STAR-QUASILINEAR EQUATIONAL THEORIES OF GROUPOIDS

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Abstract. We investigate equational theories E of groupoids with the property that every term is E-equivalent to at least one linear term.

1. Introduction

The main result of the paper [1] is that there are precisely six linear equational theories of groupoids (equational theories with the property that every term is equivalent to precisely one linear term). The present paper deals with the generalization of linear equational theories to those that are quasilinear, by which we mean equational theories such that every term is equivalent to at least one linear term. It turns out that this generalization is unwieldy: in the last section we show that there are uncountably many examples. However, the most important case in this context, as it turns out from [1], is the case when the equational theories under consideration are idempotent. The whole paper except the last section will relate to this case. Since all linear equational theories of groupoids turned out to be idempotent, all their extensions are idempotent and quasilinear. But there are also idempotent quasilinear equational theories that are not extensions of any linear ones. The main result of this paper, Theorem 11.1, states that there are precisely 28 idempotent quasilinear equational theories of groupoids. We also describe them. The corresponding varieties are all finitely generated. It will also turn out that all of them, except two that were already found to be inherently nonfinitely based in [1], are finitely based.

By a term we always mean a term in the signature of groupoids (algebras with one binary, multiplicatively denoted operation). A term is said to be *linear* if every variable has at most one occurrence in it.

Let $S(t)$ denote the set of variables occurring in a term t and let $|t|$ denote the *length* of t, i.e., the total number of occurrences of variables in t. Clearly, $|S(t)| \leq |t|$ with equality precisely when t is linear.

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In order to avoid writing too many parentheses in terms, $x_1x_2x_3 \dots x_n$ will stand for $(((x_1x_2)x_3)\dots)x_n$ (the parentheses are grouped to the left), $x \cdot yz$ will stand for $x(yz)$, etc.

By a left-associated term we mean a term $x_1x_2 \dots x_n$ where $n \geq 1$ and $x_1, \dots, x_n$ are variables.

An *equational theory* is a set of equations (ordered pairs of terms) that is closed under logical consequences; in other words, the set of all equations that are satisfied in a particular groupoid. The lattice of equational theories is antiisomorphic to the lattice of all varieties of groupoids. An equational theory E is said to be *n-linear* (for a positive integer n) if every term in at most n variables is E-equivalent to precisely one linear term (which must then be again in at most n variables). If, moreover, E is generated by its at most n-variable equations, then we say that E is *sharply n-linear.* Of course, such an equational theory is uniquely determined by its n-generated free groupoid. An equational theory is $*$-linear if it is n-linear for all n. Such equational theories were investigated in [1]. Here we will need a generalization.

An equational theory E is said to be *n-quasilinear* if every term in at most n variables is E-equivalent to at least one linear term. An equational theory is $*$-quasilinear if and only if it is n-quasilinear for all n.

A variety is said to be $*$-quasilinear (or n=quasilinear, etc.) if the corresponding equational theory has the same property.

Lemma 1.1. *Let E be a $*$-quasilinear idempotent equational theory. Then every term t is E-equivalent to a linear term t^* such that $S(t^*) \subseteq S(t)$.*

Proof. Let u be a linear term of minimal length that is E-equivalent with t, and suppose that there exists a variable $x \in S(u) - S(t)$. If $x = u$ then E is the trivial equational theory, so that every term is E-equivalent with any variable. Let $x \neq u$. Then u is not a variable and there exists a term v such that either vx or xv is a subterm of u. This subterm can be replaced by vv (use the substitution sending x to v) and then by v (use the idempotency) to obtain a linear term of smaller length also E-equivalent with t. □

It follows that every $*$-quasilinear idempotent theory defines a locally finite variety. In fact, the cardinality of the free algebra on n generators in that variety is at most the number of linear terms over $x_1, \dots, x_n$. For $n = 2, 3, 4$ we get that the cardinality is at most 4, 21, 184, respectively.

For a pair v_1, v_2 of variables denote by σ_{v_1,v_2} the substitution sending v_1 to v_2 (and fixing all variables other than v_1).

An acquaintance with the paper [1] may be useful for understanding the present paper. The standard terminology and basic facts of universal algebra can be found in the book [4].

We would like to note that most of the work on this paper was done by the first author.

2. 2-generated free algebras in 2-quasilinear idempotent equational theories

In this section we will describe all candidates for the 2-generated free algebra in a $*$-quasilinear idempotent variety. The following result can be found in [1] as Lemma 2.1. It characterizes all sharply 2-linear equational theories. They are all idempotent as it is implied by 2-linearity.

Lemma 2.1. *There are precisely twelve sharply* 2*-linear equational theories. Their* 2*-generated free groupoids are the following seven groupoids, plus their duals.* (*The first two of the seven groupoids are self-dual.*)

$\mathbf{G}_0$	x	y	xy	yx
x	x	xy	yx	y
y	yx	y	x	xy
xy	y	yx	xy	x
yx	xy	x	y	yx

$\mathbf{G}_1$	x	y	xy	yx
x	x	xy	xy	x
y	yx	y	y	yx
xy	x	xy	xy	x
yx	yx	y	y	yx

$\mathbf{G}_2$	x	y	xy	yx
x	x	xy	xy	yx
y	yx	y	xy	yx
xy	x	y	xy	x
yx	x	y	y	yx

$\mathbf{G}_3$	x	y	xy	yx
x	x	xy	xy	yx
y	yx	y	xy	yx
xy	x	y	xy	yx
yx	x	y	xy	yx

$\mathbf{G}_4$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	x	xy	x
yx	y	yx	y	yx

$\mathbf{G}_5$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

$\mathbf{G}_6$	x	y	xy	yx
x	x	xy	xy	xy
y	yx	y	yx	yx
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

The following lemma characterizes the remaining candidates for the 2-generated free groupoid in a $*$-quasilinear idempotent variety.

Lemma 2.2. *There are precisely four groupoids that serve as* 2*-generated groupoids for a* 2*-quasilinear, not* 2*-linear idempotent variety. They are the following three groupoids, plus the dual of* $\mathbf{G}_9$.

$\mathbf{G}_7$	x	y	xy
x	x	xy	y
y	xy	y	x
xy	y	x	xy

$\mathbf{G}_8$	x	y	xy
x	x	xy	xy
y	xy	y	xy
xy	xy	xy	xy

$\mathbf{G}_9$	x	y
x	x	y
y	x	y

Proof. Denote by $\mathbf{G}$ the two-generated free groupoid in the variety V corresponding to a 2-quasilinear (but not 2-linear) idempotent equational theory. Since the equational theory is not 2-linear, $\mathbf{G}$ has at most three elements x, y

and (perhaps) one of xy and yx. It is easy to see that one of the following three cases takes place.

Case 1: $xy \sim yx$. Then $\mathbf{G}$ is commutative. Note that by fixing a linear two-variable term u and stipulating that $x \cdot xy \sim u$, the 2-generated free groupoid is completely determined. Consider the following three subcases:

Subcase 1a: $x \cdot xy \sim y$. Then $\mathbf{G} = \mathbf{G}_7$.

Subcase 1b: $x \cdot xy \sim xy$. Then $\mathbf{G} = \mathbf{G}_8$.

Subcase 1c: $x \cdot xy \sim x$. Then $x \sim x \cdot xy \sim xy \cdot x \sim xy(x \cdot xy) \sim xy$, so we obtain $x \sim y$, a contradiction.

Case 2: $xy \sim y$. Then $\mathbf{G} = \mathbf{G}_9$.

Case 3: $xy \sim x$. Then $\mathbf{G}$ is the dual of $\mathbf{G}_9$.

Clearly, the varieties generated by $\mathbf{G}_7$, $\mathbf{G}_8$ and $\mathbf{G}_9$ are idempotent, 2-quasilinear and not 2-linear. □

3. Extending $\mathbf{G}_0$

In this and the next sections we are going to find which of the ten candidates (neglecting their duals) for two-generated free groupoids of an idempotent $*$-quasilinear variety do give us such a variety. The following lemmas and their proofs are based on Section 3 of [1].

Proposition 3.1. *We cannot have $\mathbf{G}_0$ as the free two-generated groupoid for a 3-quasilinear (and consequently, for a $*$-quasilinear) equational theory.*

Proof. There are 21 possibilities for a linear term equivalent with $xy \cdot zx$, and each of them is easily seen to result in an equation conflicting with the multiplication table of $\mathbf{G}_0$. □

4. Extending $\mathbf{G}_1$

Lemma 4.1. *Let $\mathbf{G}_1$ be the free two-generated groupoid for an idempotent, $*$-quasilinear equational theory E. Then $x(yz) \approx (xy)z$ and $xyz \approx xz$ belong to E.*

Proof. If E contains an equation with different leftmost variables at both sides, then we can substitute for all the remaining variables one of these two variables, and obtain an equation with the same property in just two variables, which would yield a contradiction. So, every equation of E must have the same leftmost variables and, quite similarly, also the same rightmost variables at both sides. A term both starting and ending with a variable x must be equivalent to a linear term both starting and ending with x, and therefore equal to x. In particular, the equations $xyx \approx x$, $x \cdot yx \approx x$, $xyzx \approx x$ and $x(y(zx)) \approx x$ belong to E.

Define two relations on any algebra in the variety V determined by E: $a \sim_\ell b \leftrightarrow ab = a$ and $a \sim_r b \leftrightarrow ab = b$. Then $b \sim_\ell ab \sim_r a$ and both relations are reflexive and symmetric. Reflexivity follows from the idempotence, while symmetry of $\sim_\ell$ is proved like this: if $ab = a$ then $b = b \cdot ab = ba$. The intersection of these two relations is the identity. The two relations are also transitive: if $x \sim_r y \sim_r z$ then $zx = yzx = xyzx = x$, giving $x \sim_r z$; using the identity $x(y(zx)) = x$, we can similarly prove that also $\sim_\ell$ is transitive. Now

$$x(yz) \sim_r x \sim_r xy \sim_r xyz \quad \text{and}$$
$$x(yz) \sim_\ell yz \sim_\ell z \sim_\ell xyz$$

imply that $x(yz) \approx xyz$ is valid in V. Then $xyz = (((xz)x)y)z = x(z(xy)z) = xz$ implies that $xyz \approx xz$ is valid in V. □

The variety determined by the equations $x(yz) \approx (xy)z$, $xx \approx x$ and $xyz \approx xz$ is the variety of rectangular bands. We denote it by $\mathcal{R}$.

Theorem 4.2. *There is precisely one $*$-quasilinear idempotent equational theory with 2-generated free groupoid isomorphic to $\mathbf{G}_1$; it is the equational theory of $\mathcal{R}$. The variety $\mathcal{R}$ is generated by $\mathbf{G}_1$ and its only proper subvarieties are the trivial variety, the variety of left-zero semigroups and the variety of right-zero semigroups.*

Proof. It follows from 4.1 together with Lemma 3.2 of [1]. □

5. Extending $\mathbf{G}_2$ and $\mathbf{G}_3$

Proposition 5.1. *We cannot have $\mathbf{G}_2$ or $\mathbf{G}_3$ as the free two-generated groupoid for a $*$-quasilinear idempotent equational theory.*

Proof. Suppose that either $\mathbf{G}_2$ or $\mathbf{G}_3$ serves for an idempotent, $*$-quasilinear equational theory E. Of course, the equational theory is idempotent. It is easy to see that whenever $u \approx v$ is an equation belonging to E then the terms u, v have the same rightmost variables. Thus a linear term equivalent with $x \cdot xyz$ under E must be one of the seven linear terms in the variables x, y, z that have z as the rightmost variable. However, none of the seven corresponding equations is valid in either $\mathbf{G}_2$ or $\mathbf{G}_3$. □

6. Extending $\mathbf{G}_4$

Lemma 6.1. *In any $*$-quasilinear equational theory with $\mathbf{G}_4$ as its 2-generated free groupoid, the following identities hold:*

(W1) $xx \approx x$,
(W2) $xyy \approx x$,
(W3) $x \cdot yz \approx xy$,
(W4) $xyz \approx xzy$.

Proof. The first two equations are in the multiplication table of $\mathbf{G}_4$. In the course of the proof of Lemma 3.4 in [1], it was proved that any quasilinear equational theory with $\mathbf{G}_4$ as its 2-generated free groupoid satisfies $xy \approx x \cdot yz$. Therefore, any term t, written as $xp_1p_2 \dots p_n$, is equal in $\mathbf{G}_4$ to the term $xy_1y_2 \dots y_n$, where y_i is the leftmost variable of p_i. Finally, consider a linear term in the variables x, y and z which is equal to $xyzy$ in $\mathbf{G}_4$. This term s must have x as its leftmost variable, and we may as well assume that it is left-associated. The substitution $z \mapsto y$ reveals that it can not be x, xyz or xzy, while $z \mapsto x$ shows that s is not xy. The remaining possibility, $xyzy \approx xz$, holds in $\mathbf{G}_4$, and therefore, s must be equal to xz. This yields that $xyz \approx xyzyy \approx xzy$. □

We denote by E_W the equational theory generated by the four equations (W1)–(W4) and by $\mathcal{W}$ the corresponding variety.

Lemma 6.2. *The equational theory E_W is $*$-quasilinear.*

Proof. Obviously, using the identity $(W3)$, we get that any term t is equal in E to a left-associated term t'. Then, using the identity (W4), t' is reduced to a left-associated term t'' in which the semigroup word obtained from t'' by erasing the parentheses is equal to $x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}$, where $x_i \neq x_j$ for $i \neq j$. Finally, using the idempotence and (W2), we get that t is equal to the term $y_1 y_2 \dots y_n$, where $y_1 = x_1$, $y_i \neq y_j$ for $i \neq j$, while the set $\{y_2, y_3, \dots, y_n\}$ is equal to the set $\{x_i : 2 \leq i \leq k \text{ and } m_i \text{ is odd}\}$. □

Notice that in E_W any two left-associated linear terms with the same sets of variables and the same leftmost variables must be equal.

Theorem 6.3. *There is precisely one $*$-quasilinear variety with $\mathbf{G}_4$ serving as its 2-generated free algebra; it is the variety $\mathcal{W}$. Its only proper subvarieties are the trivial variety and the variety of left-zero semigroups.*

Proof. Let V be a proper subvariety of $\mathcal{W}$. By 6.2, V is a $*$-quasilinear variety. V must satisfy an equation $t_1 \approx t_2$ such that t_1, t_2 are two $\mathcal{W}$-nonequivalent left-associated linear terms.

Consider first the case when there exists a variable y in the symmetric difference of $S(t_1)$ and $S(t_2)$. Then, by substituting all the other variables by x, one of the two terms becomes $\mathcal{W}$-equal to x while the other to one of the terms y, yx, xy. In the first two cases V is the trivial variety, while in the last case V is a subvariety of the minimal variety of left-zero semigroups.

Let $S(t_1) = S(t_2)$ and suppose that the leftmost variable x in t_1 is different from the leftmost variable y in t_2. By substituting all the other variables by x, we can reduce t_1 to xy, and t_2 to either y or yx. If $xy \approx yx$ is satisfied in V then $xy \approx xxy \approx xyx \approx yxx \approx y$, so we reduce to previous case either way. □

7. Extending $\mathbf{G}_5$

Lemma 7.1. *If $\mathbf{G}_5$ serves as the free two-generated groupoid for a 4-quasilinear equational theory then E contains a nontrivial equation in at most three variables, both sides of which are linear terms.*

Proof. This is implicit in the proof of Lemma 4.4 of [1]. □

In this section let E be a $*$-quasilinear equational theory and V be the corresponding variety, such that $\mathbf{G}_5$ is the two-generated free groupoid in V.

Thus, by 7.1, the free three-generated groupoid in V has less than 21 elements. We write $t \sim s$ if the equation $t \approx s$ belongs to E. From the multiplication table for $\mathbf{G}_5$ we get

$$x \sim xx,$$
$$x \sim x(xy),$$
$$xy \sim xyx \sim xyy \sim x(yx) \sim xy(yx).$$

Lemma 7.2. *If $u \sim v$ then u, v have the same leftmost variables.*

Proof. It is easy. □

Lemma 7.3. *Either $xyz \sim xzy$ or $x \cdot yz \sim xy$.*

Proof. By 7.1 there exist two different linear terms t_1, t_2 in the variables x, y, z such that $t_1 \sim t_2$. According to 7.2, the leftmost variables in t_1 and t_2 are the same. Without loss of generality, assume that this leftmost variable is x. Since both $xy \approx xz$ and $xy \approx x$ fail in $\mathbf{G}_5$, at least one of the two terms, say t_1, has three different variables occurring in it. Without loss of generality, we may assume that t_1 is either xyz or $x \cdot yz$. Note that xyz, $x \cdot yz$ and x are in three different $\sim$-classes, which can be seen from the substitutions $y \mapsto x$ and $z \mapsto y$. The term t_2 can be $xy, xz, x \cdot yz, x \cdot zy, xyz, xzy$. Consider two cases.

Case 1: $t_1 = xyz$. If t_2 equals one of xy, $x \cdot yz$, xz and $x \cdot zy$, we reach the contradiction in the first two cases by the substitution $y \mapsto x$, and in the last two cases by substitution $z \mapsto x$. The remaining possibility is $t_2 = xzy$.

Case 2: $t_1 = x \cdot yz$. If t_2 equals one of xz, xzy, $x \cdot zy$ and xyz, we reach the contradiction by the substitution $y \mapsto x$. The remaining possibility is $t_2 = xy$. □

Lemma 7.4. *If $xyz \sim xzy$ then $xy \cdot zy \sim xyz$.*

Proof. Let $t \sim xy \cdot zy$ where t is linear. If either $t = xy$ or $t = x \cdot yz$, we get a contradiction by the substitution $y \mapsto x$. If t is any of the terms x, xz, $x \cdot zy$, we get a contradiction by the substitution $z \mapsto x$. The only remaining possibility is $t = xyz$. □

Lemma 7.5. *If $xyz \sim xzy$ then $xy \cdot zx \sim xzy$ and $x \cdot yxz \sim x \cdot yz$.*

Proof. $xy \cdot zx \sim (x \cdot zx)y \sim xzy$ and $x \cdot yxz \sim x \cdot yzx \sim x \cdot yz$. □

Lemma 7.6. $x \cdot yz \sim xy$.

Proof. By 7.3, we can assume that $xyz \sim xzy$. There exists a term t in four variables x, y, z, u with $t \sim x \cdot yz \cdot uz$. Using 7.4, 7.5 and the two-variable equations we get

$$\sigma_{y,x}(x \cdot yz \cdot uz) \sim x \cdot uz,$$
$$\sigma_{z,x}(x \cdot yz \cdot uz) \sim xuy,$$
$$\sigma_{u,x}(x \cdot yz \cdot uz) \sim x \cdot yz,$$
$$\sigma_{z,y}(x \cdot yz \cdot uz) \sim xuy,$$
$$\sigma_{u,y}(x \cdot yz \cdot uz) \sim x \cdot yz,$$

$\sigma_{u,z}(x \cdot yz \cdot uz) \sim xzy$.

We have $x \in S(t)$, else substituting $y \mapsto z$ and $u \mapsto z$ in $x \cdot yz \cdot uz \sim t$ yields $xz \sim z$.

We have $y \in S(t)$, else substituting $z \mapsto x$ and $u \mapsto x$ yields $xy \sim x$.

We have $z \in S(t)$, else substituting $u \mapsto y$ yields $x \cdot yz \sim xy$.

We have $u \in S(t)$, else substituting $y \mapsto x$ and $z \mapsto x$ yields $xu \sim x$.

Thus $S(t) = \{x, y, z, u\}$.

Let $t = t_1 t_2$. Let v_1, v_2, v_3 represent a permutation of the variables y, z, u. One of the following four cases takes place.

Case $t_1 = x$: Then t_2 is either $v_1 \cdot v_2 v_3$ or $v_1 v_2 v_3$. The substitution $v_1 \mapsto x$, $v_2 \mapsto v_3$ makes $x \cdot yz \cdot uz$ equivalent to xv_3, while both $x(v_1 \cdot v_2 v_3)$ and $x(v_1 v_2 v_3)$ equivalent to x. We get a contradiction.

Case $t_1 = xv_1$: Then $t \sim x \cdot v_2 v_3 \cdot v_1$. By the substitution $v_2 \mapsto x$ we get that xv_1 is $\sim$-equivalent with one of the terms xv_1v_2, $x \cdot v_2 v_1$, $x \cdot v_1 v_2$. In the first case we get a contradiction by the substitution $v_1 \mapsto x$ and in the second case by $v_2 \mapsto x$. In the last case we get $x \cdot yz \sim xy$.

Case $t_1 = x \cdot v_1 v_2$ and $t_2 = v_3$: Then $t = x(v_1 v_2) v_3 \sim x v_3 (v_1 v_2)$, the previous case.

Case $t_1 = xv_1v_2$ and $t_2 = v_3$: Then the substitution $y \mapsto x$ gives $x \cdot uz \sim xuz$, which is clearly impossible.

Thus in all cases we get either a contradiction or $x \cdot yz \sim xy$. □

It follows from 7.6 that every term is $\sim$-equivalent to a left-associated linear term.

Lemma 7.7. *Either $xyzy \sim xyz$ or $xyzy \sim xzy$.*

Proof. There exists a linear term t in three variables x, y, z with $t \sim xyzy$. If t is either x or xz, we get a contradiction by the substitution $z \mapsto x$. If t is xy, we get a contradiction by $y \mapsto x$. By 7.6, the only two remaining possibilities are $t = xyz$ and $t = xzy$. □

Lemma 7.8. *Let $x, y_1, \dots, y_n$ be pairwise distinct variables. Then $xy_1 \dots y_n x \sim xy_1 \dots y_n$.*

Proof. We have $xy_1 \dots y_n x \sim xz_1 \dots z_k$ for some pairwise distinct variables $x, z_1, \dots, z_k$. If there exists a variable v in the symmetric difference of $S(xy_1 \dots y_n x)$ and $S(xz_1 \dots z_k)$, then the substitution sending all variables except v to x (and fixing v) gives $xv \sim x$, a contradiction. Thus $k = n$ and $\{y_1, \dots, y_n\} = \{z_1, \dots, z_n\}$. Let f be the substitution with $f(x) = x$ and $f(y_i) = z_i$ for all i. We have $xy_1 \dots y_n x \sim xf(y_1) \dots f(y_n)$. If $xy_1 \dots y_n x \sim xf^i(y_1) \dots f^i(y_n)$ for some positive integer i then

$xf(y_1) \dots f(y_n) x \sim xf^{i+1}(y_1) \dots f^{i+1}(y_n)$, so that
$xy_1 \dots y_n xx \sim xf^{i+1}(y_1) \dots f^{i+1}(y_n)$, i.e.,
$xy_1 \dots y_n x \sim xf^{i+1}(y_1) \dots f^{i+1}(y_n)$.

Thus, by induction, $xy_1 \dots y_n x \sim xf^i(y_1) \dots f^i(y_n)$ for all i. For i the order of the permutation f, we get $xy_1 \dots y_n x \sim xy_1 \dots y_n$. □

Denote by E_A the equational theory generated by the equations

(A1) $xx \approx x$,
(A2) $xyy \approx xy$,
(A3) $x(yz) \approx xy$,
(A4) $xyzy \approx xyz$.

Denote by E_B the equational theory generated by the equations

(B1) $xx \approx x$,
(B2) $xyy \approx xy$,
(B3) $x(yz) \approx xy$,
(B4) $xyzy \approx xzy$.

Denote by E_C the equational theory generated by the equations

(C1) $xx \approx x$,
(C2) $xyy \approx xy$,
(C3) $x(yz) \approx xy$,
(C4) $xyz \approx xzy$.

Denote by $\mathcal{V}_A$, $\mathcal{V}_B$, $\mathcal{V}_C$ the varieties corresponding to E_A, E_B, E_C, respectively.

Lemma 7.9. *The equational theory E_A is $*$-quasilinear.*

Proof. Obviously, using the identity (A3), we get that any term t is E_A-equivalent to a left-associated term t'. Then, using the identities (A1), (A2) and (A4), t' is reduced to a left-associated linear term t'' by deleting any occurrence of a variable which is not the first occurrence from the left. □

Lemma 7.10. *The equational theory E_B is $*$-quasilinear.*

Proof. Denote E_B by $\sim$. Using (B3), we get that any term is E_B-equivalent to a left-associated term. Using (B3) several times and (B1) at the last step we get

$$\begin{aligned} xy_1 \dots y_n x &\sim xy_1 \dots y_n \cdot xy_1 \\ &\sim xy_1 \dots y_n \cdot xy_1 y_2 \quad \dots \\ &\sim xy_1 \dots y_n \cdot xy_1 \dots y_n \\ &\sim xy_1 \dots y_n. \end{aligned}$$

From this it follows that every term is E_B-equivalent to a left-associated term in which the leftmost variable occurs only once. Using (B4) we get

$$\begin{aligned} xyz_1 \dots z_n y &\sim xyz_1 \dots z_{n-1} y z_n y \\ &\sim xyz_1 \dots z_{n-2} y z_{n-1} y z_n y \quad \dots \\ &\sim xyz_1 y z_2 y \dots z_n y \\ &\sim xz_1 y z_2 y \dots z_n z \quad \dots \\ &\sim xz_1 z_2 \dots z_n y. \end{aligned}$$

From this, using also (B2), it follows that every left-associated term t is E_B-equivalent to a left-associated linear term t' obtained from t like this: Let x

be the first variable from the left in t. We erase all occurrences of x except that leftmost one. Then for any variable $y \neq x$ of t we erase all occurrences of y except the rightmost one. In particular, every term is E_B-equivalent to a left-associated linear term. □

Lemma 7.11. *Every term is E_C-equivalent to a left-associated linear term. Two left-associated linear terms $x_1 \dots x_n$ and $y_1 \dots y_m$ are E_C-equivalent if and only if $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$ (so that $n = m$) and $x_1 = y_1$. E_C extends both E_A and E_B. In other words, $\mathcal{V}_C \subseteq \mathcal{V}_A \cap \mathcal{V}_B$.*

Proof. It is easy. □

Lemma 7.12. *The variety $\mathcal{V}_C$ is generated by $\mathbf{G}_5$. Its only proper subvarieties are the trivial variety and the variety of left-zero semigroups.*

Proof. Let V be a proper subvariety of $\mathcal{V}_C$. By 7.11, V satisfies an equation $x_1 \dots x_n \approx y_1 \dots y_m$ where $x_1, \dots, x_n$ are pairwise distinct variables, $y_1, \dots, y_m$ are pairwise distinct variables and either $\{x_1, \dots, x_n\} \neq \{y_1, \dots, y_m\}$ or $x_1 \neq y_1$. If the sets are different then the equation reduces to either $x \approx y$ or $xy \approx x$ or $xy \approx y$; in the last case it also reduces to $x \approx y$. If $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$ and $x_1 \neq x_2$ then the equation reduces to $x_1 x_2 \approx x_2 x_1$, which then also reduces to $x \approx y$. Thus V is a subvariety of the minimal variety of left-zero semigroups. Since $\mathbf{G}_5$ satisfies the equations (C1)–(C4) and is not a left-zero semigroup, it follows that $\mathcal{V}_C$ is generated by $\mathbf{G}_5$. □

Lemma 7.13. *All proper subvarieties of $\mathcal{V}_A$ are subvarieties of $\mathcal{V}_C$.*

Proof. Note that $\mathcal{V}_A$ satisfies $xyx \approx xxyx \approx xxy \approx xy$. Let V be a proper subvariety of $\mathcal{V}_A$. Then V satisfies an equation $t_1 \approx t_2$ not satisfied in $\mathcal{V}_A$, such that both t_1 and t_2 are left-associated linear terms. If $S(t_1) \neq S(t_2)$, then this equation reduces to either $xy \approx x$ or $xy \approx y$. In the first case V is contained in the variety of left-zero semigroups, and in the second case it is the trivial variety. If $S(t_1) = S(t_2)$ and the leftmost variable x of t_1 is different from the leftmost variable y of t_2, then by replacing all the other variables with x we reduce t_1 to xy and t_2 to yx, so that V is the trivial variety. Finally, let $S(t_1) = S(t_2)$, let the leftmost variable x of t_1 be also the leftmost variable in t_2, and let i be the least number such that the i-th variable y in t_1 from the left differs from the i-th variable z in t_2 from the left. Replace all variables occurring before the i-th variable by x, and all other variables except y and z by y. We obtain $xyz \approx xzy$, so that V is contained in $\mathcal{V}_C$. □

Lemma 7.14. *All proper subvarieties of $\mathcal{V}_B$ are subvarieties of $\mathcal{V}_C$.*

Proof. It is similar to the proof of 7.13. □

Theorem 7.15. *There are precisely three $*$-quasilinear varieties with $\mathbf{G}_5$ serving as their 2-generated free algebra: the varieties $\mathcal{V}_A$, $\mathcal{V}_B$ and $\mathcal{V}_C$. Their only subvarieties, except themselves, are the variety of left-zero semigroups* *and the trivial variety.* $\mathcal{V}_A$ *is generated by the groupoid* $\mathbf{A}$ *and* $\mathcal{V}_B$ *is generated by the groupoid* $\mathbf{B}$*; the multiplication tables of* $\mathbf{A}$ *and* $\mathbf{B}$ *are given below.* $\mathcal{V}_C$ *is generated by the groupoid* $\mathbf{G}_5$*.*

$\mathbf{A}$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	d	e	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

$\mathbf{B}$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	d	c	c	c	c
d	d	e	d	d	d
e	d	e	e	e	e

Proof. It is easy to check that $\mathbf{A}$ belongs to $\mathcal{V}_A$ but not to $\mathcal{V}_B$ and that $\mathbf{B}$ belongs to $\mathcal{V}_B$ but not to $\mathcal{V}_A$. The rest follows from the lemmas of this section. □

8. Extending $\mathbf{G}_6$

In this section let E be a $*$-quasilinear equational theory and V be the corresponding variety, such that $\mathbf{G}_6$ is the two-generated free groupoid in V. We write $t \sim s$ if the equation $t \approx s$ belongs to E. From the multiplication table for $\mathbf{G}_6$ we get

$$x \sim xx,$$
$$xy \sim x(xy) \sim x(yx) \sim xyx \sim xyy \sim xy(yx).$$

Lemma 8.1. *If* $t \sim s$ *then* $S(t) = S(s)$ *and the terms* t, s *have the same leftmost variables.*

Proof. This follows easily from the properties of $\mathbf{G}_6$. □

It has been proved in Lemma 5.1 of [1] that if E is 3-linear then the three-generated free groupoid in V must be one of the seven 21-element groupoids $\mathbf{Q}_1, \dots, \mathbf{Q}_7$. We do not need to repeat the definitions of these seven groupoids. Also, three $*$-linear varieties $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ were constructed in [1] with three-generated free algebras $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_4$, respectively.

Lemma 8.2. *If one of the seven groupoids* $\mathbf{Q}_1, \dots, \mathbf{Q}_7$ *is the three-generated free groupoid in* V *then* V *is a subvariety of either* $\mathcal{L}_1$ *or* $\mathcal{L}_2$ *or* $\mathcal{L}_3$*.*

Proof. In Lemma 6.1 of [1] it was proved that four of the seven groupoids, namely $\mathbf{Q}_3$, $\mathbf{Q}_5$, $\mathbf{Q}_6$ and $\mathbf{Q}_7$, are impossible in this context. (The proof was done for the linear case, but the same proof can serve in our quasilinear case.) Three possibilities for the three-generated free groupoid in V remain.

Let the free groupoid be $\mathbf{Q}_1$. The base for the equations of $\mathcal{L}_1$ found in Theorem 9.1 of [1] consists of three equations. Since these are equations in three variables, they are satisfied in V and thus $V \subseteq \mathcal{L}_1$.

Now let the free groupoid be either $\mathbf{Q}_2$ or $\mathbf{Q}_4$. According to Theorem 16.2 of [1], the varieties $\mathcal{L}_2$ and $\mathcal{L}_3$ are generated by $\mathbf{Q}_2$ and $\mathbf{Q}_4$, respectively. Thus $V \subseteq \mathcal{L}_2$ in the first case and $V \subseteq \mathcal{L}_3$ in the second. □

Thus is remains to consider the case when E is not 3-linear. It follows from 8.1 that at least one of the following four cases takes place:

(a) $xyz \sim x \cdot yz$,
(b) $x \cdot yz \sim xzy$,
(c) $xyz \sim xzy$,
(d) $x \cdot yz \sim x \cdot zy$.

Lemma 8.3. *If either* (a) *or* (b) *is satisfied then* $V \subseteq \mathcal{L}_1$.

Proof. If (a) is satisfied then all the three equations in the base for $\mathcal{L}_1$ (given by Theorem 9.1 of [1]) belong to E, so that $V \subseteq \mathcal{L}_1$. If (b) is satisfied then $xyz \sim x \cdot zy \sim x \cdot zy \cdot x \sim xy \cdot z \cdot x \sim xy \cdot xz \sim x \cdot xzy \sim x(x \cdot yz) \sim x \cdot yz$, so that we are in case (a). (The identities of $\mathbf{G}_6$ were also used.) □

Lemma 8.4. *Let both* (c) *and* (d) *take place. Then* (a) *takes place.*

Proof. We have $xy \cdot z \sim xy(xy \cdot z) \sim (x \cdot xy)(xy \cdot z) \sim_{(c)} x \cdot (xy \cdot z) \cdot xy \sim_{(d)} x \cdot (z \cdot xy) \cdot xy \sim_{(d)} x \cdot (z \cdot yx) \cdot xy \sim_{(d)} x \cdot yxz \cdot xy \sim_{(c)} x \cdot yzx \cdot xy \sim x \cdot yz \cdot xy \sim_{(c)} x \cdot xy \cdot yz \sim xy \cdot yz \sim_{(c)} x \cdot yz \cdot y \sim (x \cdot yz)(y \cdot (x \cdot yz)) \sim_{(d)} (x \cdot yz)(y \cdot (yz \cdot x)) \sim_{(d)} (x \cdot yz)(yz \cdot x \cdot y) \sim_{(c)} (x \cdot yz)(yz \cdot y \cdot x) \sim (x \cdot yz)(yz \cdot x) \sim x \cdot yz$. □

Lemma 8.5. *Let* (d) *take place, but neither* (a) *nor* (b) *nor* (c). *Then* $V \subseteq \mathcal{L}_1$.

Proof. If $xy \cdot xz \sim x \cdot yz$ then the substitution $x \mapsto yx$ yields (a), a contradiction. If $xy \cdot xz \sim xzy$, we obtain (a) similarly by $y \mapsto xy$. It remains that $xy \cdot xz \sim xyz$.

Case $x \cdot xyz \sim x \cdot yz$: All equations of the base for $\mathcal{L}_1$ (Theorem 9.1 of [1]) belong to E and thus $V \subseteq \mathcal{L}_1$.

Case $x \cdot xyz \sim xyz$: Then $xzy \sim x(xzy) \sim x(xz \cdot xy) \sim_{(d)} x(xy \cdot xz) \sim x(xyz) \sim xyz$. We get (c), a contradiction.

Case $x \cdot xyz \sim xzy$: Then $xzy \sim x \cdot xyz \sim x(xy \cdot xz) \sim_{(d)} x(xz \cdot xy) \sim x \cdot xzy \sim xyz$, a contradiction again. □

Lemma 8.6. *Let* (c) *hold, but neither* (a) *nor* (b) *nor* (d). *Then:*

(1) $x \cdot xyz \sim xyz$,
(2) $x \cdot yz \cdot zy \sim xyz$,
(3) $xy \cdot zy \sim xyz$.
(4) $x(y \cdot xz) \sim x \cdot yz$,
(5) $x(y \cdot zx) \sim x \cdot yz$,

Proof. (1) The term $x \cdot xyz$ must be $\sim$-equivalent with one of xyz, $x \cdot yz$, $x \cdot zy$. If either $x \cdot xyz \sim x \cdot yz$ or $x \cdot xyz \sim x \cdot zy$ then (because $x \cdot xyz \sim x \cdot xzy$) we would get $x \cdot yz \sim x \cdot zy$, a contradiction since (d) is not satisfied.

(2) We have $x \cdot yz \cdot zy \sim_{(c)} x \cdot zy \cdot yz$. If $x \cdot yz \cdot zy$ is equivalent with either $x \cdot yz$ or $x \cdot zy$ then we get (d), a contradiction. Thus $x \cdot yz \cdot zy$ must be equivalent with xyz.

(3) $xy \cdot zy \sim_{(2)} x \cdot (y \cdot zy) \cdot (zy \cdot y) \sim x \cdot yz \cdot zy \sim_{(2)} xyz$

(4) Suppose that $x(y \cdot xz) \sim x \cdot zy$. Then $x \cdot zy \sim x(y \cdot xz) \sim x(y(x \cdot xz)) \sim x \cdot xzy \sim_{(1)} xzy$, a contradiction since (a) is not satisfied.

Now suppose that $x(y \cdot xz) \sim xyz$ (6). Then $xy \cdot yz \sim_{(6)} x(y(x \cdot yz)) \sim_{(6)} x \cdot yxz \sim x \cdot yzx \sim x \cdot yz$ (7).

Now $x \cdot yz \sim x \cdot yzx \sim_{(3)} x \cdot (yz \cdot xz) \sim_{(7)} (x \cdot yz)(yz \cdot xz) \sim_{(c)} (x \cdot yz)(y \cdot xz \cdot z) \sim_{(7)} (x \cdot yz)(yx \cdot xz \cdot z) \sim_{(c)} (x \cdot yz)(yx \cdot z \cdot xz) \sim_{(6)} (x \cdot yz)(y \cdot (x \cdot yz) \cdot xz) \sim_{(c)} (x \cdot yz)((y \cdot xz)(x \cdot yz)) \sim (x \cdot yz)(y \cdot xz) \sim_{(c)} x \cdot (y \cdot xz) \cdot yz \sim_{(6)} xy \cdot z \cdot yz \sim_{(c)} xy \cdot yz \cdot z \sim_{(7)} x \cdot yz \cdot z \sim_{(c)} xz \cdot yz \sim_{(3)} xzy \sim xyz$. We get a contradiction.

So it must be that $x(y \cdot xz) \sim x \cdot yz$.

(5) $x \cdot yz \sim_{(4)} x(y \cdot xz) \sim x(y(x \cdot zx)) \sim x(y \cdot zx)$. □

Lemma 8.7. *If* (c) *takes place then also either* (a) *or* (b) *or* (d) *takes place.*

Proof. Suppose that (c) takes place but neither (a) nor (b) nor (d). Then we will prove that the term $x(yz)(uz)$ is not $\sim$-equivalent to any liner term. Let $x(yz)(uz) \sim t$ where t is linear. By 8.1, $S(t) = \{x, y, z, u\}$. Using 8.6(3), $xyz \sim xzy$ and the two-variable equations of $\mathbf{G}_6$ it is easy to see that

$$\begin{aligned}
&\sigma_{y,x}(x \cdot yz \cdot uz) \sim xzu,\\
&\sigma_{z,x}(x \cdot yz \cdot uz) \sim xuy,\\
&\sigma_{u,x}(x \cdot yz \cdot uz) \sim xzy,\\
&\sigma_{z,y}(x \cdot yz \cdot uz) \sim xyu,\\
&\sigma_{u,y}(x \cdot yz \cdot uz) \sim x \cdot yz,\\
&\sigma_{u,z}(x \cdot yz \cdot uz) \sim xzy.
\end{aligned}$$

Let $t = t_1 t_2$. Let v_1, v_2, v_3 represent a permutation of the variables y, z, u. One of the following three cases must take place:

<u>Case $t_1 = x$:</u> Then t_2 contains the other three variables. If $t_2 = v_1 \cdot v_2 v_3$, we obtain a contradiction by the substitution $\sigma_{v_1,x}$. If $t_2 = v_1 v_2 v_3$, a contradiction can be obtained using $\sigma_{v_3,x}$.

<u>Case $t_1 = xv_1$:</u> Use $\sigma_{v_1,x}$ to obtain a contradiction.

<u>Case $t_2 = v_3$:</u> If $t_1 = x \cdot v_1 v_2$ then $t = x(v_1 v_2) v_3 \sim_{(c)} x v_3 (v_1 v_2)$, the previous case. If $t_1 = x v_1 v_2$, we get a contradiction by $\sigma_{u,y}$. □

Proposition 8.8. *Let E be a $*$-quasilinear equational theory and V be the corresponding variety, such that $\mathbf{G}_6$ is the two-generated free groupoid in V. Then V is a subvariety of either $\mathcal{L}_1$ or $\mathcal{L}_2$ or $\mathcal{L}_3$.*

Proof. It follows from the previous lemmas in this section. □

9. Extending $\mathbf{G}_7$

In this section let E be a $*$-quasilinear equational theory and V be the corresponding variety, such that $\mathbf{G}_7$ is the two-generated free groupoid in V. We write $t \sim s$ if the equation $t \approx s$ belongs to E. Thus we have $xy \sim yx$ and $xyx \sim y$.

Lemma 9.1. $x(y \cdot xz) \sim z \cdot xy$.

Proof. There is a linear term t in the variables x, y, z with $t \sim x(y \cdot xz)$. If $x \notin S(t)$, we get a contradiction by the substitution $y \mapsto z$. If $y \notin S(t)$, we get a contradiction by $z \mapsto x$. If $z \notin S(t)$, we get a contradiction by $y \mapsto x$. Thus $S(t) = \{x, y, z\}$. Using the substitution $y \mapsto x$, we obtain that t cannot be $\sim$-equivalent with either $x \cdot zy$ or $y \cdot xz$. The only remaining possibility is $t \sim z \cdot xy$. □

Lemma 9.2. $xy \cdot xz \sim x \cdot yz$.

Proof. There is a linear term t in the variables x, y, z with $t \sim xy \cdot xz$. If $x \notin S(t)$, we get a contradiction by the substitution $y \mapsto z$. If $y \notin S(t)$, we get a contradiction by $z \mapsto x$. If $z \notin S(t)$, we get a contradiction by $y \mapsto x$. Thus $S(t) = \{x, y, z\}$. Using the substitution $z \mapsto y$, we obtain that t cannot be $\sim$-equivalent with either $y \cdot xz$ or $z \cdot xy$. The only remaining possibility is $t \sim x \cdot yz$. □

Lemma 9.3. $xyzux \sim yuz$.

Proof. There is a linear term t in the variables x, y, z, u with $t \sim xyzux$. We are going to show that $S(t) = \{y, z, u\}$.

If $y \notin S(t)$, we get a contradiction by the substitution $z, u \mapsto x$. If $z \notin S(t)$, we get a contradiction by $y, u \mapsto x$. If $u \notin S(t)$, we get a contradiction by $y, z \mapsto x$.

Suppose that $x \in S(t)$. Then one of the following cases takes place for three variables v_1, v_2, v_3.

<u>Case $t = x(v_1 \cdot v_2v_3)$:</u> Using 9.1, we get a contradiction in all subcases by the substitution $v_1 \mapsto x$, $v_2 \mapsto v_3$.

<u>Case $t = xv_1 \cdot v_2v_3$:</u> We will use 9.1 and 9.2. If $v_1 = y$, we get a contradiction by $y, u \mapsto z$. If $v_2 = y$ or $v_3 = y$, then by the substitution by $y \mapsto x$, $u \mapsto z$ the term t becomes $\sim$-equivalent to xz, while $xyzux$ to x.

<u>Case $t = x \cdot v_1v_2 \cdot v_3$:</u> Using 9.1, we get a contradiction in all subcases by the substitution $v_1 \mapsto x$, $v_2 \mapsto v_3$.

<u>Case $t = xv_1v_2v_3$:</u> We get a contradiction in all subcases by the substitution $v_2 \mapsto x$, $v_1 \mapsto v_3$.

Thus $S(t) = \{y, z, u\}$ If either $t \sim yzu$ or $t \sim uzy$, then using 9.1 the substitution $u \mapsto y$ yields $yz \sim z$, a contradiction. Now $t \sim yuz$ is the only remaining case. □

Lemma 9.4. $xzyu \sim xuyz$.

Proof. $xzyu \sim zxyu \sim zxyuzz \sim xuyz$ by 9.3. □

Denote by E_D the equational theory generated by the equations

(D1) $xx \approx x$,
(D2) $xy \approx yx$,
(D3) $x \cdot xy \approx y$,
(D4) $xzyu \approx xuyz$.

Denote by $\mathcal{D}$ the variety corresponding to E_D.

Lemma 9.5. *E_D contains the following equations:*

(D5) $xzyx \approx xyz$,
(D6) $xy \cdot zu \approx xz \cdot yu$.

Proof. $xzyx \approx xxyz \approx xyz$ by (D4) and (D2). Using (D5), (D4), (D2) and (D5) once more, we have $xy \cdot zu \approx (x \cdot zu \cdot y)x \approx zuxyx \approx uzxyx \approx uyxzx \approx xz \cdot uy$. □

Lemma 9.6. *The equational theory E_D is $*$-quasilinear.*

Proof. We will write $t \equiv s$ if the equation $t \approx s$ belongs to E_D and every variable has the same number of occurrences in t as in s.

Let us first prove that if t is a term, $x \in S(t)$ and $t \neq x$, then there exist terms v, w such that either $t \equiv vx$ or $t \equiv vxw$. Since $x \in S(t)$, it follows from (D2) that there exist terms $u_1, \dots, u_n$ (for some $n \geq 1$) such that $t \equiv xu_1 \dots u_n$.

Let n be odd. Using (D4) we have

$$\begin{aligned} t &\equiv u_1 x u_2 \dots u_n \\ &\equiv u_1 u_3 u_2 x u_4 \dots u_n \quad \dots \\ &\equiv u_1 u_3 u_2 \dots u_{n-2} u_{n-3} x u_{n-1} u_n \\ &\equiv u_1 u_3 u_2 \dots u_{n-2} u_{n-3} u_n u_{n-1} x, \end{aligned}$$

so that $t \equiv vx$ where $v = u_1 u_3 u_2 \dots u_{n-2} u_{n-3} u_n u_{n-1}$.

If n is even, we can start in the same way but we end up with $t \equiv u_1 u_3 u_2 \dots u_{n-1} u_{n-2} x u_n$, so that $t \equiv vxw$ where $t = u_1 u_3 u_2 \dots u_{n-1} u_{n-2}$ and $w = u_n$.

We claim that every term which is not E_D-equivalent to any term of smaller length must be linear. Let t be a counterexample to this claim which has minimal length and let x occur twice in t. Then $t = pq$ and by minimality of t, each of p, q is linear and x occurs once in each of p, q. Applying the above observation to p and q and taking commutativity into account, we get the following six cases (for some terms u, v, u', v'): $t \equiv xx$, $t \equiv uxx$, $t \equiv uxvx$, $t \equiv ux \cdot u'x$, $t \equiv uxv \cdot u'x$ and $t \equiv uxv \cdot u'xv'$. The first two are obviously E_D-equivalent to terms of smaller length, and the remaining cases are dealt with in the following way:

<u>Case $t \equiv uxvx$</u>: A contradiction, since $uxvx \approx xuvx \approx xvu$ in E_D.

<u>Case $t \equiv ux \cdot u'x$</u>: A contradiction, since $ux \cdot u'x \approx uu' \cdot xx \approx uu'x$ in E_D.

<u>Case $t \equiv uxv \cdot u'x$</u>: A contradiction, since $uxv \cdot u'x \approx (v \cdot ux) \cdot u'x \approx vu' \cdot uxx \approx vu'u$ in E_D.

<u>Case $t \equiv uxv \cdot u'xv'$</u>: A contradiction, since $uxv \cdot u'xv' \approx ux \cdot u'x \cdot vv' \approx uu'x \cdot vv'$ in E_D. □

Theorem 9.7. *There is precisely one $*$-quasilinear variety with $\mathbf{G}_7$ serving as its 2-generated free algebra: the variety $\mathcal{D}$. It is the variety generated by $\mathbf{G}_7$ and its only proper subvariety is the trivial variety. It is also based on the following equations:*

$xx \approx x,$
$xy \approx yx,$
$x \cdot xy \approx y,$
$xy \cdot zu \approx xz \cdot yu.$

Proof. Denote by V the variety determined by these last four equations. By 9.5 we have $\mathcal{D} \subseteq V$. It was proved in [2], and also in [3], that V is a minimal variety. Since $\mathcal{D}$ obviously contains the groupoid $\mathbf{G}_7$, $\mathcal{D}$ is nontrivial and thus $\mathcal{D} = V$. It also follows that $\mathcal{D}$ is generated by $\mathbf{G}_7$. By 9.6, $\mathcal{D}$ is $*$-quasilinear. By 9.4, $\mathcal{D}$ is the only $*$-quasilinear variety with $\mathbf{G}_7$ serving as its 2-generated free algebra. □

Groupoids satisfying $xy \cdot zu \approx xz \cdot yu$ were called abelian in some papers, entropic in some other papers, but we prefer to call them medial. Thus $\mathcal{D}$ is the variety of idempotent commutative medial groupoids satisfying $x \cdot xy \approx y$.

10. Extending $\mathbf{G}_8$ and $\mathbf{G}_9$

Proposition 10.1. *There exists precisely one $*$-quasilinear variety with $\mathbf{G}_8$ serving as its 2-generated free algebra: the variety of semilattices.*

Proof. Let $\sim$ be a $*$-quasilinear variety with $\mathbf{G}_8$ serving as its 2-generated free algebra. Clearly, it is sufficient to prove $xyz \sim x \cdot yz$. The term $x(y \cdot xz)$ is equivalent to a linear term in variables x, y, z. By using one of the substitutions $y \mapsto x$, $z \mapsto x$ or $z \mapsto y$, it is easy to see that $x(y \cdot xz)$ cannot be $\sim$-equivalent to any of the terms x, y, z, xy, xz, yz. It remains to consider the following three cases.

Case 1: $x(y \cdot xz) \sim x \cdot yz$. Then $x \cdot yz \sim x \cdot yz \cdot yz \sim (x \cdot yz)(y((x \cdot yz)z)) \sim (x \cdot yz)(y(z \cdot xy)) \sim (x \cdot yz)(y \cdot xz)$. Similarly we have $y \cdot xz \sim (y \cdot xz)(x \cdot yz)$ and we get the associative law.

Case 2: $x(y \cdot xz) \sim z \cdot xy$. Then $y \cdot xz \sim x(z \cdot xy) \sim x(x(y \cdot xz)) \sim x(y \cdot xz) \sim z \cdot xy \sim yx \cdot z$.

Case 3: $x(y \cdot xz) \sim y \cdot xz$. Obviously, the term $xy \cdot xz$ is not $\sim$-equivalent to any of the terms x, y, z, xy, xz, yz. Thus it remains to consider the following three subcases.

Subcase $xy \cdot xz \sim x \cdot yz$: Then $x \cdot yz \sim xy \cdot xz \sim xy(x \cdot xz) \sim x(y \cdot xz) \sim y \cdot xz$.

Subcase $xy \cdot xz \sim y \cdot xz$: Then $y \cdot xz \sim xy \cdot xz \sim xz \cdot xy \sim z \cdot xy$.

Subcase $xy \cdot xz \sim z \cdot xy$: This is similar to the previous subcase. □

Proposition 10.2. *There exists precisely one $*$-quasilinear variety with $\mathbf{G}_9$ serving as its 2-generated free algebra: the variety of right-zero semigroups.*

Proof. It is obvious. □

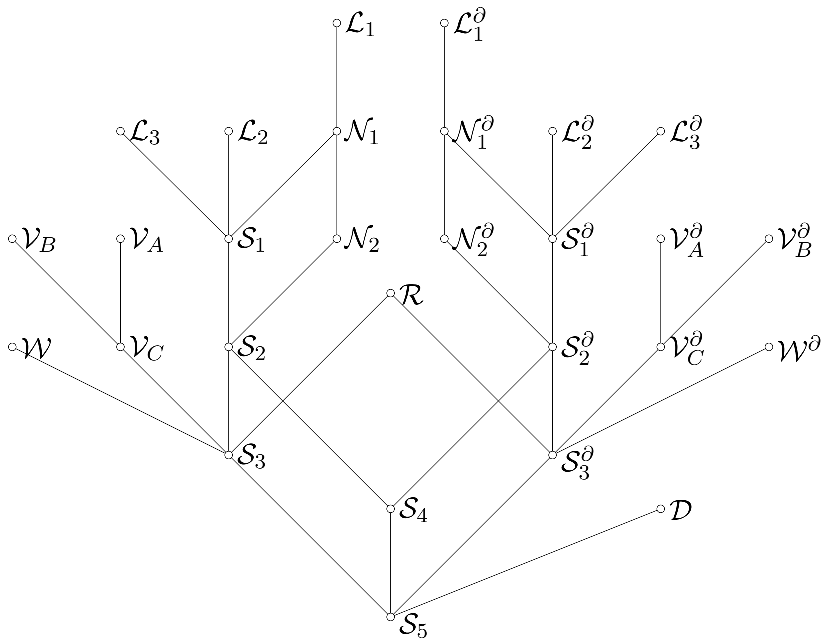
11. All idempotent $*$-quasilinear varieties

Recall from [1] the following notation:

$\mathcal{N}_1$ is the variety of $\mathcal{L}_1$-algebras satisfying $w(xy \cdot z) \approx w(x \cdot yz)$;
$\mathcal{N}_2$ is the variety of $\mathcal{L}_1$-algebras satisfying $w \cdot xy \approx w \cdot yx$;
$\mathcal{S}_1$ is the variety of idempotent semigroups satisfying $xyx \approx xy$;
$\mathcal{S}_2$ is the variety of idempotent semigroups satisfying $wxy \approx wyx$;
$\mathcal{S}_3$ is the variety of semigroups satisfying $xy \approx x$;
$\mathcal{S}_4$ is the variety of semilattices;
$\mathcal{S}_5$ is the trivial variety.

It was proved in Theorem 15.5 of [1] that the list consisting of these varieties, the varieties $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and all their duals, is the list of all $*$-linear varieties and their subvarieties. (Only $\mathcal{S}_4$ and $\mathcal{S}_4$ are self-dual.) Combining this with the previous results of this paper, we obtain

Theorem 11.1. *There are precisely* 28 *idempotent $*$-quasilinear varieties, namely:* $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{R}$, $\mathcal{W}$, $\mathcal{V}_A$, $\mathcal{V}_B$, $\mathcal{V}_C$, $\mathcal{D}$, $\mathcal{L}_1^\partial$, $\mathcal{L}_2^\partial$, $\mathcal{L}_3^\partial$, $\mathcal{N}_1^\partial$, $\mathcal{N}_2^\partial$, $\mathcal{S}_1^\partial$, $\mathcal{S}_2^\partial$, $\mathcal{S}_3^\partial$, $\mathcal{W}^\partial$, $\mathcal{V}_A^\partial$, $\mathcal{V}_B^\partial$, $\mathcal{V}_C^\partial$. *While* $\mathcal{L}_3$ *and* $\mathcal{L}_3^\partial$ *are inherently nonfinitely based, all the remaining* 26 *varieties are finitely based. All inclusions between these* 28 *varieties can be found from the following picture, representing a down-set in the lattice of varieties of groupoids.*



12. Non-idempotent $*$-quasilinear equational theories

We have seen that there are not many idempotent $*$-quasilinear varieties of groupoids and that all of them are locally finite (or even finitely generated).

Arbitrary $*$-quasilinear varieties are not necessarily locally finite, as shown by the following example.

Example 12.1. Let $\mathbf{G}$ be the groupoid with the underlying set ω (the set of nonnegative integers) and multiplication $\circ$ defined by $a \circ b = a + 1$. In the equational theory of $\mathbf{G}$, every term $xt_1 \dots t_n$ is equivalent to the linear term $xy_1 \dots y_n$ where y_i are pairwise distinct variables different from x. Thus the equational theory of $\mathbf{G}$ is $*$-quasilinear. The corresponding variety generated by $\mathbf{G}$ is not locally finite. It has infinitely many subvarieties, although only countably many, since it is term equivalent to the variety of all monounary algebras.

We are now going to find a $*$-quasilinear equational theory such that the corresponding variety has $2^{\aleph_0}$ subvarieties.

The set of *slender* terms is the smallest set of terms containing all variables and such that whenever t is a slender term and x is a variable then both tx and xt are slender terms. Denote by U the set of all terms that are either not linear or not slender. Denote by E the equivalence on the set of terms such that U is the only non-singleton block of E. It is easy to see that E is a $*$-quasilinear equational theory.

Let A be an infinite countable set. Denote by B the set of all nonempty words $a_1 \dots a_n$ over A such that the elements $a_i \in A$ are pairwise different. For $u \in B$ let $S(u)$ denote the set of the elements of A occurring in u. Denote by $\mathbf{C}$ the groupoid with the underlying set $C = B \cup \{0\}$ and multiplication $\circ$ defined in the following way. $u \circ v = 0$ for all $u, v \in C$ except these cases:

if $a, b \in A$ and $a \neq b$ then $a \circ b = ab$;

if $a \in A$, if $u \in B$ is of length 2 and if $a \notin S(u)$ then $a \circ u = au$;

if $u \in B$ is of length at least 3, if $a \in A$ and if $a \notin S(u)$ then $u \circ a = ua$.

In particular, $a \circ 0 = 0 \circ a = a \circ a = (a \circ b) \circ (d \circ e) = 0$ for all $a, b, c, d \in C$. Thus every non-slender term evaluates to 0 and it is easy to check that also every non-linear term evaluates to 0 under any interpretation of variables in $\mathbf{C}$. Thus $\mathbf{C}$ is a model of E.

For every subset K of the set of nonnegative integers define a binary relation R_K on C as follows: for $u, v \in C$ we have $(u, v) \in R_K$ if and only if either $u = v$ or else there exist $n \geq 1$ and elements a, b, c, d_i, e_i of A such that $u = abcd_1 \dots d_n \in B$, $v = abce_1 \dots e_n \in B$ and for any i with $0 \leq 2i < n$ either $d_{2i} = e_{2i}$ & $d_{2i+1} = e_{2i+1}$ or $i \in K$ & $d_{2i} = e_{2i+1}$ & $d_{2i+1} = e_{2i}$. Clearly, R_K is an equivalence and it is easy to check that it is a congruence of $\mathbf{C}$.

Let us prove that the factor $\mathbf{C}/R_K$ satisfies $t_1 \approx t_2$, where

$$t_1 = (x \cdot yz)x_0 \dots x_{2i-1}x_{2i}x_{2i+1} \quad \text{and} \quad t_2 = (x \cdot yz)x_0 \dots x_{2i-1}x_{2i+1}x_{2i},$$

if and only if $i \in K$. If $i \notin K$ then t_1, t_2 evaluate to two different elements of $\mathbf{C}/R_K$ if we interpret the variables by pairwise different elements of A. Now let $i \in K$. Take any mapping h of the set $x, y, z, x_0, \dots, x_{2i+1}$ into C and denote by H the extension of h to a homomorphism of the groupoid of terms into $\mathbf{C}$. We need to prove that $(H(t_1), H(t_2)) \in R_K$. It is not difficult to see that if the range of h is not contained in A then $H(t_1) = H(t_2) = 0$. Thus we can assume that $a = h(x)$, $b = h(y)$, $c = h(z)$, $a_i = h(x_i)$ are elements of A. It is easy to see that if these elements of A are not pairwise distinct then again $H(t_1) = H(t_2) = 0$. Thus we can assume that all the elements $a, b, c, a_0, \dots, a_{2i+1}$ are pairwise distinct. Then $H(t_1) = abca_0 \dots a_{2i+1} \in B$ and $H(t_2) = abca_0 \dots a_{2i-1}a_{2i+1}a_{2i} \in B$. But then, by the definition of R_K, $(H(t_1), H(t_2)) \in R_K$.

It follows that for different subsets K the groupoids $\mathbf{C}/R_K$ generate different varieties. Thus we obtain:

Theorem 12.2. *There are $2^{\aleph_0}$ different $*$-quasilinear equational theories of groupoids.*

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