\gamma) = \mathbf{t}(\beta)$.

1.4. **Proposition.**

(i) β *is totally antitransitive.*
(ii) $\beta \subseteq \gamma \subseteq \alpha$*.*
(iii) γ *is an ordering of* L*.*
(iv) *If* $(a, b) \in \alpha$ *and* $\mathrm{Int}_\alpha(a, b)$ *is finite then* $(a, b) \in \gamma$*.*

Proof. It is obvious. □

We say that the lattice L is resuscitable if so is the ordering α (i.e., $\alpha = \gamma$).

1.5. **Proposition.** *The lattice* L *is resuscitable, provided that the following two conditions are satisfied:*

(1) *no right (left, resp.) directed infinite* $\mathbf{i}(\alpha)$*-sequence is right (left, resp.) bounded in* $L(\alpha)$*;*
(2) *no left (right, resp.) directed infinite* β*-sequence is left (right, resp.) bounded in* $L(\alpha)$*.*

Proof. See II.1.8. □

1.6. **Corollary.** *The lattice* L *is resuscitable, provided that it is finite.*

1.7. **Example.** The boolean lattice of all subsets of an infinite set is not resuscitable.

1.8. **Example.** A chain is resuscitable if and only if it can be embedded into the chain of integers (with respect to the usual ordering of integers).

1.9. **Example.** Consider the lattice $L_1 = \{1, a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots\}$ with $(x, y) \in \alpha$ if and only if either $x = y$, or $x = a_0$, or $y = 1$, or $(x, y) = (a_i, a_j)$ where $i \leq j$, or $(x, y) = (a_i, b_j)$ where $i \leq j$. This infinite lattice L_1 is resuscitable, while its sublattice $\{1, a_0, a_1, a_2, \dots\}$ is not. It follows that the class of resuscitable lattices is not closed under sublattices.

2. On when the covering relation is right/left confluent (or weakly semimodular lattices)

The lattice L is called

- upwards (downwards, resp.) weakly semimodular if the semilattice $L(+)$ ($L(\cdot)$, resp.) is weakly semimodular;
- weakly semimodular if it is both upwards and downwards weakly semimodular.

2.1. **Lemma.** *The lattice L is upwards (downwards, resp.) weakly semimodular if and only if the relation β is right (left, resp.) confluent.*

Proof. See II.2.1. □

2.2. **Lemma.** *Assume that L is upwards (downwards, resp.) weakly semimodular. If $a, b, c \in L$ are such that $(a, b) \in \gamma$ and $(a, c) \in \gamma$ ($(b, a) \in \gamma$ and $(c, a) \in \gamma$, resp.) then $(b, b + c) \in \gamma$ and $(c, b + c) \in \gamma$ ($(bc, b) \in \gamma$ and $(bc, c) \in \gamma$, resp.).*

Proof. See II.2.3. □

2.3. **Corollary.** *If L is upwards (downwards, resp.) weakly semimodular then the ordering γ is right (left, resp.) strictly confluent.*

2.4. **Lemma.** *Assume that L is upwards (downwards, resp.) weakly semimodular. If $(a, b) \in \gamma$ then there exists no right (left, resp.) directed infinite $\mathbf{i}(\gamma)$-sequence $(a_0, a_1, a_2, \dots)$ ($(\dots, b_2, b_1, b_0)$, resp.) such that $a_0 = a$ ($b_0 = b$, resp.) and $(a_i, b) \in \alpha$ ($(a, b_i) \in \alpha$, resp.) for every $i \geq 1$.*

Proof. See II.2.6. □

2.5. **Lemma.** *Assume that L is weakly semimodular. If $(a, b) \in \gamma$ then:*

(i) *$K = \mathrm{Int}_\gamma(a, b)$ is a sublattice of L, $a = 0_K$ and $b = 1_K$.*
(ii) *K is resuscitable.*
(iii) *If $c \in \mathrm{Int}_\alpha(a, b)$ and either $(a, c) \in \gamma$ or $(c, b) \in \gamma$ then $c \in K$.*

Proof. See II.2.7. □

2.6. **Example.** Consider the lattice L_2 with seven elements $0, 1, a, b, c, d, e$ and the covering relation $\beta = \{(0, a), (0, b), (a, c), (a, d), (b, d), (b, e), (c, 1), (d, 1), (e, 1)\}$. (A finite lattice is uniquely determined by its covering relation.) Clearly, L_2 is upwards weakly semimodular but not downwards weakly semimodular.

2.7. **Example.** Consider the lattice $\mathbf{N}$ with five elements $0, 1, a, b, c$ and the covering relation $\beta = \{(0, a), (0, b), (a, 1), (b, c), (c, 1)\}$. Clearly, $\mathbf{N}$ is neither upwards nor downwards weakly semimodular.

3. Semimodular lattices

The lattice L is called

- upwards (downwards, resp.) semimodular if the semilattice $L(+)$ ($L(\cdot)$, resp.) is semimodular;
- semimodular if it is both upwards and downwards semimodular.

3.1. **Lemma.**

(i) *If L is (upwards, downwards) semimodular then it is (upwards, downwards) weakly semimodular.*
(ii) *If L is semimodular then γ is a stable ordering of L.*

Proof. See II.3.2. □

3.2. **Proposition.** *Assume that L is resuscitable. Then L is (upwards, downwards) semimodular if and only if it is (upwards, downwards) weakly semimodular.*

Proof. See II.3.3. □

3.3. **Corollary.** *If L is finite then L is (upwards, downwards) semimodular if and only if it is (upwards, downwards) weakly semimodular.*

3.4. **Proposition.** *Assume that L is weakly semimodular. Let $(a,b) \in \gamma$ and $K = \mathrm{Int}_\gamma(a,b)$. Then:*

(i) *K is a sublattice of L, $a = 0_K$ and $b = 1_K$.*
(ii) *K is semimodular and resuscitable.*
(iii) *Every subchain of $K(\alpha)$ is finite and of length at most $\mathrm{dist}_\gamma(a,b)$.*
(iv) *$K \subseteq \mathrm{Int}_\alpha(a,b)$ and $c \in K$, provided that $c \in \mathrm{Int}_\alpha(a,b)$ and either $(a,c) \in \gamma$ or $(c,b) \in \gamma$.*
(v) *If L is upwards or downwards semimodular then $K = \mathrm{Int}_\alpha(a,b)$.*

Proof. Combine 2.5 and II.6.3. □

3.5. **Proposition.** *The following four conditions are equivalent:*

(i) *L is upwards (downwards, resp.) weakly semimodular, no right directed infinite $\mathbf{i}(\alpha)$-sequence is right bounded in $L(\alpha)$ and no left directed infinite β-sequence is left bounded in $L(\alpha)$.*
(ii) *L is upwards (downwards, resp.) weakly semimodular, no left directed infinite $\mathbf{i}(\alpha)$-sequence is left bounded in $L(\alpha)$ and no right directed infinite β-sequence is right bounded in $L(\alpha)$.*
(iii) *L is upwards (downwards, resp.) semimodular and resuscitable.*
(iv) *L is upwards (downwards, resp.) weakly semimodular and every right and left bounded subchain of $L(\alpha)$ is finite.*

Proof. See II.6.4. □

3.6. **Example.** The lattice L_2 from 2.6 is upwards semimodular but not downwards weakly semimodular.

3.7. **Example.** Consider the lattice $L_3 = \{0, 1, a, b_1, b_2, \dots\}$ with $(x,y) \in \alpha$ if and only if either $x = y$ or $x = 0$ or $y = 1$ or $(x,y) = (b_i, b_j)$ where $i < j$. This infinite lattice L_3 is weakly semimodular but neither upwards nor downwards semimodular. Moreover, $(0,1) \in \gamma$, $\mathrm{dist}_\gamma(0,1) = 2$ and $\mathrm{Int}_\gamma(0,1) = \{0, a, 1\} \neq L_3 = \mathrm{Int}_\alpha(0,1)$.

4. Modular lattices

The lattice L is called modular if no sublattice of L is a copy of the pentagon (the lattice $\mathbf{N}$ from 2.7).

4.1. **Proposition.** *If L is modular then it is semimodular.*

Proof. It is obvious. □

4.2. **Proposition.** *A resuscitable lattice is modular if and only if it is weakly semimodular.*

Proof. The direct implication follows from 4.1. Let L be a resuscitable, weakly semimodular lattice. By 3.2, L is semimodular. Let $x < y$ stand for $(x,y) \in \mathbf{i}(\alpha)$ and $x \prec y$ stand for $(x,y) \in \beta$. Suppose that L is not modular, so that it contains a subpentagon $\{o,a,b,c,i\}$ (o is its smallest element, i is the largest, and $b < c$). Choose these five elements in such a way that the interval $\mathrm{Int}(o,i)$ has minimal possible length. (Since L is resuscitable and semimodular, every interval I of L has a finite length n and every maximal chain in I is of length n.)

Suppose $o \prec b$. Then $a \prec i$ by the upwards semimodularity, from which we get $o \prec c$ by the downwards semimodularity, a contradiction. Thus o is not covered by b and there exists an element $d \in L$ with $o \prec d < b$. Put $e = a + d$. By the upwards semimodularity we have $a \prec e$; since a is not covered by i, we get $a \prec e < i$. Thus $b \nleq e$ and e is incomparable with both b and c. By the minimality of $\mathrm{Int}(o,i)$, the elements d,e,b,c,i do not form a subpentagon. Since $e+b=i$, we get $ec \nleq d$. Put $f = ec$. Thus $d < f < e$. But then the elements o,a,d,f,e form a subpentagon of L, a contradiction with the minimality of $\mathrm{Int}(o,i)$. □

4.3. **Corollary.** *A finite lattice is modular if and only if it is semimodular.*

4.4. **Example.** Proposition 4.2 cannot be generalized to arbitrary lattices. Let L be any infinite lattice such that its covering relation is empty. Then L is semimodular. Of course, such a lattice need not to be modular. Thus a semimodular lattice is not necessarily modular.

The lattice L is called

- upwards (downwards, resp.) strongly modular if the semilattice $L(+)$ ($L(\cdot)$, resp.) is strongly modular;
- strongly modular if it is both upwards and downwards strongly modular.

4.5. **Example.** For every cardinal number $\kappa > 0$ denote by M_κ the (unique up to isomorphism) lattice of length 2 with κ atoms (elements covering the least element). Clearly, each M_κ is a strongly modular lattice. We see that a strongly modular lattice is not necessarily distributive.

4.6. **Example.** Denote by L_4 the lattice with six elements a,b,c,d,e,f, such that $\beta = \{(a,b),(b,c),(c,f),(a,d),(d,e),(b,e),(e,f)\}$. (The product of the two-element chain with the three-element chain.) Clearly, L_4 is neither downwards nor upwards strongly modular. On the other hand, it is distributive.

4.7. **Proposition.** *The following conditions are equivalent:*

(i) *L is upwards strongly modular;*
(ii) *L is downwards strongly modular;*
(iii) *L is strongly modular;*
(iv) *neither* $\mathbf{N}$ *nor* L_4 *can be embedded into* L.

Proof. By 4.6, each of the first three conditions implies (iv). Thus it is sufficient to prove that (iv) implies (i). Let L be a modular lattice not containing a sublattice isomorphic with L_4 and suppose that L is not upwards strongly modular, so that it contains four distinct elements a, b, c, i such that a is incomparable with b, $i = a + b$ abd $b < c < i$. If $ac < i$ then these four elements together with ac form a subpentagon, a contradiction. Thus ac is incomparable with b. Put $d = ac$ and $e = ab = db$; we have $e < d < a < i$. Also, put $f = d + b$, so that $b < f \leq c < i$. It can be easily checked that the elements e, d, a, b, f, i form a sublattice isomorphic with L_4, a contradiction. □

4.8. **Example.** For two finite lattices P and Q we define a lattice $L = P \oplus Q$, called their glued ordinal sum, as follows. We can assume that $P \cap Q = \{1_P\} = \{0_Q\}$. In that case put $L = P \cup Q$ and $\alpha_L = \alpha_P \cup \alpha_Q \cup (P \times Q)$. Similarly, we can define $R_1 \oplus \cdots \oplus R_n$ for any finite nonempty sequence of lattices $R_1, \ldots, R_n$. It follows from 4.7 that a finite lattice is strongly modular if and only if it can be expressed as the glued ordinal sum of a finite sequence of finite lattices, each of which is either a chain or isomorphic to M_n for some $n \geq 2$.

5. On when the covering relation is regular

5.1. **Proposition.** *If L is upwards or downwards weakly semimodular then its covering relation β is regular.*

Proof. See II.5.1. □

References

[1] V. Flaška, J. Ježek, T. Kepka and J. Kortelainen, *Transitive closures of binary relations I.* To appear.

[2] V. Flaška, J. Ježek and T. Kepka, *Transitive closures of binary relations II.* To appear.

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