ON SEPARATING SETS OF WORDS II

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Abstract. Special replacement relation in free monoids is studied with particular interest in antisymmetry and antitransitivity.

1. Introduction

This article is an immediate continuation of [1]. References like I.3.3 lead to the corresponding section and result of [1] and all definitions and preliminaries are taken from the same source.

2. More results on separated pairs of words

Throughout this section, let $u, v \in A^*$ be such that $u \neq v$, $|u| = |v|$ and both the pairs (u, v) and (v, u) are separated. According to I.3.3, these two pairs are strongly separated (clearly, $u \neq \varepsilon \neq v$).

Lemma 2.1. *$uvx = xuv$ iff $x = (uv)^m$ for some $m \geq 0$.*

Proof. We will proceed by induction on $|x|$. If $x = \varepsilon$, then $m = 0$. If $|x| < |u|$, then $u = xr$, $v = sx$, and so $x = \varepsilon$ and $m = 0$ again. Finally, if $|u| \leq |x|$, then $up = x = qv$, $uvqv = uvx = xuv = upuv$, $vq = pu$, $p = vt$, $q = tu$ and $uvt = up = x = qv = tuv$. If $|t| = |x|$, then $u = \varepsilon = v$, a contradiction. Thus $|t| < |x|$, $t = (uv)^{m_1}$ by induction and $x = uvt = (uv)^m$, $m = m_1 + 1$. □

Lemma 2.2. *If $pux = xvq$ and $|x| \leq |pu|$, then just one of the following two cases takes place:*

(1) *$p = vt$, $q = tu$ and $x = vtu$ (then $|x| = |pu| = |vq|$);*
(2) *$p = xvt$ and $q = tux$ (then $|x| < |p| = |q|$).*

Proof. We have $pu = xz$ and $vq = zx$. If $|z| \leq |u|$, then $u = u_1z$, $v = zv_1$, and hence $z = \varepsilon$. Consequently, $pu = x = vq$ and it follows that $p = vt$, $q = tu$ and $x = vtu$, so that (1) is true. On the other hand, if $|u| < |z|$, then $u_2u = z = vv_2$, $u_2 = vt$, $v_2 = tu$ and $z = vtu$. From this, $pu = xz = xvtu$, $p = xvt$, $vq = zx = vtux$, $q = tux$ and $|x| < |p|$. □

Lemma 2.3. *$pux = xvq$ iff $p = yvt$, $q = tuy$ and $x = (yvtu)^m y$ $(= y(vtuy)^m)$, $m \geq 0$.*

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Proof. Only the direct implication needs a proof and we will proceed by induction on $|x|$.

If $|x| \leq |pu|$, then either 2.2 (1) is true and we put $y = \varepsilon$, $m = 1$, or 2.2 (2) is true and we put $y = x$, $m = 0$.

If $|pu| < |x|$, then $pux_1 = x = x_1vq$, $1 \leq |x_1| < |x|$, and we use induction hypothesis. □

Lemma 2.4. *$puyv = uyvq$ iff at least one (and then just one) of the following two cases takes place:*

(1) $p = \varepsilon = q$;
(2) $p = uzvt$, $q = tuzv$ *and* $y = (zvtu)^m z$, $m \geq 0$.

Proof. Again, only the direct implication needs a proof.

If $|p| < |u|$, then $u = pr$, $v = sq$, $ryv = uys$ and, by I.3.7, $r = uu_1$, $s = v_1v$. Now, $u = puu_1$, $v = v_1vq$ and $p = \varepsilon = q$.

If $|u| \leq |p|$, then $p = uu_2$, $q = v_2v$ and $yvv_2 = u_2uy$. It remains to use 2.3 □

Lemma 2.5. *Let $p, q, x, y \in A^*$ be such that $|x| \leq |p|$. Then $puyvx = xuyvq$ iff at least one (and then just one) of the following two cases takes place:*

(1) $p = x = q$;
(2) $p = xuzvt$ *and* $q = tuzvx$ *and* $y = (zvtu)^m z$, $m \geq 0$.

Proof. As usual, only the direct implication needs a proof. We have $p = xp_1$, $q = q_1x$, $|p_1| = |q_1|$ and $p_1uyv = uyvq_1$. The rest follows from 2.4. □

Lemma 2.6. *Let $p, q, x, y \in A^*$ be such that $|p| < |x|$. Then $puyvx = xuyvq$ iff $x = puzvt = tuzvq$ and $y = (zvtu)^m z$, $m \geq 0$.*

Proof. Standard (use 2.4). □

3. Auxiliary results (a)

Throughout this section, let Z be a strongly separating set of words, $Z \neq \{\varepsilon\}$, and let $p, q, r, s, t, w, z \in A^*$ be such that $ptq = w = rzs$, $z \in Z$ and p, q are (Z-) reduced.

Lemma 3.1. *Just one of the following nine cases takes place:*

(a1) $r = pg$, $t = gh$, $q = ks$, $z = hk$, $g \neq \varepsilon \neq h$, $k \neq \varepsilon$ *and* h, k, s *are reduced;*
(a2) $r = pg$, $t = gz$, $q = s$, $g \neq \varepsilon$ *and* s *is reduced;*
(a3) $r = pg$, $t = gzh$, $s = hq$, $g \neq \varepsilon \neq h$;
(a4) $r = p$, $z = th$, $q = hs$, $h \neq \varepsilon$ *and* h, s, r, t *are reduced;*
(a5) $r = p$, $z = t$, $s = q$ *and* r, s *are reduced;*
(a6) $r = p$, $t = zh$, $s = hq$, $h \neq \varepsilon$ *and* r *is reduced;*
(a7) $p = rg$, $z = gh$, $t = hf$, $s = fq$, $g \neq \varepsilon \neq f$, $h \neq \varepsilon$ *and* r, g, h *are reduced;*

(a8) *$p = rg$, $z = gt$, $q = s$, $g \neq \varepsilon \neq t$ and r, g, t, s are reduced;*
(a9) *$p = rg$, $z = gh = gtf$, $h = tf$, $q = fs$, $g \neq \varepsilon \neq f$ and r, g, h, t, f, s are reduced;*

Proof. It will be divided into three parts:

(i) Let $|p| < |r|$. Then $r = pg$, $g \neq \varepsilon$, $ptq = pgzs$ and $tq = gzs$. Since q is reduced, we have $|g| < |t|$, $t = gh$, $h \neq \varepsilon$, $ghq = gzs$, $hq = zs$ and $pt = pgh = rh$.

If $|h| < |z|$, then $z = hk$, $k \neq \varepsilon$, $hq = zs = hks$, $q = ks$ and (a1) is fulfilled.

If $|h| = |z|$, then $h = z$, $q = s$, $t = gz$ and (a2) is satisfied.

If $|h| > |z|$, then $h = zh_1$, $h_1 \neq \varepsilon$, $h_1 q = s$, $t = gzh_1$ and (a3) is true.

(ii) Let $|p| = |r|$. Then $p = r$ and $tq = zs$.

If $|t| < |z|$, then $z = th$, $h \neq \varepsilon$, $tq = zs = ths$, $q = hs$ and (a4) is valid.

If $|t| = |z|$, then $z = t$, $q = s$ and (a5) holds.

If $|t| > |z|$, then $t = zh$, $h \neq \varepsilon$, $zhq = tq = zs$, $hq = s$ and (a6) follows.

(iii) Let $|p| > |r|$. Then $p = rg$, $g \neq \varepsilon$, $rgtq = ptq = rzs$ and $gtq = zs$. Since g is reduced, we have $|g| < |z|$, $z = gh$, $h \neq \varepsilon$. Moreover, $gtq = zs = ghs$ and $tq = hs$.

If $|h| < |t|$, then $t = hf$, $f \neq \varepsilon$, $hfq = tq = hs$, $fq = s$ and (a7) is clear.

If $|h| = |t|$, then $t = h$, $q = s$, $z = gt$ and (a8) is evident.

If $|h| > |t|$, then $h = tf$, $f \neq \varepsilon$, $tfs = tq = hs$, $q = fs$ and (a9) is visible.

□

Lemma 3.2. *Assume that (a1) is true. Then:*

(i) *$w = pgzs = pghks$, $z = hk$, $t = gh$, $q = ks$, $g \neq \varepsilon \neq h$, $k \neq \varepsilon$, $|z| \geq 2$, $|t| \geq 2$, h, k, s, p, ks are reduced and the pair (t, z) is not separated.*
(ii) *If pg is reduced, then* $\mathrm{tr}(w) = 1$.
(iii) *If t is reduced, then g is reduced.*
(iv) *If g is reduced and pg is not reduced, then $p = p_1 u$, $g = vq_1$, $t = vq_1 h$, $w = p_1 uvq_1 zs$, $u \neq \varepsilon \neq v$, $uv \in Z$, p_1, q_1, u, v are reduced and* $\mathrm{tr}(w) = 2$.

Proof.

(i) The assertion follows easily from (a1).
(ii) Combine (i) and I.5.4.
(iii) Obvious from $t = gh$.
(iv) Since p, g are reduced and pg is not, we have $pg = p_1 z_1 q_1$, $p = p_1 u$, $g = vq_1$, $z_1 = uv \in Z$, $u \neq \varepsilon \neq v$, p_1, q_1 reduced and $|z_1| \geq 2$. Thus $w = p_1 uvq_1 zs$ and $\mathrm{tr}(w) = 2$ by I.5.4.

□

Lemma 3.3. *Assume that (a2) is true. Then:*

(i) $w = pgzs$, $t = gz$, $q = s$, $g \neq \varepsilon$, $|t| \geq 2$, s *is reduced and* t *is not reduced.*
(ii) *If* pg *is reduced, then* $\mathrm{tr}(w) = 1$.
(iii) *If* g *is reduced and* pg *is not reduced, then* $p = p_1u$, $g = vq_1$, $t = vq_1z$, $w = p_1uvq_1zs$, $u \neq \varepsilon \neq v$, $uv \in Z$, p_1, q_1, u, v *are reduced and* $\mathrm{tr}(w) = 2$.

Proof. We can proceed similarly as in the proof of 3.2. □

Lemma 3.4. *Assume that (a3) is true. Then:*

(i) $w = pgzs = pgzhq$, $t = gzh$, $s = hq$, $g \neq \varepsilon \neq h$, $|t| \geq 3$ *and* t *is not reduced.*
(ii) *If* pg *and* s *are reduced, then* $\mathrm{tr}(w) = 1$.

Proof. Similar to the proof of 3.2. □

Lemma 3.5. *Assume that (a4) is true. Then:*

(i) $w = pzs = pths$, $z = th$, $q = hs$, $t \neq \varepsilon \neq h$, $|z| \geq 2$ *and* h, s, t, hs *are reduced.*
(ii) $\mathrm{tr}(w) = 1$.

Proof. Easy. □

Lemma 3.6. *Assume that (a5) is true. Then:*

(i) $w = pzs = pts$, $z = t$, $q = s$, s *is reduced and* t *is not reduced.*
(ii) $\mathrm{tr}(w) = 1$.

Proof. Easy. □

Lemma 3.7. *Assume that (a6) is true. Then:*

(i) $w = pzhq$, $t = zh$, $s = hq$, $h \neq \varepsilon$, $|t| \geq 2$ *and* t *is not reduced.*
(ii) *If* hq *is reduced, then* $\mathrm{tr}(w) = 1$.
(iii) *If* h *is reduced and* hq *is not reduced, then* $w = pzp_1uvq_1$, $h = p_1u$, $q = vq_1$, $t = zp_1u$, $u \neq \varepsilon \neq v$, $uv \in Z$, p_1, q_1, u, v *are reduced and* $\mathrm{tr}(w) = 2$.

Proof. Similar to the proof of 3.2. □

Lemma 3.8. *Assume that (a7) is true. Then:*

(i) $w = rzfq = rghfq$, $z = gh$, $t = hf$, $s = fq$, $g \neq \varepsilon \neq f$, $h \neq \varepsilon$, $|z| \geq 2$, $|t| \geq 2$, h, g, r, rg *are reduced and the pair* (z, t) *is not separated.*
(ii) *If* fq *is reduced, then* $\mathrm{tr}(w) = 1$.
(iii) *If* t *is reduced, then* f *is reduced.*
(iv) *If* f *is reduced and* fq *is not reduced, then* $f = p_1u$, $q = vq_1$, $t = hp_1u$, $w = rzp_1uvq_1$, $u \neq \varepsilon \neq v$, $uv \in Z$, p_1, q_1, u, v *are reduced and* $\mathrm{tr}(w) = 2$.

Proof. Similar to the proof of 3.2. □

Lemma 3.9. *Assume that (a8) is true. Then:*

(i) $w = rgts$, $z = gt$, $q = s$, $g \neq \varepsilon \neq t$, $|z| \geq 2$ *and* r, g, t, s, rg *are reduced.*
(ii) $\mathrm{tr}(w) = 1$.

Proof. Easy. □

Lemma 3.10. *Assume that (a9) is true. Then:*

(i) $w = rgtfs$, $z = gtf$, $q = fs$, $g \neq \varepsilon \neq f$, $|z| \geq 2$ *and* r, g, t, f, s, tf, rg, fs *are reduced.*
(ii) $\mathrm{tr}(w) = 1$.

Proof. Easy. □

Lemma 3.11. *If* $\mathrm{tr}(w) \geq 2$, *then just one of the five conditions (a1), (a2), (a3), (a6) and (a7) holds.*

Proof. Combine the preceding lemmas of this section. □

Lemma 3.12.

(i) *If at least one of (a2), (a3), (a5) and (a6) holds, then* t *is not reduced.*
(ii) *If* t *is reduced, then just one of (a1), (a4), (a7), (a8), (a9) holds.*
(iii) *If* t *is reduced and* $\mathrm{tr}(w) \geq 2$, *then just one of (a1), (a7) holds and* $\mathrm{tr}(w) = 2$.

Proof. Combine the preceding lemmas of this section. □

Lemma 3.13.

(i) *If* t *is reduced then* $\mathrm{tr}(w) \leq 2$.
(ii) *If* $t = \varepsilon$, *then (a9) is satisfied.*
(iii) *If* $t \in A$ *(i. e.,* $|t| = 1$*), then just one of (a4), (a5), (a8), (a9) is true (if (a5) is true, then* $z = t \in A$*) and* $\mathrm{tr}(w) = 1$.
(iv) *If* $|t| \leq 1$, *then* $\mathrm{tr}(w) = 1$.
(v) *If* $z \in A$ *(i. e.,* $|z| = 1$*), then just one of (a2), (a3), (a5), (a6) is true (if (a5) is true, then* $t = z \in A$*).*
(vi) *If* $z \in A$ *and* $\mathrm{tr}(w) \geq 2$, *then either (a2) or (a6) holds and* t *is not reduced.*

Proof. Combine the preceding lemmas of this section. □

4. Auxiliary results (b)

In this section, let Z be a strongly separating set of words, $Z \neq \{\varepsilon\}$ and let $p_1, q_1, p_2, q_2, t_1, t_2, w_1, w_2 \in A^*$ and $z_1, z_2 \in Z$ be such that $p_1 z_1 q_1 = w_1 = p_2 t_2 q_2$, $p_1 t_1 q_1 = w_2 = p_2 z_2 q_2$ and p_1, q_1 are (Z-) reduced.

Lemma 4.1. *Assume that* $|p_1| = |p_2|$. *Then* $p_1 = p_2$, $z_1q_1 = t_2q_2$ *and* $t_1q_1 = z_2q_2$. *Moreover:*

(i) *If* $|t_2| < |z_1|$, *then* $z_1 = t_2r_1$, $t_1 = z_2r_1$, $q_2 = r_1q_1$, $r_1 \neq \varepsilon$, $|t_1| \geq 2$ *and* t_1 *is not reduced.*

(ii) *If* $|t_2| = |z_1|$, *then* $z_1 = t_2$, $t_1 = z_2$ *and* $q_1 = q_2$.

(iii) *If* $|t_2| > |z_1|$, *then* $t_2 = z_1s_1$, $z_2 = t_1s_1$, $q_1 = s_1q_2$, $s_1 \neq \varepsilon$, $|t_2| \geq 2$ *and* t_2 *is not reduced.*

Proof. Easy. □

Lemma 4.2. *Assume that* $|p_1| < |p_2|$. *Then* $p_2 = p_1u_1$, $z_1q_1 = u_1t_2q_2$, $t_1q_1 = u_1z_2q_2$, $u_1 \neq \varepsilon$, $|u_1| < |t_1|$, $t_1 = u_1u_2$, $u_2q_1 = z_2q_2$, $u_2 \neq \varepsilon$, $|t_1| \geq 2$. *Moreover:*

(i) *If* $|q_1| \leq |q_2|$, *then* $q_2 = r_2q_1$, $u_2 = z_2r_2$, $t_1 = u_1z_2r_2$ *and* t_1 *is not reduced.*

(ii) *If* $|q_1| > |q_2|$, *then* $q_1 = v_1q_2$, $t_1v_1 = u_1z_2$, $z_1v_1 = u_1t_2$, $z_2 = u_2v_1$, $v_1 \neq \varepsilon$ *and* u_2, v_1 *are reduced.*

(iii) *If* $|q_1| > |q_2|$ *and* $|z_1| \leq |u_1|$, *then* $u_1 = z_1s_2$, $v_1 = s_2t_2$, $t_1 = z_1s_2u_2$, $z_2 = u_2s_2t_2$ *and neither* u_1 *nor* p_2 *nor* t_1 *is reduced.*

(iv) *If* $|q_1| > |q_2|$ *and* $|z_1| > |u_1|$, *then* $z_1 = u_1v_2$, $t_2 = v_2v_1$, $v_2 \neq \varepsilon$ *and* v_2 *is reduced.*

Proof. Easy. □

Lemma 4.3. *Assume that* $|p_1| > |p_2|$. *Then* $p_1 = p_2u_3$, $t_2q_2 = u_3z_1q_1$, $z_2q_2 = u_3t_1q_1$, $u_3 \neq \varepsilon$ *and* p_2, u_3 *are reduced. Moreover:*

(i) *If* $|t_2| \leq |u_3|$, *then* $q_2 = r_3z_1q_1$, $u_3 = t_2r_3$, $p_1 = p_2t_2r_3$, $t_2r_3t_1 = z_2r_3z_1$ *and* t_2, r_3 *are reduced. Further,* $|t_2| < |z_2|$, $z_2 = t_2s_3$, $s_3 \neq \varepsilon$, $r_3t_1 = s_3r_3z_1$, $|z_1| < |t_1|$, $t_1 = kz_1$, $r_3k = s_3r_3$, $k \neq \varepsilon$, $|t_1| \geq 2$ *and* t_1 *is not reduced.*

(ii) *If* $|t_2| > |u_3|$, *then* $t_2 = u_3u_4$, $z_1q_1 = u_4q_2$, $u_4 \neq \varepsilon$ *and* $|t_2| \geq 2$.

(iii) *If* $|t_2| > |u_3|$ *and* $|q_2| \leq |q_1|$, *then neither* u_4 *nor* t_2 *is reduced.*

(iv) *If* $|t_2| > |u_3|$ *and* $|q_2| > |q_1|$, *then* $q_2 = v_3q_1$, $z_1 = u_4v_3$, $u_3t_1 = z_2v_3$, $v_3 \neq \varepsilon$, v_3, u_4 *are reduced,* $|u_3| < |z_2|$, $z_2 = u_3v_4$, $t_1 = v_4v_3$, $v_4 \neq \varepsilon$ *and* v_4 *is reduced.*

Proof. Easy. □

Lemma 4.4. *Assume that either* $|t_1| \leq 1$ *or* t_1 *is reduced and the same is true for* t_2. *Then at least one of the following three cases takes place:*

(i) $z_1 = t_2$, $z_2 = t_1$, $p_1 = p_2$ *and* $q_1 = q_2$.

(ii) $z_1 = u_1v_2$, $z_2 = u_2v_1$, $t_1 = u_1u_2$, $t_2 = v_2v_1$, $p_2 = p_1u_1$, $q_1 = v_1q_2$, $u_1, u_2, v_1, v_2 \in A^+$ *and all* u_1, u_2, v_1, v_2 *are reduced.*

(iii) $z_1 = u_4v_3$, $z_2 = u_3v_4$, $t_1 = v_4v_3$, $t_2 = u_3u_4$, $p_1 = p_2u_3$, $q_2 = v_3q_1$, $u_3, u_4, v_3, v_4 \in A^+$ *and all* u_3, u_4, v_3, v_4 *are reduced.*

Proof. It follows from 4.1, 4.2 and 4.3 that only the cases 4.1 (ii), 4.2 (iv) and 4.3 (iv) come into account. □

5. Disturbing pairs

Let Z be a strongly separating set of words, $Z \neq \{\varepsilon\}$, and let $\psi : Z \to A^*$ be a mapping. Consider the relations σ, ρ, λ, τ, ξ, ν and μ defined in I.6 and I.7.

An ordered pair $(z_1, z_2) \in Z \times Z$ will be called *disturbing* if there exist words $u, v, r, s \in A^+$ such that $z_1 = ur$, $z_2 = sv$, $\psi(z_1) = us$ and $\psi(z_2) = rv$.

An ordered pair $(z_1, z_2) \in Z \times Z$ will be called *paradisturbing* if $\psi(z_1) = z_2$ and $\psi(z_2) = z_1$.

Lemma 5.1. *Let $(z_1, z_2) \in Z \times Z$ be a disturbing pair, $z_1 = ur$, $z_2 = sv$, $\psi(z_1) = us, \psi(z_2) = rv$, $u, v, r, s \in A^+$. Put $w_1 = urv$ and $w_2 = usv$. Then:*

(i) $|z_1| \geq 2$, $|z_2| \geq 2$, $|\psi(z_1)| \geq 2$, $|\psi(z_2)| \geq 2$.
(ii) *The words u, v, r and s are reduced.*
(iii) $(w_1, w_2) \in \nu$.
(iv) $\mathrm{tr}(w_1) = 1 = \mathrm{tr}(w_2)$.
(v) *Both w_1 and w_2 are pseudoreduced.*
(vi) $w_1 = w_2$ *iff* $r = s$.
(vii) *If $w_1 = w_2$, then w_1 is strongly pseudoreduced.*

Proof. Easy. □

Lemma 5.2. *Let $(z_1, z_2) \in Z \times Z$ be a paradisturbing pair. Then:*

(i) $(z_1, z_2) \in \nu$.
(ii) $\mathrm{tr}(z_1) = 1 = \mathrm{tr}(z_2)$.
(iii) *Both z_1 and z_2 are weakly pseudoreduced.*

Proof. Obvious. □

Proposition 5.3. *There exist no disturbing pairs, provided that either $Z \subseteq A$ or $\psi(Z) \subseteq A$.*

Proof. Obvious. □

Proposition 5.4. *Suppose that for every $z \in Z$, either $|\psi(z)| \leq 1$ or $\psi(z)$ is reduced. Then the following conditions are equivalent:*

(i) *There exist no disturbing and no paradisturbing pairs in $Z \times Z$.*
(ii) *Every pseudoreduced meagre word is reduced.*

Proof.

(i) implies (ii). Let, on the contrary w_1 be a weakly pseudoreduced with $\mathrm{tr}(w_1) = 1$. Then $w_1 = p_1 z_1 q_1$, where $z_1 \in Z$ and p_1, q_1 are reduced (use I.6.6). If $w_2 = p_1 t_1 q_1$, $t_1 = \psi(z_1)$, then $(w_1, w_2) \in \rho$, and hence $(w_2, w_1) \in \rho$, since w_1 is weakly pseudoreduced. Consequently, $w_2 = p_2 z_2 q_2$, $z_2 \in Z$, and $w_1 = p_2 t_2 q_2$, $t_2 = \psi(z_2)$. Now, 4.4 applies. If 4.4 (i) is true, then (z_1, z_2) is paradisturbing. If 4.4 (ii) is true, then (z_1, z_2) is disturbing. Finally, if 4.4 (iii) is true, then (z_2, z_1) is disturbing.

(ii) implies (i). See 5.1 and 5.2. □

6. Meagre and pseudomeagre words

Let Z be a strongly separating set of words such that $Z \neq \{\varepsilon\}$ (except for 6.9) and let $\psi : Z \to A^*$ be a mapping. Consider the relations σ, ρ, λ, τ, ξ, ν and μ defined in I.6 and I.7.

A word w is called *meagre* if $\mathrm{tr}(w) \leq 1$.

A word w is called *pseudomeagre* if $(w, x) \in \rho$ for at most one $x \in A^*$.

Lemma 6.1. *Every meagre word is pseudomeagre.*

Proof. Obvious. □

Lemma 6.2. *Let $z \in Z$ be such that $\psi(z) \in \{\varepsilon, z\}$. Then the word z^n, $n \geq 2$, is pseudomeagre but not meagre.*

Proof. It follows from I.6.6 that $\mathrm{tr}(z^n) = n \geq 2$, and so z^n is not meagre. On the other hand, if $(z^n, x) \in \rho$, then $x = z^{n-1}$ for $\psi(z) = \varepsilon$ and $x = z^n$ for $\psi(z) = z$. □

Lemma 6.3. *Let $z_1, z_2, z \in Z$ and $u, v, x \in A^*$ be such that $z_1 x z_2 = uzv$.*

(i) *If $u = \varepsilon$, then $z = z_1$ and $v = xz_2$.*
(ii) *If $v = \varepsilon$, then $z = z_2$ and $u = z_1 x$.*
(iii) *If $u \neq \varepsilon \neq v$, then $u = z_1 u_1$, $v = v_1 z_2$ and $x = u_1 z v_1$.*

Proof.

(i) Easy to see.
(ii) Easy to see.
(iii) If $|u| < |z_1|$, then $z_1 = uy$, $y \neq \varepsilon$, $uyxz_2 = z_1 x z_2 = uzv$, $yxz_2 = zv$, a contradiction. Thus $|u| \geq |z_1|$ and, similarly, $|v| \geq |z_2|$. The rest is clear.

□

Lemma 6.4. *Let $z \in Z$ and $x \in A^*$ be such that $\psi(z) = zxz$. Then:*

(i) *$\mathrm{tr}(zxz) \geq 2$ and zxz is not meagre.*
(ii) *zxz is pseudomeagre iff $\psi(z_1) = z_1 v z u z_1$ whenever $z_1 \in Z$ and $x = u z_1 v$ (or $\psi(z) = z u z_1 v z$).*

Proof.

(i) Obvious.
(ii) Clearly, $(\varepsilon, z, xz), (zx, z, \varepsilon) \in \mathrm{Tr}(zxz)$, $\varepsilon\psi(z)xz = zxzxz = zx\psi(z)\varepsilon$ and $(zxz, zxzxz) \in \rho$. If x is reduced, then $\mathrm{tr}(zxz) = 2$ by I.6.6, and hence zxz is pseudomeagre (and the other condition is satisfied trivially).

Now, let $(u_1, z_1, v_1) \in \mathrm{Tr}(zxz)$, $u_1 \neq \varepsilon \neq v_1$. According to 6.3, $u_1 = zu$, $v_1 = vz$ and $x = uz_1 v$. We have $zxz = zuz_1vz$ and $(zxz, zu\psi(z_1)vz) \in \rho$. Consequently, $zu\psi(z_1)vz = zxzxz$

iff $u\psi(z_1)v = xzx = uz_1vzuz_1v$ and iff $\psi(z_1) = z_1vzuz_1$. The rest is clear.

□

Lemma 6.5. *Let $z_1, z_2 \in Z$ and $x, y \in A^*$ be such that $\psi(z_1) = yxz_1$ and $\psi(z_2) = z_2xy$. Then:*

(i) $\mathrm{tr}(z_2xz_1) \geq 2$ *and z_2xz_1 is not meagre.*
(ii) *z_2xz_1 is pseudomeagre iff $\psi(z_3) = z_3vyuz_3$ whenever $z_3 \in Z$ and $x = uz_3v$ (or $\psi(z_1) = yuz_3vz_1$ or $\psi(z_2) = z_2uz_3vy$).*

Proof.

(i) Obvious.
(ii) Clearly, $(\varepsilon, z_2, xz_1), (z_2x, z_1, \varepsilon) \in \mathrm{Tr}(z_2xz_1)$, $\varepsilon\psi(z_2)xz_1 = z_2xyxz_1 = z_2x\psi(z_1)\varepsilon$ and $(z_2xz_1, z_2xyxz_1) \in \rho$. If x is reduced, then $\mathrm{tr}(z_2xz_1) = 2$ by I.6.6, and hence z_2xz_1 is pseudomeagre (and the other condition is satisfied trivially).

Now, let $(u_1, z_3, v_1) \in \mathrm{Tr}(z_2xz_1)$, $u_1 \neq \varepsilon \neq v_1$. According to 6.3, $u_1 = z_2u$, $v_1 = vz_1$ and $x = uz_3v$. We have $z_2xz_1 = z_2uz_3vz_1$ and $(z_2xz_1, z_2u\psi(z_3)vz_1) \in \rho$. Consequently, $z_2u\psi(z_3)vz_1 = z_2xyxz_1$ iff $u\psi(z_3)v = xyx = uz_3vyuz_3v$ and iff $\psi(z_3) = z_3vyuz_3$. The rest is clear.

□

Proposition 6.6. *Suppose that every pseudomeagre word is meagre. Then the following three conditions are satisfied:*

(b1) *$\varepsilon \neq \psi(z) \neq z$ for every $z \in Z$;*
(b2) *If $z_1, z_2 \in Z$ and $x, y \in A^*$ are such that $\psi(z_1) = yxz_1$ and $\psi(z_2) = z_2xy$, then $x \neq \varepsilon \neq y$ and x is not reduced;*
(b3) *If $z_1, z_2, z_3 \in Z$ and $u, v, y \in A^*$, then either $\psi(z_1) \neq yuz_3vz_1$ or $\psi(z_2 \neq z_2uz_3vy$ or $\psi(z_3) \neq z_3vyuz_3$*

Proof. The condition (b1) follows from 6.2. Further, if $\psi(z_1) = yxz_1$ and $\psi(z_2) = z_2xy$, then x is not reduced due to 6.5, and hence $x \neq \varepsilon$. Moreover, if $y = \varepsilon$, then z_2z_1 is pseudomeagre, but not meagre, and therefore $x \neq \varepsilon \neq y$ and we have shown (b2). Finally, (b3) follows from 6.5. □

Proposition 6.7. *Suppose that the following two conditions are satisfied:*

(c1) *$\varepsilon \neq \psi(z) \neq z$ and $\psi(z) \neq zxz$ for all $z \in Z$ and $x \in A^*$;*
(c2) *If $z_1, z_2 \in Z$ and $x, y \in A^*$ are such that $\psi(z_1) \neq \psi(z_2)$, then either $\psi(z_1) \neq yxz_1$ or $\psi(z_2) \neq z_2xy$.*

Then every pseudomeagre word is meagre.

Proof. Let, on the contrary, w be pseudomeagre word, but not meagre. Then $\mathrm{tr}(w) \geq 2$, and therefore $pz_1q = w = rz_2s$, where $(p, z_1, q) \neq$

(r, z_2, s) and $z_1, z_2 \in Z$; we will assume $|rz_2| \le |pz_1|$, the other case being similar.

Assume, for a moment, that $z_1 = z = z_2$. Then $|r| < |p|$ and we get a contradiction by easy combination of (c1) and 3.11. Consequently, $z_1 \ne z_2$ and it follows easily that $|r| < |p|$. Then $\psi(z_1) \ne \psi(z_2)$ and we get a contradiction with (c2). □

Proposition 6.8.

(i) *Suppose that $\psi(z) \ne \varepsilon$ and that z is neither a prefix nor a suffix of $\psi(z)$ for every $z \in Z$. Then every pseudomeagre word is meagre.*

(ii) *Suppose that $|\psi(z)| \le |z|$ for every $z \in Z$. Then every pseudomeagre word is meagre if and only if $\varepsilon \ne \psi(z) \ne z$ for every $z \in Z$.*

Proof. See 6.6 and 6.7 □

Remark 6.9. Let $Z = \{\varepsilon\}$. Then ε is the only meagre word. Moreover:

(i) If $\psi(\varepsilon) = \varepsilon$, then all words are pseudomeagre (and hence there exist pseudomeagre words that are not meagre).

(ii) If $\psi(\varepsilon) = t$ and $|\operatorname{var}(t)| = 1$, $t = a^m$, $a \in A$, $m \ge 1$, then a word w is pseudomeagre iff $w = a^n$, $n \ge 0$. Consequently, there exist pseudomeagre words that are not meagre.

(iii) If $\psi(\varepsilon) = t$ and $|\operatorname{var}(t)| \ge 2$, then ε is the only pseudomeagre word (and hence all pseudomeagre words are meagre).

7. Disturbing triples

This section is an immediate continuation of the preceding one.

An ordered triple $(z_1, z_2, z_3) \in Z \times Z \times Z$ will be called *disturbing* if there exist $u, v, g, h \in A^+$ and $p \in A^*$ such that $z_1 = uv$, $z_3 = gh$ and $\psi(z_2) = vpg$.

Lemma 7.1. *Let $(z_1, z_2, z_3) \in Z \times Z \times Z$ be a disturbing triple, $z_1 = uv$, $z_3 = gh$, $\psi(z_2) = vpg$, $u, v, g, h \in A^+$, $p \in A^*$. Then:*

(i) *$|z_1| \ge 2$, $|z_3| \ge 2$ and $|\psi(z_2)| \ge 2$.*

(ii) *The words u, v, g, h are reduced.*

(iii) *$(u_1, v_1) \in \rho$, $\operatorname{tr}(u_1) = 1$ and $\operatorname{tr}(v_1) \ge 2$, where $u_1 = uz_2h$ and $v_1 = uvpgh$.*

Proof. Easy (use I.6.6). □

Proposition 7.2. *There exist no disturbing triples, provided that either $Z \subseteq A$ or $\psi(Z) \subseteq A$.*

Proof. Obvious. □

Proposition 7.3. *Suppose that for every $z \in Z$, either $|\psi(z)| \le 1$ or $\psi(z)$ is reduced. Then the following conditions are equivalent:*

(i) *There exist no disturbing triples in* $Z \times Z \times Z$.
(ii) *If* $(w_1, w_2) \in \rho$ *and* $\mathrm{tr}(w_1) = 1$, *then* $\mathrm{tr}(w_2) \le 1$.
(iii) *If* $(w_1, w_2) \in \rho$ *and* w_1 *is meagre, then* w_2 *is meagre.*
(iv) *If* $(w_1, w_2) \in \tau$ *and* $\mathrm{tr}(w_1) = 1$, *then* $\mathrm{tr}(w_2) \le 1$.
(v) *If* $(w_1, w_2) \in \xi$ *and* w_1 *is meagre, then* w_2 *is meagre.*

Proof.
(i) implies (ii). We have $w_1 = pz_2q$, $z_2 \in Z$, p, q reduced, and $w_2 = ptq$, $t = \psi(z_2)$. Now, assume that $w_2 = rz_3s$ and 3.1 applies. If $|t| \le 1$, then $\mathrm{tr}(w_2) = 1$ by 3.13 (iv), and therefore we will assume that $|t| \ge 2$. Then t is reduced and, according to 3.12 (iii) we can assume that (a1) holds, the case (a7) being similar.

By 3.2 $w_2 = pghks$, $z_3 = hk$, $t = gh$, $q = ks$, $g \ne \varepsilon \ne h$, $k \ne \varepsilon$ and, moreover, g is reduced, since t is so. If pg is reduced, then $\mathrm{tr}(w_2) = 1$ by 3.2 (ii). If pg is not reduced, then, by 3.2 (iv), $pg = p_1z_1q_1$, $z_1 = uv$, $p = p_1u$, $g = vq_1$, $t = vq_1h$, $u \ne \varepsilon \ne v$ and the triple (z_1, z_2, z_3) is disturbing.

(ii) implies (iii), (iii) implies (iv), (iv) implies (v). Obvious.

(v) implies (i). See 7.1 (iii). □

8. On when the relation ρ is antisymmetric

As usual, let Z be a strongly separating set of words such that $Z \ne \{\varepsilon\}$ (except for 8.7, 9.11) and let $\psi : Z \to A^*$ be a mapping.

Proposition 8.1. *The relation* ρ $(= \rho_{Z,\psi})$ *is irreflexive if and only if* $\psi(z) \ne z$ *for every* $z \in Z$.

Proof. Obvious from the definition of ρ. □

Proposition 8.2. *The relation* ρ *is antisymmetric (i. e.,* $u = v$, *whenever* $(u, v) \in \rho$ *and* $(v, u) \in \rho$*) if and only if the following three conditions hold:*

(1) *If* $z_1, z_2 \in Z$ *and* $x, y \in A^*$ *are such that* $z_2 = x\psi(z_1)y$ *and* $\psi(z_2) = xz_1y$, *then* $\psi(z_2) = z_2$ *(and hence* $\psi(z_1) = z_1$ *as well);*
(2) *If* $z_1, z_2 \in Z$ *and* $x, y \in A^*$ *are such that* $z_2 = yx\psi(z_2)$ $(z_2 = \psi(z_2)xy$, *resp.) and* $\psi(z_1) = z_1xy$ $(\psi(z_1) = yxz_1$, *resp.), then* $x = \varepsilon = y$ *(and hence* $\psi(z_1) = z_1$, $\psi(z_2) = z_2$*);*
(3) *If* $z_1, z_2 \in Z$ *and* $x, y, u, v \in A^+$ *are such that* $z_1 = uy$, $z_2 = xv$, $\psi(z_1) = vy$ *and* $\psi(z_2) = xu$, *then* $u = v$ *(and hence* $\psi(z_1) = z_1$, $\psi(z_2) = z_2$*).*

Proof. Use I.5.4. □

Corollary 8.3. *Assume that for every* $z \in Z$, *either* $|\psi(z)| \le 1$ *or* $\psi(z)$ *is reduced. Then:*

(i) *The relation* ρ *is antisymetric if and only the following two conditions hold:*

(i1) *If* $(z_1, z_2) \in (Z \times Z) \cap (A \times A)$ *is a paradisturbing pair, then* $z_1 = z_2$*;*
(i2) *There exist no disturbing pairs in* $Z \times Z$*.*
(ii) *The relation* ρ *is both irreflexive and antisymmetric if and only if there exist no disturbing nor paradisturbing pairs in* $Z \times Z$*.*

Proposition 8.4. *The following conditions are equivalent:*
(i) *If* $(u, v) \in \rho$ *and* $(v, v) \in \rho$*, then* $u = v$*.*
(ii) *If* $(u, v) \in \rho$ *and* $(u, u) \in \rho$*, then* $u = v$*.*
(iii) *Either* $\psi(z) \neq z$ *for every* $z \in Z$ *or* $\psi(z) = z$ *for every* $z \in Z$*.*

Proof. Easy to check. □

Proposition 8.5. *Assume that* $|z_1| - |\psi(z_1)| \neq |\psi(z_2)| - |z_2|$ *for all* $z_1, z_2 \in Z$*. Then the relation* ρ *is both irreflexive and antisymmetric (i. e., it is strictly antisymmetric).*

Proof. Use I.5.4. □

Proposition 8.6. *The relation* ρ *is weakly antisymmetric (i. e.,* $u = v$*, whenever* $(u, v) \in \rho$*,* $(v, u) \in \rho$*,* $(u, u) \in \rho$*) if and only if* $\psi(z_1) = z_1$*, whenever* $z_1, z_2, z_3 \in Z$ *and* $p, q, r, s, x, y \in A^*$ *are such that* $pz_1q = rz_2s = x\psi(z_3)y$ *and* $p\psi(z_1)q = xz_3y$*.*

Proof. Obvious. □

Remark 8.7. Let $Z = \{\varepsilon\}$. If $\psi(\varepsilon) = \varepsilon$, then $\rho = \mathrm{id}_{A^*}$, and hence ρ is antisymmetric, but not irreflexive. If $\psi(\varepsilon) \neq \varepsilon$, then ρ is both irreflexive and antisymmetric. Moreover, 8.4 is true in both cases.

9. On when the relation ρ is antitransitive

This section is an immediate continuation of preceding one.

Proposition 9.1. *The relation* ρ *is weakly antitransitive (i. e.,* $(w, v) \notin \rho$*, whenever* $u, v, w \in A^*$ *are such that* $u \neq v \neq w \neq u$*,* $(w, u) \in \rho$ *and* $(u, v) \in \rho$*) if and only if the following condition is satisfied:*
(1) *If* $z_1, z_2 \in Z$ *and* $x, y, k \in A^*$ *are such that* $\psi(z_1) \neq z_1$*,* $\psi(z_2) \neq z_2$ *and* $z_1k\psi(z_2) \neq \psi(z_1)kz_2$*, then* $(u, v) \notin \rho$ *and* $(v, u) \notin \rho$*, where* $u = xz_1k\psi(z_2)y$ *and* $v = x\psi(z_1)kz_2y$

Proof. See I.7.1. □

Lemma 9.2. *Let* $z \in Z$ *and* $k \in A^*$*. Then* $zk\psi(z) \neq \psi(z)kz$ *iff* $\psi(z) \neq z$ *and either* $\psi(z) = \varepsilon$ *and* $k \neq z^n$ *for every* $n \geq 0$ *or* $\varepsilon \neq \psi(z) \neq (zu)^m z$ *for all* $u \in A^*$ *and* $m \geq 1$ *or* $\psi(z) = (zv)^t z$ *and* $k \neq (vz)^n v$ *for some* $v \in A^*$*,* $t \geq 1$ *and every* $n \geq 0$*.*

Proof. Easy. □

Lemma 9.3. *Let* $z \in Z$ *be such that* $\psi(z)$ *is reduced and let* $k \in A^*$*. Then* $zk\psi(z) \neq \psi(z)kz$ *iff either* $\psi(z) \neq \varepsilon$ *or* $\psi(z) = \varepsilon$ *and* $k \neq z^n$ *for every* $n \geq 0$*.*

Proof. This follows from 9.2. □

Lemma 9.4. *Let $z_1, z_2 \in Z$, $z_1 \neq z_2$, and $k \in A^*$. Then $z_1 k\psi(z_2) \neq \psi(z_1)kz_2$ iff at least one of the following three conditions is satisfied:*

(1) *$\psi(z_1) \neq z_1$ and $\psi(z_2) = z_2$;*
(2) *$\psi(z_2) \neq z_2$, $\psi(z_1) = z_1 uv$ for some $u, v \in A^*$ and either $\psi(z_2) \neq vuz_2$ or $\psi(z_2) = vuz_2$ and $k \neq (uv)^n u$ for every $n \geq o$;*
(3) *$\psi(z_2) \neq z_2$, $\psi(z_1) \neq z_1 xy$ for all $x, y \in A^*$.*

Proof. Easy. □

Lemma 9.5. *Let $z_1, z_2 \in Z$ be such that $z_1 \neq z_2$ and both $\psi(z_1)$, $\psi(z_2)$ are reduced. Then $z_1 k\psi(z_2) \neq \psi(z_1)kz_2$ for every $k \in A^*$.*

Proof. This follows easily from 9.4 □

Proposition 9.6. *Assume that for every $z \in Z$, either $|\psi(z)| \leq 1$ or $\psi(z)$ is reduced. Then the relation ρ is weakly antitransitive if and only if $(u, v) \notin \rho$ and $(v, u) \notin \rho$, whenever $u = xz_1 k\psi(z_2)y$, $v = x\psi(z_1)kz_2 y$ and z_1, z_2 are such that:*

(1) *If $z_1, \psi(z_1) \in A \cap Z$, then $\psi(z_1) \neq z_1$;*
(2) *If $z_2, \psi(z_2) \in A \cap Z$, then $\psi(z_2) \neq z_2$;*
(3) *If $z_1 = z_2 = z$ and $\psi(z) = \varepsilon$, then $k \neq z^n$ for every $n \geq 0$.*

Proof. Combine 9.1, 9.2 and 9.4. □

Corollary 9.7. *Assume that for every $z \in Z$, $\psi(z) \neq z$ and either $|\psi(z)| \leq 1$ or $\psi(z)$ is reduced (equivalently, either $\psi(z)$ is reduced or $\psi(z) = \varepsilon$ or $\psi(z) \in A$ and $\psi(z) \neq z$). Then the relation ρ is weakly antitransitive if and only if $(u, v) \notin \rho$ and $(v, u) \notin \rho$ (i. e., u, v are incomparable in ρ), whenever $u = xz_1 k\psi(z_2)y$, $v = x\psi(z_1)kz_2 y$ and $z_1, z_2 \in Z$ are such that either $z_1 \neq z_2$ or $z_1 = z_2$ and $\psi(z_1) \neq \varepsilon$ or $z_1 = z_2$ and $\psi(z_1) = \varepsilon$ and $k \neq z_1^n$ for every $n \geq 0$.*

Proposition 9.8. *Assume that $\psi(z_0) \neq z_0$ for at least one $z_0 \in Z$. Then the following conditions are equivalent:*

(i) *The relation ρ is irreflexive and weakly antitransitive.*
(ii) *The relation ρ is strictly antitransitive (i. e., $(w, v) \notin \rho$ whenever $(w, u) \in \rho$ and $(u, v) \in \rho$).*
(iii) *The relation ρ is antitransitive (i. e., $u = v = w$, whenever $(w, u) \in \rho$, $(u, v) \in \rho$ and $(w, v) \in \rho$).*
(iv) *The condition 9.1 (1) is satisfied and $\psi(z) \neq z$ for every $z \in Z$.*

Proof.

(i) implies (ii). Let $(w, u), (u, v), (w, v) \in \rho$. Since ρ is weakly antitransitive, either $w = u$ or $u = v$ or $w = v$. On the other hand, since ρ is irreflexive, we have $w \neq u \neq v \neq w$, a contradiction.

(ii) implies (iii). Obvious.

(iii) implies (iv). Clearly, ρ is weakly antitransitive, and hence 9.1 (1) follows from 9.1. Moreover, $\psi(z) \neq z$ follows from 8.4.

(iv) implies (i). Use 8.1 and 9.1. □

Proposition 9.9. *Assume that* $|z_1| + |z_2| - |z_3| \neq |\psi(z_1)| + |\psi(z_2)| - |\psi(z_3)|$ *for all* $z_1, z_2, z_3 \in Z$. *Then the relation* ρ *is strictly antitransitive.*

Proof. Let $(w,u),(u,v),(w,v) \in \rho$. Then $pz_1q = w = rz_3s$, $p\psi(z_1)q = u = xz_2y$, $r\psi(z_3)s = v = x\psi(z_2)y$. Consequently, $|w| - |u| = |z_1| - |\psi(z_1)|$, $|w| - |v| = |z_3| - |\psi(z_3)|$, $|u| - |v| = |z_2| - |\psi(z_2)|$. From this we get $|z_3| - |\psi(z_3)| = |w| - |v| = |w| - |u| + |u| - |v| = |z_1| - |\psi(z_1)| + |z_2| - |\psi(z_2)|$ and $|z_1| + |z_2| - |z_3| = |\psi(z_1)| + |\psi(z_2)| - |\psi(z_3)|$, a contradiction. □

Remark 9.10. The condition from 9.9 is satisfied e. g. if $|z| - |\psi(z)|$ is odd for every $z \in Z$.

Remark 9.11. Let $Z = \{\varepsilon\}$. If $\psi(\varepsilon\} = \varepsilon$, then $\rho = \mathrm{id}_{A^*}$, and hence ρ is antitransitive, but not strictly antitransitive. If $\psi(\varepsilon) \neq \varepsilon$, then ρ is strictly antitransitive.

Proposition 9.12. *Assume that* $\varepsilon \notin Z$ *and for every* $z \in Z$ $zx \neq \psi(z) \neq yz$, $x, y \in A^*$. *Then* ρ *is antitransitive.*

Proof. According to I.7.1, we have to prove that for all $z_1, z_2 \in Z$ and $w \in A^*$ such that $z_1 w\psi(z_2) \neq \psi(z_1)wz_2$ we have $(z_1w\psi(z_2), \psi(z_1)wz_2) \notin \rho$ and $(\psi(z_1)wz_2, z_1w\psi(z_2)) \notin \rho$. Suppose, for a contradiction, that there are $z_1, z_2 \in Z$ and $w \in A^*$ such that $(z_1w\psi(z_2), \psi(z_1)wz_2) \in \rho$ (the other case is similar). This means that there exist $u, v \in A^*$ and $z \in Z$ such that $z_1w\psi(z_2) = uzv$ and $\psi(z_1)wz_2 = u\psi(z)v$. If $u = \varepsilon$ then $z = z_1$, $v = w\psi(z_2)$ and $\psi(z_1)wz_2 = \psi(z_1)w\psi(z_2)$, thus $z_2 = \psi(z_2)$, a contradiction. Hence we may assume that $u = z_1u'$ and hence $w\psi(z_2) = u'zv$ and $\psi(z_1)wz_2 = z_1u'\psi(z)v$. Since $z_1x \neq \psi(z_1)$, $z_1 = \psi(z_1)s$ for a proper $s \in A^*$ (s is a suffix of z_1), $w\psi(z_2) = u'zv$ and $wz_2 = su'\psi(z)v$. Now, let $w = s^nw'$, $u' = s^mu''$, w', u'' be such that s is not a prefix of either one of them. Then $s^nw'\psi(z_2) = s^mu''zv$ and $s^nw'z_2 = s^{m+1}u''\psi(z)v$. If $n \leq m$ then $w'z_2 = s^{m-n+1}u''\psi(z)v$ and (s is not a prefix of w') there exists a suffix of z_1 which is a prefix of z_2, a contradiction. If $n > m$ then $s^{n-m}w'\psi(z_2) = u''zv$ and (s is not a prefix of u'') there exists a suffix of z_1 which is a prefix of z, a contradiction again. □

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