COMMUTATIVE ZEROPOTENT SEMIGROUPS WITH FEW INVARIANT CONGRUENCES

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Abstract. Commutative semigroups satisfying the equation $2x + y = 2x$ and having only two G-invariant congruences for an automorphism group G are considered. Some classes of these semigroups are characterized and some other examples are constructed.

Every congruence-simple (i.e., possesing just two congruence relations) commutative semigroup is finite and either two-element or a group of prime order. The class of (non-trivial) commutative semigroups having only trivial invariant congruences is considerably more opulent. These semigroups are easily divided into four pair-wise disjoint subclasses (see 1.3). The fourth one contains commutative semigroups that are nil of index two and have no irreducible elements. This subclass is enigmatic a bit and it is the purpose of the present note to construct various examples of the indicated semigroups (called zs-semigroups in the sequel). Among others, we show that if S is a non-trivial commutative zs-semigroup without non-trivial invariant congruences, then the group of automorphisms of S contains a non-commutative free subsemigroup.

1. Introduction

Let G be a multiplicative group. By a (unitary left G-) semimodule we mean a commutative semigroup S $(= S(+))$ together with a G-scalar multiplication $G \times S \to S$ such that $a(x+y) = ax+ay$, $a(bx) = (ab)x$ and $1x = x$ for all $a, b \in G$ and $x, y \in S$.

Let S be a semimodule. An element $w \in S$ is called absorbing if $Gw = w = S+w$. There exists at most one absorbing element in S and, if it exists, it will usually be denoted by the symbol o_S (or only o); we will also write $o \in S$.

A non-empty subset I of S is an ideal if $GI \subseteq I$ and $S+I \subseteq I$. The semimodule S will be called ideal-simple (or only id-simple) if $|S| \geq 2$ and $I = S$ whenever I is an ideal of S such that $|I| \geq 2$.

Lemma 1.1. *Let S be a semimodule and $w \in S$. The one-element set $\{w\}$ is an ideal of S if and only if $w = o_S$ is an absorbing element of S.*

Proof. Obvious. □

A semimodule S will be called congruence-simple (or only cg-simple) if S has just two congruence relations (i.e., equivalences compatible with the addition and the scalar multiplication).

Proposition 1.2. *Every cg-simple semimodule is id-simple.*

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Proof. If S is cg-simple, then S is non-trivial and, if I is an ideal of S, then $r = (I \times I) \cup \mathrm{id}_S$ is a congruence of S. Now, either $r = \mathrm{id}_S$ and $|I| = 1$ (see 1.1) or $r = S \times S$ and $I = S$. Thus S is id-simple. □

Let S be a (commutative) semigroup/semimodule. We will say that S is

- a semigroup/semimodule with zero addition (a za-semigroup/za-semimodule) if $|S + S| = 1$ (then $o \in S$ and $S + S = o$);
- a zeropotent semigroup/semimodule (a zp-semigroup/zp-semimodule) if $2x + y = 2x$ for all $x, y \in S$ (then $o \in S$ and $2x = o$);
- a zp-semigroup/zp-semimodule without irreducible elements (a zs-semigroup/zs-semimodule) if S is a zp-semigroup/zp-semimodule and S=S+S;
- idempotent if $x + x = x$ for every $x \in S$;
- cancellative if $x + y \neq x + z$ for all $x, y, z \in S$, $y \neq z$.

The following basic classification of cg-simple semimodules is given in [1]:

Theorem 1.3. *Let S be a cg-simple semimodule. Then just one of the following four cases takes place:*
(1) *S is a two-element za-semimodule;*
(2) *S is idempotent;*
(3) *S is cancellative;*
(4) *S is a zs-semimodule.*

There exists only one two-element za-semimodule up to isomorphism. Cg-simple idempotent semimodules over a commutative group are fully characterized in [1] (see also [3], [4] and [5]) and cg-simple chains (and the corresponding groups) are studied in [6] and [7]. Some information on cg-simple cancellative semimodules is also available from [1] and various examples of non-trivial commutative zs-semigroups are collected in [2]. The aim of this note is to initiate a study of cg-simple zs-semimodules. The following starting result restricts our choice of groups in the zeropotent case:

Proposition 1.4. *Let no subsemigroup of a group G be a free semigroup of rank (at least) 2. Then there exist no cg-simple zs-semimodules over G.*

Proof. Let S be a non-trivial zs-semimodule and let $x, y, z \in S$ be such that $x = y + z \neq o$. Denote by A (B, resp.) the set of $a \in G$ ($b \in G$, resp.) such that $ax = y$ or $ax + v = y$, $v \in S$ ($bx = z$ or $bx + v = z$, resp.). Then $A \cap B = \emptyset$, $AA \cup AB \subseteq A$ and $BB \cup BA \subseteq B$. Now, if $a \in A$ and $b \in B$, then the subsemigroup of G generated by $\{a, b\}$ is free, a contradiction. Thus either $A = \emptyset$ or $B = \emptyset$ and we will assume $A = \emptyset$, the other case being similar.

Put $I = Gx \cup (Gx + S)$. Then I is an ideal of S, $y \notin I$ and $I \neq S$. On the other hand, $\{x, o\} \subseteq I$ and $|I| \geq 2$. Consequently, the semimodule S is not id-simple and, according to 1.2, it is not cg-simple either. □

Notice that among the groups from 1.4 we find all periodic groups and all locally nilpotent groups (but not all metabelian groups).

Now, let R be a subsemigroup of a group G and let $\mathbf{M} = \{A \mid A \subseteq G, A \neq \emptyset, AR \subseteq A\}$. The set $\mathbf{M}$ is closed under unions and non-empty intersections, $R \in \mathbf{M}$ and $G \in \mathbf{M}$. Now, we define an addition $+$ on $\mathbf{M}$ by $A + B = A \cup B$ if $A \cap B = \emptyset$ and $A + B = G$ otherwise.

Lemma 1.5. $\mathbf{M}(+)$ *is a commutative zp-semigroup and* $o_{\mathbf{M}} = G$.

Proof. Easy to check. □

Moreover, we define a scalar multiplication on $\mathbf{M}$ by $(a, A) \to aA = \{ax \mid x \in A\}$, $a \in G$, $A \in \mathbf{M}$.

Lemma 1.6. $\mathbf{M}$ *is a zp-semimodule over the group* G.

Proof. Easy to check. □

Define a relation ξ on $\mathbf{M}$ by $(A, B) \in \xi$ iff $\{M \in \mathbf{M} \mid A \cap M = \emptyset\} = \{M \in \mathbf{M} \mid B \cap M = \emptyset\}$.

Lemma 1.7. ξ *is a congruence of the semimodule* $\mathbf{M}$.

Proof. Easy to check. □

Lemma 1.8. *Let* η *be a congruence of* $\mathbf{M}$ *such that* $\xi \subseteq \eta$ *and* $(R, G) \in \eta$. *Then* $\eta = \mathbf{M} \times \mathbf{M}$.

Proof. Clearly, $(xR, G) = (xR, xG) \in \eta$ for every $x \in G$. Let $A \in \mathbf{M}$ and $a \in A$. If $aR \cap B \neq \emptyset$ for every $B \in \mathbf{M}$ such that $B \subseteq A$, then $(aR, A) \in \xi \subseteq \eta$, and so $(A, G) \in \eta$. On the other hand, if $B \in \mathbf{M}$ is maximal with respect to $B \subseteq A$ and $aR \cap B = \emptyset$, then $(A, B \cup aR) \in \xi$. Since $(G, B \cup aR) \in \eta$, we get $(A, G) \in \eta$ again. □

Lemma 1.9. $(R, G) \in \xi$ *if and only if* $G = RR^{-1}$ *(then* R *is right uniform).*

Proof. If $(R, G) \in \xi$, then $R \cap A \neq \emptyset$ for every $A \in \mathbf{M}$. In particular, $R \cap xR \neq \emptyset$ for every $x \in G$, and hence $x \in RR^{-1}$. To show the other implication, we just proceed conversely. □

Lemma 1.10. (i) *If* R *is not right uniform, then* $(R, G) \notin \xi$.
(ii) *If* G *is not generated by* R, *then* $(R, G) \notin \xi$.

Proof. (i) There exist $a, b \in R$ such that $aR \cap bR = \emptyset$. Then $R \cap a^{-1}bR = \emptyset$, $ab^{-1}R \in \mathbf{M}$ and, of course, $G \cap a^{-1}bR = a^{-1}bR \neq \emptyset$. Thus $(R, G) \notin \xi$.
(ii) Use 1.9. □

Lemma 1.11. *Assume that* R *is not right uniform. Then* $(R, G) \notin \xi$ *and, if* κ *is a congruence of* $\mathbf{M}$ *maximal with respect to* $\xi \subseteq \kappa$ *and* $(R, G) \notin \kappa$, *then* $\mathbf{N} = \mathbf{M}/\kappa$ *is a cg-simple zs-semimodule.*

Proof. $\mathbf{N}$ is non-trivial and it follows readily from 1.8 that $\mathbf{N}$ is a cg-simple zp-semimodule. Since R is not right uniform, there are right ideals A and B of R such that B is maximal with respect to $A \cap B = \emptyset$. Then $A + B = A \cup B$, $(A \cup B, R) \in \xi \subseteq \kappa$, $(A \cup B, G) \notin \kappa$ and $A/\kappa + B/\kappa \neq o_{\mathbf{N}}$. Thus $\mathbf{N}$ is not a za-semimodule, and hence $\mathbf{N}$ is a zs-semimodule by 1.3. □

Proposition 1.12. *If* R *is not right uniform, then a factorsemimodule of* $\mathbf{M}$ *is a congruence-simple zs-semimodule.*

Proof. See 1.11. □

Theorem 1.13. *There exists at least one cg-simple zs-semimodule over* G *if and only if the group* G *contains at least one subsemigroup that is a free semigroup of rank (at least) 2.*

Proof. The direct implication is shown in 1.4. As concerns the inverse implication, the existence of cg-simple zs-semimodule is shown in 1.12. □

2. Basic properties of zeropotent semimodules

Throughout this secion, let S be a zp-semimodule over a group G. Firstly, define a relation $\preceq_S$ on S by $x \preceq_S y$ if and only if $x = y$ or $y = x + v$ for some $v \in S$.

Lemma 2.1. (i) *The relation $\preceq_S$ is an ordering compatible with the addition and scalar multiplication.*

(ii) *o_S is a greatest element of the ordered set $(S, \preceq_S)$.*

(iii) *If $|S| \geq 2$, then $S \setminus (S + S)$ is the set of minimal elements of $(S, \preceq_S)$.*

(iv) *If $x, y, z \in S$ are such that $x \preceq_S y$ and $x \preceq_S z$, then $y + z = o$.*

Proof. Easy. □

Proposition 2.2. *Assume that S is a non-trivial zs-semimodule. Then:*

(i) *The ordered set $(S, \preceq_S)$ has no minimal elements.*

(ii) *$S(+)$ is not finitely generated (and hence S is infinite).*

Proof. (i) This follows immediately from 2.1(iii).

(ii) If $S(+)$ were generated by s finite number m of elements, then S should contain at most 2^m elements, a contradiction with (i). □

Lemma 2.3. *The following conditions are equivalent:*

(i) *If $x, y, z, u, v \in S$ are such that $x + y \neq o \neq z$ and $x + u = z = y + v$, then either $z = x + y$ or $z = x + y + w$ for some $w \in S$.*

(ii) *If $x, y, z \in S$ are such that $x+y \neq o \neq z$ and $x \preceq_S z$, $y \preceq_S z$, then $x+y \preceq_S z$.*

(iii) *If $x, y \in S$ are such that $x + y \neq o$, then $x + y = \sup(x, y)$ in $(S, \preceq_S)$.*

Proof. Easy. □

The semimodule S will be called downwards-regular if the equivalent conditions of 2.3 are satisfied.

For every $x \in S$, let $\mathrm{Ann}_S(x) = \{y \in S \mid x + y = o\}$. Further, let $\mathrm{Ann}_S(S) = \{x + S \mid S + x = o\}$.

Lemma 2.4. (i) *For every $x \in S$, the annihilator $\mathrm{Ann}_S(x)$ is an ideal of the additive semigroup $S(+)$.*

(ii) *$\mathrm{Ann}_S(S)$ is an ideal of the semimodule S.*

Proof. Obvious. □

Define a relation $\dashv_S$ on S by $x \dashv_S y$ if and only if $\mathrm{Ann}_S(x) \subseteq \mathrm{Ann}_S(y)$.

Lemma 2.5. (i) *The relation $\dashv_S$ is a quasiordering compatible with the addition and scalar multiplication.*

(ii) *If $x \preceq_S y$, then $x \dashv_S y$.*

(iii) *$\pi_S = \ker(\dashv_S)$ is a congruence of the semimodule S.*

(iv) *$\pi_S = S \times S$ if and only if S is a za-semimodule.*

Proof. Easy. □

The semimodule S will be called separable if $\pi_S = \mathrm{id}_S$.

The semimodule S will be called upwards-regular if $\mathrm{Ann}_S(x+y) \subseteq \mathrm{Ann}_S(z)$ whenever $x, y, z \in S$ are such that $x+y \neq o \neq z$ and $\mathrm{Ann}_S(x) \cup \mathrm{Ann}_S(y) \subseteq \mathrm{Ann}_S(z)$.

In the sequel, let $\tau_S = \{(x,y) \in S \times S \mid x+y \neq o\}$ and $\sigma_S = \{(x,y) \mid x+y = o\} = S \times S \setminus \tau_S$. Further, define μ_S (ν_S, resp.) by $(x,y) \in \mu_S$ ($(x,y) \in \nu_S$, resp.) if and only if $z \preceq_S x$, $z \preceq_S y$ ($z \dashv_S x$, $z \dashv_S y$, resp.) for at least one $z \in S$.

Lemma 2.6. (i) *The relations τ_S, σ_S, μ_S and ν_S are symmetric.*

(ii) *The relations σ_S, μ_S and ν_S are reflexive.*

(iii) *τ_S is irreflexive.*

(iv) *$\pi_S \subseteq \sigma_S$.*

(v) *$\mu_S \subseteq \nu_S \subseteq \sigma_S$.*

Proof. Easy. □

The semimodule S will be called (strongly) balanced if $\sigma_S = \nu_S$ ($\sigma_S = \mu_S$).

The semimodule S will be called transitive if the group G operates transitively on the set $S \setminus \{o_S\}$.

Proposition 2.7. *If S is non-trivial and transitive, then S is id-simple.*

Proof. Easy. □

Proposition 2.8. *Assume that S is id-simple and either $S+S \neq S$ or $\mathrm{Ann}_S(S) \neq \{o_S\}$. Then:*

(i) *$S+S = \{o_S\}$, $\mathrm{Ann}_S(S) = S$ and S is a za-semimodule.*

(ii) *$x \preceq_S y$ if and only if either $x = y$ or $y = o_S$.*

(iii) *$\pi_S = S \times S = \dashv_S$.*

(iv) *G operates transitively on $R = S \setminus \{o_S\}$ (i.e., S is transitive).*

(v) *$\nu_S = (R \times R) \cup \mathrm{id}_S$ is a congruence of S.*

(vi) *G operates primitively on R if and only if id_S, ν_S and $S \times S$ are the only congruences of S.*

Proof. Easy. □

Proposition 2.9. *Assume that S is cg-simple and $|S| \geq 3$. Then:*

(i) *$\mathrm{Ann}_S(S) = \{o_S\}$ and S is separable.*

(ii) *$\dashv_S$ is a compatible ordering of S.*

Proof. It follows from 2.5(iii) that either $\pi_S = S \times S$ or $\pi_S = \mathrm{id}_S$. If $\pi_S = S \times S$, then S is a za-semimodule by 2.5(iv) and S is id-simple by 1.2. Now, it follows from 2.8(v) that $|R| = 1$ and $|S| = 2$, a contradiction. Consequently, $\pi_S = \mathrm{id}_S$ and $\dashv_S$ is transitive. The rest follows from 2.5. □

Proposition 2.10. *Assume that $|S| \geq 3$. Then S is cg-simple if and only if S is separable and id-simple.*

Proof. The direct implication follows from 1.2 and 2.9. Now, assume that S is separable and id-simple.

Let r be a congruence of S and $I = \{x \mid (x,o) \in r\}$. Then I is an ideal of S and $r = S \times S$, provided that $I = S$.

Let $(x,y) \in r$, $x \neq y$. Since S is separable, $(x,y) \notin \pi_S$ and we can assume that $x \dashv_S y$ is not true. Then $\mathrm{Ann}_S(x) \nsubseteq \mathrm{Ann}_S(y)$ and there is $z \in S$ such that $x + z = o \neq y + z$. Now, $y + z \in I$, $I \neq \{o\}$, $I = S$, since S is id-simple, and $r = S \times S$. □

Proposition 2.11. *Assume that S is transitive and $|S| \geq 3$. The following conditions are equivalent:*
(i) *S is cg-simple.*
(ii) *S is separable.*

Proof. (i) implies (ii) by 2.9(i) and (ii) implies (i) by 2.7 and 2.10. □

Proposition 2.12. *Assume that S is id-simple, take $w \in S$, $w \neq o$, and consider a congruence r of S maximal with respect to $(w, o) \notin r$. Then S/r is a cg-simple zp-semimodule.*

Proof. Clearly, $T = S/r$ is a non-trivial zp-semimodule. Now, let s be a congruence of S such that $r \subseteq s$, $r \neq s$, and put $I = \{x \in S \mid (x, o) \in s\}$. Then I is an ideal of S and $\{o, w\} \subseteq I$. Thus $I = S$, since S is id-simple, and we conclude that $s = S \times S$. It follows easily that T is cg-simple. □

Corollary 2.13. *Assume that S is id-simple and $S + S \neq \{o_S\}$. Then at least one factorsemimodule of S is a cg-simple zs-semimodule.*

Corollary 2.14. *Assume that S is transitive and $S + S \neq \{o_S\}$. Then at least one factorsemimodule of S is a cg-simple zs-semimodule.*

3. Examples of congruence-simple zs-semimodules

Example 3.1. Let S be a non-trivial commutative zs-semigroup and $G = \mathrm{Aut}(S)$ (the automorphism group of S). Then S becomes a G-semimodule. If S is separable and G operates transitively on $S \setminus \{o_S\}$, then S is cg-simple semimodule.

Example 3.2. Let $(R, \leq)$ be a non-empty ordered set together with an irreflective and symmetric relation τ defined on R. For $x, y \in R$, let $x \vee y = \sup(x, y)$, provided that this supremum exists. Now, assume that the following three conditions are satisfied:

(α) If $x, y \in R$ are such that $(x, y) \in \tau$, then $x \vee y$ exists;
(β) If $(x, y) \in \tau$ and $(z, x \vee y) \in \tau$, then $(x, z) \in \tau$ and $(y, x \vee z) \in \tau$;
(γ) For every $x \in R$ there exist $y, z \in R$ such that $(y, z) \in \tau$ and $x = y \vee z$.

Further, let $o \notin R$, $S = R \cup \{o\}$, $x + y = x \vee y$ if $x, y \in R$, $(x, y) \in \tau$ and $x + y = o$ otherwise. Then S $(= S(+))$ becomes a commutative zs-semigroup.

Let G be a group operating on R (i.e., a mapping $G \times R \to R$ is defined such that $a(bx) = (ab)x$ and $1x = x$) and assume that $(ax, ay) \in \tau$ for every $(x, y) \in \tau$ and that $u \leq v$ implies $au \leq av$. Then $ax \vee ay = a(x \vee y)$ for $(x, y) \in \tau$ and S becomes a G-semimodule $(ao = o)$. If G operates transitively on R, then S is a transitive semimodule. In such a case, by 2.14, at least one factorsemimodule of S is a cg-simple zs-semimodule. Furthermore, if S is transitive, then S is cg-simple iff it is separable (2.11). Finally, S is separable iff the following two conditions are satisfied:

(δ) For every $x \in R$ there exists $y \in R$ with $(x, y) \in \tau$;
(ϵ) For all $x, y \in R$, $x \neq y$, $(x, y) \notin \tau$, there exists $z \in R$ such that either $(x, z) \in \tau$, $(y, z) \notin \tau$ or $(x, z) \notin \tau$, $(y, z) \in \tau$.

(Notice that (δ) is true, provided that S is transitive.)

Example 3.3. (cf. 3.2). Let T $(= T(\wedge, \vee))$ be a distributive lattice with a smallest element 0_T and a greatest element 1_T such that $|T| \geq 3$. Consider the basic order $\leq$ defined on T and also the ordered set $(R, \leq)$, $R = T \setminus \{0_T, 1_T\}$. Assume that the following two conditions are satisfied:

(μ) If $x, y \in R$ and $x \wedge y = 0_T$, then $x \vee y \neq 1_T$;
(ν) For every $x \in R$ there exist $y, z \in R$ such that $y \wedge z = 0_T$ and $y \vee z = x$.

Put $S = T \setminus \{1_T\}$ and define an addition on S by $x + y = x \vee y$ if $x \wedge y = 0_T$ and $x + y = 1_T$ otherwise. Then S $(= S(+))$ is a commutative zs-semigroup. Further, let a group G operate on R ($a(bx) = (ab)x$ and $1x = x$) in such a way that $x \leq y$ implies $ax \leq ay$. Then S becomes a G-semimodule ($a1_T = 1_T$). If G operates transitively on R, then S is a cg-simple zs-semimodule iff the following is true:

(σ) For all $x, y \in R$, $x \neq y$, $x \wedge y \neq 0_T$, there exists $z \in R$ such that either $x \wedge z = 0_T \neq y \wedge z$ or $x \wedge z \neq 0_T = y \wedge z$.

Example 3.4. Let I be an infinite set with $|I| \geq \aleph_1$ and let $\aleph$ be an infinite cardinal number such that $\aleph < |I|$. Denote by $\mathbf{J}$ the set $\{A \mid A \subseteq I, |A| = \aleph\} \cup \{I\}$ and define an operation $\oplus$ on $\mathbf{J}$ by $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ and $A \oplus B = I$ otherwise. Then $\mathbf{J}$ is a non-trivial commutative zs-semigroup and $\mathbf{J}$ becomes a G-semimodule, $G = \mathrm{Aut}(\mathbf{J}(\oplus))$. It is easy to check that the semimodule $\mathbf{J}$ is transitive, separable and upwards-regular, but neither downwards-regular non balanced. By 2.11, $\mathbf{J}$ is cg-simple.

Example 3.5. Let I be an infinite set, $\mathbf{K}$ a (non-principal) maximal ideal of the Boolean algebra of subsets of I such that $K \in \mathbf{K}$ for every $K \subseteq I$, $|K| = |I|$, and let $\mathbf{L} = \{A \in \mathbf{K} \mid |A| = |I|\} \cup \{I\}$. Define an addition $\oplus$ on $\mathbf{L}$ by $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ and $A \oplus B = I$ otherwise and put $G = \mathrm{Aut}(\mathbf{L}(\oplus))$. Then $\mathbf{L}$ $(= \mathbf{L}(\oplus))$ is a non-trivial separable commutative zs-semigroup and G operates transitively on $\mathbf{L} \setminus \{o\}$. Consequently, $\mathbf{L}$ is a cg-simple zs-semimodule over G.

Example 3.6. Let I be an infinite set and $\mathbf{I}$ the set of infinite subsets of I. Define an operation $\boxplus$ on $\mathbf{I}$ by $A \boxplus B = A \cup B$ if $A \cap B$ is finite and $A \boxplus B = I$ otherwise. Then $\mathbf{I}$ $(= \mathbf{I}(\boxplus))$ is a non-trivial commutative zs-semigroup and r is a congruence of $\mathbf{I}$, where $(A, B) \in r$ iff the symmetric difference $(A \cup B) \setminus (A \cap B)$ is finite. Then $\mathbf{J} = \mathbf{I}/r$ is a non-trivial (commutative) zs-semigroup. Moreover, if $|I| = \aleph_0$ and $G = \mathrm{Aut}(\mathbf{J})$, then $\mathbf{J}$ is a separable, upwards- and downwards-regular transitive G-semimodule ($\mathbf{J}$ is not balanced). Consequently, $\mathbf{J}$ is a cg-simple zs-semimodule.

Assume that $|I| \geq \aleph_1$ and put $\mathbf{P} = \{A \in \mathbf{I} \mid |A| = \aleph_0\} \cup \{I\}$. Then $\mathbf{P}$ is a subsemigroup of $\mathbf{I}$ and $\mathbf{Q} = \mathbf{P}/r$ is a non-trivial (commutative) zs-semigroup. Moreover, if $H = \mathrm{Aut}(\mathbf{Q})$, then $\mathbf{Q}$ is a transitive H-semimodule and it is easy to check that $\mathbf{Q}$ is an upwards- and downwards-regular strongly balanced cg-simple zs-semimodule.

4. Fractional left ideals and zeropotent semimodules

In this section, let R be a subsemigroup of a group G such that $1 \in R$. We denote by $\mathbf{F}$ $(= \mathbf{F}(G, R))$ the set of fractional left R-ideals of G. That is, $A \in \mathbf{F}$ iff $A \subseteq G$, $A \neq \emptyset$, $RA \subseteq A$ and $A \subseteq Ra$ for some $a \in G$. The set ($\mathbf{G}(G, R) =$) $\mathbf{G} = \mathbf{F} \cup \{\emptyset\}$ is closed under arbitrary intersections and G operates on $\mathbf{G}$ via $a * A = Aa^{-1}$, $A \in \mathbf{G}$, $a \in G$. The set ($\mathbf{P}(G, R) =$) $\mathbf{P} = \{Ra \mid a \in G\}$ of principal fractional left R-ideals is contained in $\mathbf{F}$ and we put $(\mathbf{Q}(G,R) =)\ \mathbf{Q} = \mathbf{P} \cup \{\emptyset\}$. Notice that G operates transitively on $\mathbf{P}$.

Construction 4.1. Assume that the following condition is satisfied:

(f1) If $a \in G$ is such that $R \cap aR = \emptyset$, then $R \cap Ra = Rb$ for some $b \in G$ (then $b \in R$).

Now, define an addition on the set $\mathbf{Q}$ in the following way:

(1) $Ra + Rb = Ra \cap Rb$ for all $a, b \in G$ such that $R \cap ab^{-1}R = \emptyset$ (by (f1), we have $Ra \cap Rb \in \mathbf{P}$);
(2) $Ra + Rb = \emptyset$ for all $a, b \in G$ such that $R \cap ab^{-1}R \neq \emptyset$;
(3) $Ra + \emptyset = \emptyset = \emptyset + Ra$ for every $a \in G$;
(4) $\emptyset + \emptyset = \emptyset$.

Now, we have obtained a groupoid $\mathbf{Q} = \mathbf{Q}(+)$.

Lemma 4.1.1. *$A + B = B + A$, $A + A = \emptyset$ and $A + \emptyset = \emptyset$ for all $A, B \in \mathbf{Q}$.*

Proof. Obvious. □

Lemma 4.1.2. *For every $a \in G$, the transformation $A \to a * A$ $(= Aa^{-1})$ is an automorphism of $\mathbf{Q}(+)$.*

Proof. Easy to check. □

Lemma 4.1.3. *$\mathbf{Q}$ is a semigroup if and only if the following condition is satisfied:*
(f2) *If $a, b, c \in G$ are such that $R \cap aR = \emptyset = R \cap bc^{-1}R$ and $R \cap Ra = Rc$, then $R \cap dR = \emptyset = R \cap ab^{-1}R$, where $Ra \cap Rb = Rd$.*

Proof. (i) Let $\mathbf{Q}(+)$ be associative. Then $(R + Ra) + Rb = R + (Ra + Rb)$. But $(R + Ra) + Rb = (R \cap Ra) + Rb = Rc + Rb = Rc \cap Rb = R \cap Ra \cap Rb \neq \emptyset$, and hence $R \cap ab^{-1}R = \emptyset$, $Ra \cap Rb = Rd$ by (f1), $R + Rd \neq \emptyset$ and $R \cap dR = \emptyset$.

(ii) Let (f2) be satisfied. Firstly, if $a, b \in G$ are such that $(R + Ra) + Rb \neq \emptyset$, then (f2) implies $(R + Ra) + Rb = R + (Ra + Rb)$. Next, if $a, b, c \in G$ are such that $(Ra + Rb) + Rc \neq \emptyset$, then $(R + Rba^{-1}) + Rca^{-1} = a * ((Ra + Rb) + Rc) \neq \emptyset$, and hence $(R + Rba^{-1}) + Rca^{-1} = R + (Rba^{-1} + Rca^{-1}) = a * (Ra + (Rb + Rc))$. Consequently, $(Ra+Rb)+Rc = a^{-1}*(a*((Ra+Rb)+Rc)) = a^{-1}*(a*(Ra+(Rb+Rc))) = Ra + (Rb + Rc)$. Finally, if $a, b, c \in G$ are such that $Ra + (Rb + Rc) \neq \emptyset$, then $(Rc + Rb) + Ra = Ra + (Rb + Rc) \neq \emptyset$, and therefore $Ra + (Rb + Rc) = (Rc + Rb) + Ra = Rc + (Rb + Ra) = (Ra + Rb) + Rc$ by the commutativity of the addition and the preceding part of the proof. The rest is clear. □

Assume that (f2) is true. It follows from 4.1.1, 4.1.2 and 4.1.3 that $\mathbf{Q}$ becomes non-trivial transitive zp-semimodule over the group G.

Lemma 4.1.4. *$\mathbf{Q}$ is a (non-trivial) zs-semimodule if and only if the following condition is satisfied:*
(f3) *$R \cap aR = \emptyset$ for at least one $a \in G$.*

Proof. Use the transitivity of $\mathbf{Q}$. □

Proposition 4.1.5. *Assume that the conditions* (f1), (f2) *and* (f3) *are satisfied. Then:*
(i) *$\mathbf{Q} = \mathbf{Q}(+, *)$ is a non-trivial transitive zs-semimodule over G.*
(ii) *$\mathbf{Q}$ is ideal-simple.*

(iii) *If* $Ra \preceq_{\mathbf{Q}} Rb$*, then* $Rb \subseteq Ra$.
(iv) $\mathrm{Ann}_{\mathbf{Q}}(Ra) = \{Rb \mid R \cap ab^{-1}R \neq \emptyset\} \cup \{\emptyset\}$.
(v) *If* $Rb \subseteq Ra$*, then* $Ra \dashv_{\mathbf{Q}} Rb$.
(vi) $\mathrm{Ann}_{\mathbf{Q}}(\mathbf{Q}) = \{\emptyset\}$.
(vii) $\mathbf{Q}$ *is balanced.*

Proof. See 4.1.1, 4.1.2, 4.1.3 and 4.1.4 to show (i), ..., (vi). Finally, if $R \cap ab^{-1}R \neq \emptyset$, then $ab^{-1}r \in R$ for some $r \in R$ and we have $Ra \cup Rb \subseteq Rr^{-1}b$. Now, using (v), we show easily that $\mathbf{Q}$ is balanced. □

Finally, assume that the conditions (f1), (f2) and (f3) are satisfied (see 4.1.5) and consider two more conditions:

(f4) For every $a \in R \setminus R^{-1}$ there exists $b \in G$ such that $R \cap bR = \emptyset$ and $Ra = R \cap Rb$;
(f5) For every $a \in (RR^{-1}) \setminus R^{-1}$ there exists $b \in G$ such that $R \cap bR = \emptyset \neq R \cap abR$.

Lemma 4.1.6. *The following conditions are equivalent:*
(i) *If* $a, b \in G$*, then* $Ra \preceq_{\mathbf{Q}} Rb$ *if and only if* $Rb \subseteq Ra$ *(see 4.1.5(iii)).*
(ii) *The condition* (f4) *is satisfied.*

Proof. Easy to check. □

Lemma 4.1.7. *If* (f4) *is true, then* $\mathbf{Q}$ *is downwards-regular and strongly balanced.*

Proof. Use 4.1.6. □

Lemma 4.1.8. $\mathrm{Ann}_{\mathbf{Q}}(R) = \{Rb \mid b \in RR^{-1}\} \cup \{\emptyset\}$.

Proof. Easy to check. □

Lemma 4.1.9. *The following conditions are equivalent:*
(i) *If* $a, b \in G$*, then* $Ra \dashv_{\mathbf{Q}} Rb$ *if and only if* $Rb \subseteq Ra$ *(see 4.1.5(v)).*
(ii) *The condition* (f5) *is satisfied.*

Proof. (i) implies (ii). Let $a \in G$ be such that $R \cap abR = \emptyset$ whenever $b \in G$ is such that $R \cap bR = \emptyset$. It follows from 4.1.7(iv) and 4.1.8 that $R_a \dashv_{\mathbf{Q}} R$. Now, by (i), $R \subseteq Ra$, and hence $a \in R^{-1}$.

(ii) implies (i). Let $a, b \in G$ be such that $Ra \dashv_{\mathbf{Q}} Rb$ and $Ra \neq Rb$. Then $Rc \dashv_{\mathbf{Q}} R$, $c = ab^{-1}$ and $Rc \neq R$. Now, assume that $R \nsubseteq Rc$. Then $c \notin R^{-1}$ and, by (f5), $R \cap dR = \emptyset \neq R \cap cdR$ for some $d \in G$. Consequently, $Rd^{-1} \in \mathrm{Ann}_{\mathbf{Q}}(Rc) \subseteq \mathrm{Ann}_{\mathbf{Q}}(R)$ and $Rd^{-1} = Re$ for some $e \in RR^{-1}$ (4.1.8). Thus $d^{-1} = rs^{-1}$, $r, s \in R$, $dR = sr^{-1}R$ and $s \in R \cap dR$, a contradiction. It follows $R \subseteq Rc$ and $Rb \subseteq Ra$. □

Lemma 4.1.10. *If* (f5) *is true, then* $\mathbf{Q}$ *is separable and upwards-regular.*

Proof. Use 4.1.9. □

4.2. Consider the conditions (f1), ..., (f5) defined in 4.1.

Lemma 4.2.1. (i) *If* (f1) *is true, then* $G = RR^{-1} \cup R^{-1}R$ *(and hence the group* G *is generated by* R*).*
(ii) *If* $G = RR^{-1} \cup R^{-1}R$ *and every left ideal of* R *is principal, then* (f1) *is true.*
(iii) (f3) *is true if and only if* $G \neq RR^{-1}$.

Proof. Easy to see. □

Corollary 4.2.2. *If G is generated by R, R is left uniform, not right uniform and every left ideal of R is principal, then the conditions* (f1) *and* (f3) *are satisfied.*

5. Zeropotent semimodules and fractional left ideals

In this section, let S be an ideal-simple zeropotent G-semimodule such that $\mathrm{Ann}_S(S) = \{o_S\}$ (or, equivalently, $S + S \neq \{o_S\}$). For $u, v \in S$, put $(u : v) = \{a \in G \mid au \preceq_S v\}$ and $[u : v] = \{a \in G \mid au \dashv_S v\}$.

Lemma 5.1. (i) $(u : v) \subseteq [u : v]$ *for all* $u, v \in S$.
(ii) $(u : o) = [u : o] = G$ *for every* $u \in S$.
(iii) $(o : w) = [o : w] = \emptyset$ *for every* $w \in S$, $w \neq o$.
(iv) $(u : av) = a(u : v)$ *and* $(au : v) = (u : v)a^{-1}$ *for all* $a \in G$ *and* $u, v \in S$.
(v) $[u : av] = a[u : v]$ *and* $[au : v] = [u : v]a^{-1}$ *for all* $a \in G$ *and* $u, v \in S$.
(vi) $(au : au) = a(u : u)a^{-1}$ *for all* $a \in G$ *and* $u \in S$.
(vii) $[au : au] = a[u : u]a^{-1}$ *for all* $a \in G$ *and* $u \in S$.

Proof. The inclusion $(u : v) \subseteq [u : v]$ follows from 2.5(ii) and the remaining assertions can be checked readily. □

Lemma 5.2. (i) $(u : v_1)(v_2 : u) \subseteq (v_2 : v_1)$ *and* $[u : v_1][v_2 : u] \subseteq [v_2 : v_1]$ *for all* $u, v_1, v_2 \in S$.
(ii) $(u : u)(v : u) \subseteq (v : u)$ *and* $[u : u][v : u] \subseteq [v : u]$ *for all* $u, v \in S$.
(iii) $(u : u)(u : u) \subseteq (u : u)$ *and* $[u : u][u : u] \subseteq [u : u]$ *for every* $u \in S$.

Proof. Easy to check directly. □

Lemma 5.3. *Let* $u_1, u_2, u, v_1, v_2, v \in S$.
(i) *If* $u_1 \preceq_S u_2$, *then* $(u_2 : v) \subseteq (u_1 : v)$.
(ii) *If* $v_1 \preceq_S v_2$, *then* $(u : v_1) \subseteq (u : v_2)$.
(iii) *If* $v_2 \preceq_S u_1$, *then* $(u_1 : v_1)(u_2 : v_2) \subseteq (u_2 : v_1)$.

Proof. Easy to check directly. □

Lemma 5.4. *Let* $u_1, u_2, u, v_1, v_2, v \in S$.
(i) *If* $u_1 \dashv_S u_2$, *then* $[u_2 : v] \subseteq [u_1 : v]$.
(ii) *If* $v_1 \dashv_S v_2$, *then* $[u : v_1] \subseteq [u : v_2]$.
(iii) *If* $v_2 \dashv_S u_1$, *then* $[u_1 : v_1][u_2 : v_2] \subseteq [u_2 : v_1]$.

Proof. Easy to check directly. □

Lemma 5.5. $(u : v) \neq \emptyset \neq [u : v]$ *for all* $u, v \in S$, $u \neq o$.

Proof. Denote by I the set of $z \in S$ such that $au \preceq_S z$ for some $a \in G$. Then $\{o, u\} \subseteq I$ and I is an ideal of S. Since S is id-simple, we get $I = S$, $v \in I$, and therefore $(u : v) \neq \emptyset$. Since $(u : v) \subseteq [u : v]$, we have $[u : v] \neq \emptyset$, too. □

In the remaining part of this section, fix an element $w \in S$, $w \neq o_S$. It follows from 5.1(i), 5.2(iii) and 5.5 that both $R_1 = (w : w)$ and $R_2 = [w : w]$ are subsemigroups of G and $1 \in R_1 \subseteq R_2$. We put $\mathbf{F}_i = \mathbf{F}(G, R_i)$, $\mathbf{G}_i = \mathbf{G}(G, R_i)$, $\mathbf{P}_i = \mathbf{P}(G, R_i)$ and $\mathbf{Q_i} = \mathbf{Q}(G, R_i)$, $i = 1, 2$ (see the preceding section).

Lemma 5.6. (i) $R_1^* = R_1 \cap R_1^{-1} = \{a \in G \mid aw = w\}$.
(ii) $R_2^* = R_2 \cap R_2^{-1} = \{a \in G \mid (w, aw) \in \pi_S\}$.
(iii) *If S is separeble, then $R_1^* = R_2^*$.*

Proof. (i) If $aw = w$, then $a^{-1}w = w$, $a, a^{-1} \in R_1$ and $a \in R_1^*$. Conversely, if $a \in R_1^*$, then $a, a^{-1} \in R_1$. Now, if $w \neq aw$, then $w = aw + u = a^{-1}w + v$, $u, v \in S$, and we get $aw = w + av$, $w = w + z$, $z = av + u$, $w = w + 2z = w + o = o$, a contradiction.
(ii) Easy to check.
(iii) Since S is separable, we have $\pi_S = \mathrm{id}_S$ and the assertion follows by combination of (i) and (ii). □

Lemma 5.7. *Let $v \in S$. Then:*
(i) $R_1(v : w) \subseteq (v : w)$.
(ii) $R_2[v : w] \subseteq [v : w]$.
(iii) $(w : v) \neq \emptyset = [w : v]$.
(iv) $(v : w) \subseteq R_1 a^{-1}$ *for every* $a \in (w : v)$.
(v) $[v : w] \subseteq R_2 a^{-1}$ *for every* $a \in [w : v]$.
(vi) *If $v \neq o_S$, then $(v : w) \neq \emptyset \neq [v : w]$.*
(vii) $(v : w)(v : v) \subseteq (v : w)$.
(viii) $[v : w][v : v] \subseteq [v : w]$.

Proof. (i) If $a \in R_1$ and $b \in (v : w)$, then $aw = w$, $bv \preceq_S w$, and so $abv \preceq_S aw = w$ and $ab \in (v : w)$.
(ii) Similar to (i).
(iii) See 5.5.
(iv) By 5.2(i), $(v : w)(w : v) \subseteq (w : w) = R_1$, and so $(v : w) \subseteq R_1(w : v)^{-1}$.
(v) Similar to (iv).
(vi) See 5.5.
(vii) Use 5.2(i).
(viii) Similar to (vii). □

Using the foregoing lemma, we get mappings $(\varphi_w =)\ \varphi : S \to \mathbf{G}_1$ and $(\psi_w =)\ \psi : S \to \mathbf{G}_2$ defined by $\varphi(v) = (v : w)$ and $\psi(v) = [v : w]$ for every $v \in S$ (5.7(i), (ii)).

Lemma 5.8. (i) $\varphi(S \setminus \{o\}) \subseteq \mathbf{F}_1$.
(ii) $\varphi(av) = \varphi(v)a^{-1} = a * \varphi(v)$ *for all* $a \in G$ *and* $v \in S$.
(iii) *If $u \preceq_S v$, then $\varphi(v) \subseteq \varphi(u)$.*

Proof. (i) This follows from 5.5.
(ii) We have $\varphi(av) = (av : w) = (v : w)a^{-1} = \varphi(v)a^{-1} = a * \varphi(v)$ by 5.1(iv).
(iii) If $a \in \varphi(v)$, then $av \preceq_S w$, and, of course, $au \preceq_S av$. Thus $au \preceq_S w$ and $a \in \varphi(u)$. □

Lemma 5.9. (i) $\psi(S \setminus \{o\}) \subseteq \mathbf{F}_2$.
(ii) $\psi(av) = \psi(v)a^{-1} = a * \psi(v)$ *for all* $a \in G$ *and* $v \in S$.
(iii) *If $u \dashv_S v$, then $\psi(v) \subseteq \psi(u)$.*

Proof. Similar to that of 5.8. □

Lemma 5.10. (i) $\varphi(v) \subseteq \psi(v)$ *for every* $v \in S$.
(ii) $\varphi(w) = R_1$ *and* $\psi(w) = R_2$.
(iii) $\varphi(o_S) = \emptyset = \psi(o_S)$.

Proof. Obvious. □

Lemma 5.11. *Assume that S is transitive. Then:*
(i) *φ is a bijection of S onto $\mathbf{Q}_1$.*
(ii) *$u \preceq_S v$ if and only if $\varphi(v) \subseteq \varphi(u)$.*

Proof. (i) Let $u, v \in S$ be such that $\varphi(u) = \varphi(v)$. If $u = o$ or $v = o$, then $\varphi(u) = \emptyset = \varphi(v)$ and $u = o = v$ by 5.8(i). Hence, assume that $u \neq o \neq v$. Then $u = aw$ and $v = bw$ for some $a, b \in G$. Now, $R_1a^{-1} = \varphi(aw) = \varphi(u) = \varphi(v) = \varphi(bw) = R_1b^{-1}$, $R_1a^{-1}b = R_1$, $a^{-1}b \in R_1^*$, $w = a^{-1}bw$ and, finally, $u = aw = bw = v$ (use 5.8(ii) and 5.6(i)). We have proved that φ is an injective mapping.

If $a \in G$, then $\varphi(a^{-1}w) = (w : w)a = R_1a$ by 5.1(iv). It follows that φ is a projective mapping. Consequently, $\varphi : S \to \mathbf{Q}_1$ is a bijection.

(ii) If $u \preceq_S v$, then $\varphi(v) \subseteq \varphi(u)$ by 5.8(iii). Conversely, if $\varphi(v) \subseteq \varphi(u)$, $v \neq o$, $u = aw$, $v = bw$, then $R_1b^{-1} = \varphi(v) \subseteq \varphi(u) = R_1a^{-1}$, $R_1 \subseteq R_1a^{-1}b = \varphi(b^{-1}aw) = (b^{-1}aw : w)$, $1 \in (b^{-1}aw : w)$, $b^{-1}aw \preceq_S w$ and, finally, $u = aw \preceq_S bw = v$. □

Lemma 5.12. *Assume that S is transitive. Then:*
(i) *ψ is a projection of S onto $\mathbf{Q}_2$.*
(ii) *$\ker(\psi) = \pi_S$.*
(iii) *$u \dashv_S v$ if and only if $\psi(v) \subseteq \psi(u)$.*

Proof. Similar to that of 5.11. □

Corollary 5.13. *Assume that S is transitive. Then $\psi : S \to \mathbf{Q}_2$ is a bijection if and only if S is separable.*

Lemma 5.14. *If S is downwards-regular, then $\varphi(u + v) = \varphi(u) \cap \varphi(v)$ for all $u, v \in S$ such that $u + v \neq o_S$.*

Proof. The inclusion $\varphi(u+v) \subseteq \varphi(u) \cap \varphi(v)$ is clear from the definitions. Conversely, if $a \in \varphi(u) \cap \varphi(v)$, then $au \preceq_S w$, $av \preceq_S w$, and hence $a(u + v) \preceq_S w$, since S is downwards-regular. Thus $a \in \varphi(u + v)$. □

Lemma 5.15. *If S is upwards-regular, then $\psi(u+v) = \psi(u) \cap \psi(v)$ for all $u, v \in S$ such that $u + v \neq o_S$.*

Proof. Similar to that of 5.14. □

Theorem 5.16. *Let S be a transitive zeropotent G-semimodule such that $S + S \neq \{o_S\}$ (see 2.8). Let $w \in S$, $w \neq o_S$, $R_1 = \{a \in G \mid aw \preceq_S w\}$ and $R_2 = \{a \in G \mid aw \dashv_S w\}$. Then:*
(i) *S is ideal-simple, $S + S = S$ and $\mathrm{Ann}_S(S) = \{o_S\}$.*
(ii) *Both R_1 and R_2 are subsemigroups of G and $1 \in R_1 \subseteq R_2$.*
(iii) *The mapping $\varphi : v \to \{a \in G \mid av \preceq_S w\}$ is a bijection of S onto $\mathbf{Q}(G, R_1)$ such that $u \preceq_S v$ if and only if $\varphi(v) \subseteq \varphi(u)$.*
(iv) *If S is downwards-regular, then $\varphi(u + v) = \varphi(u) \cap \varphi(v)$ for all $u, v \in S$ such that $u + v \neq o_S$.*
(v) *The mapping $\psi : v \to \{a \in G \mid av \dashv_S w\}$ is a projection of S onto $\mathbf{Q}(G, R_2)$ such that $\ker(\psi) = \pi_S$ and $u \dashv_S v$ if and only if $\psi(v) \subseteq \psi(u)$.*
(vi) *If S is separable, then ψ is a bijection of S onto $\mathbf{Q}(G, R_2)$.*
(vii) *If S is upwards-regular, then $\psi(u + v) = \psi(u) \cap \psi(v)$ for all $u, v \in S$ such that $u + v \neq o_S$.*

Proof. See 2.7, 2.8, 5.1(i), 5.2(iii), 5.5, 5.11, 5.14, 5.12 and 5.15. □

Lemma 5.17. *Let $a, b \in G$, $u = aw$ and $v = bw$. Then:*
(i) $\varphi(u) \cap \varphi(v) \neq \emptyset$ *if and only if* $R_1 \cap R_1 a^{-1} b \neq \emptyset$.
(ii) $R_1 \cap a^{-1} b R_1 \neq \emptyset$ *if and only if there exists* $c \in G$ *with* $cw \preceq_S u$ *and* $cw \preceq_S v$.

Proof. (i) We have $\varphi(u) = R_1 a^{-1}$ and $\varphi(v) = R_1 b^{-1}$. The rest is clear.

(ii) If $d = a^{-1} be$, where $d, e \in R_1$, then $ad = c = be$, $cw = adw \preceq_S aw = u$ and $cw = bew \preceq_S bw = v$. Similarly the converse implication. □

Lemma 5.18. *Assume that S is strongly balanced. If $a, b \in G$ are such that $aw + bw = o$, then $R_1 \cap a^{-1} b R_1 \neq \emptyset$.*

Proof. Use 5.17(ii). □

Lemma 5.19. *Let $a, b \in G$, $u = aw$ and $v = bw$. Then:*
(i) $\psi(u) \cap \psi(v) \neq \emptyset$ *if and only if* $R_2 \cap R_2 a^{-1} b \neq \emptyset$.
(ii) $R_2 \cap a^{-1} b R_2 \neq \emptyset$ *if and only if there exists* $c \in G$ *with* $cw \dashv_S u$ *and* $cw \dashv_S v$.

Proof. Similar to that of 5.17. □

Lemma 5.20. *Assume that S is balanced. If $a, b \in G$ are such that $aw + bw = o$, then $R_2 \cap a^{-1} b R_2 \neq \emptyset$.*

Proof. Use 5.19(ii). □

Lemma 5.21. *Assume that S is transitive. If $a \in G$ is such that $w + aw \neq o$, then $a \in R_1^{-1} R_1$.*

Proof. We have $w + aw = bw$ for some $b \in G$, $aw \preceq_S bw$, $b^{-1} a \in R_1$, $w \preceq_S bw$, $b^{-1} \in R_1$. Consequently, $a \in R_1^{-1} R_1$. □

Lemma 5.22. *Assume that S is strongly balanced. If $a \in G$ is such that $w + aw = o$, then $a \in R_1 R_1^{-1}$.*

Proof. This follows immediately from 5.18. □

Lemma 5.23. *If S is transitive and strongly balanced, then $G = R_1^{-1} R_1 \cup R_1 R_1^{-1}$.*

Proof. Combine 5.21 and 5.22. □

6. A few consequences

6.1. Let S be a non-trivial transitive zp-semimodule over a group G such that S is downwards-regular and strongly balanced. By 2.7 and 2.8, S is ideal-simple, $\mathrm{Ann}_S(S) = \{o_S\}$ and $S + S = S$, i.e., S is a zs-semimodule.

Now, choose $w \in S$, $w \neq o_S$, and put $R = R_{1,w} = \{a \in G \mid aw \preceq_S w\}$ and $\varphi = \varphi_w$, where $\varphi_w(v) = \{a \in G \mid av \preceq_S w\}$ for every $v \in S$. According to 5.2(iii), 5.5 and 5.11, R is a subsemigroup of the group G, $1 \in R$ and φ is a bijection of S onto $\mathbf{Q} = \mathbf{Q}(G, R)$ such that $u \preceq_S v$ iff $\varphi(v) \subseteq \varphi(u)$. Moreover, by 5.8 and 5.14, $\varphi(av) = \varphi(v) a^{-1}$, $\varphi(aw) = Ra^{-1}$, $a \in G$, and if $u, v \in S$ are such that $u + v \neq o_S$, then $\varphi(u + v) = \varphi(u) \cap \varphi(v)$.

Lemma 6.1.1. *The condition* (f1) *(see 4.1) is satisfied.*

Proof. Let $a \in G$ be such that $R \cap Ra = \emptyset$. It follows from 5.18 that $a^{-1}w + w \neq o$, and hence $a^{-1}w+w = b^{-1}w$ for some $b \in G$. Now, $Rb = \varphi(b^{-1}w) = \varphi(a^{-1}w+w) = \varphi(a^{-1}w) \cap \varphi(w) = Ra \cap R$. □

The condition (f1) is true, and so we get groupoid $\mathbf{Q} = \mathbf{Q}(+)$ due to 4.1.

Lemma 6.1.2. *φ is an isomorphism of $S(+)$ onto $\mathbf{Q}(+)$.*

Proof. Since φ is a bijection, we have to show that φ is a homomorphism of the additive structures. For, let $u, v \in S$. We have $\varphi(o_S) = \emptyset$, and hence $\varphi(u + v) = \emptyset = \varphi(u) + \varphi(v)$, provided that either $u = o$ or $v = o$. Now, assume $u \neq o \neq v$. Then $u = aw$ and $v = bw$, $a, b \in G$.

Firstly, let $u + v \neq o$. If $R \cap a^{-1}bR \neq \emptyset$, then $cw \preceq_S u$ and $cw \preceq_S v$ for some $c \in G$ by 5.17(ii) and it follows that $u+v = o$, a contradiction. Thus $R \cap a^{-1}bR = \emptyset$, $Ra^{-1} + Rb^{-1} = Ra^{-1} \cap Rb^{-1} \neq \emptyset$ in $\mathbf{Q}(+)$ and we get $\varphi(u + v) = \varphi(u) \cap \varphi(v) = Ra^{-1} \cap Rb^{-1} = Ra^{-1} + Rb^{-1} = \varphi(u) + \varphi(v)$.

Next, let $u + v = o$. Then $R \cap a^{-1}bR \neq \emptyset$ by 5.18, and therefore $\varphi(u + v) = \varphi(o) = \emptyset = Ra^{-1} + Rb^{-1} = \varphi(u) + \varphi(v)$, too. □

Lemma 6.1.3. *The condition* (f2) *is satisfied.*

Proof. By 6.1.2, $S(+)$ is isomorphic to $\mathbf{Q}(+)$. Consequently, $\mathbf{Q}(+)$ is a semigroup and (f2) follows by 4.1.3. □

Lemma 6.1.4. *$\mathbf{Q}$ is a non-trivial transitive zs-semimodule and $\varphi : S \to \mathbf{Q}$ is an isomorphism of the semimodules.*

Proof. See 4.1, 6.1.2 and 6.1.3. □

Lemma 6.1.5. *The conditions* (f3) *and* (f4) *are satisfied.*

Proof. By 6.1.4, $\mathbf{Q}$ ($\cong S$) is a non-trivial zs-semimodule. Now, (f3) follows from 4.1.4 and (f4) is clear from 4.1.6 and 5.11(ii). □

Theorem 6.1.6. *The conditions* (f1), (f2), (f3) *and* (f4) *are satisfied (see 4.1) and the semimodules S and $\mathbf{Q}(G, R)$ are isomorphic.*

Proof. See 6.1.2, ..., 6.1.5. □

6.2. Let S be a non-trivial transitive zp-semimodule over a group G such that S is upwards-regular and balanced. By 2.7 and 2.8, S is ideal-simple, $\mathrm{Ann}_S(S) = \{o_S\}$ and $S + S = S$, i.e., S is a zs-semimodule.

Now, choose $w \in S$, $w \neq o_S$, and put $R = R_{2,w} = \{a \in G \mid aw \dashv_S w\}$ and $\psi = \psi_w$, where $\psi_w(v) = \{a \in G \mid av \dashv_S w\}$ for every $v \in S$. According to 5.2(iii), 5.5 and 5.12, R is a subsemigroup of the group G, $1 \in R$ and ψ is a projection of S onto $\mathbf{Q} = \mathbf{Q}(G, R)$ such that $\ker(\psi) = \pi_S$ and $u \dashv_S v$ iff $\psi(v) \subseteq \psi(u)$. Moreover, by 5.9 and 5.15, $\psi(av) = \psi(v)a^{-1}$, $\psi(aw) = Ra^{-1}$, $a \in G$, and if $u, v \in S$ are such that $u + v \neq o_S$, then $\psi(u + v) = \psi(u) \cap \psi(v)$.

Lemma 6.2.1. *The condition* (f1) *(see 4.1) is satisfied.*

Proof. Similar to that of 6.1.1 (use 5.20). □

The condition (f1) is true, and so we get the groupoid $\mathbf{Q} = \mathbf{Q}(+)$ due to 4.1.

Lemma 6.2.2. *ψ is a homomorphism of $S(+)$ onto $\mathbf{Q}(+)$.*

Proof. We have to show that ψ is a homomorphism of the additive structures. For, let $u, v \in S$. We have $\psi(o_S) = \emptyset$, and hence $\psi(u+v) = \emptyset = \psi(u) + \psi(v)$, provided that either $u = o$ or $v = o$. Now, assume that $u \neq o \neq v$. Then $u = aw$ and $v = bw$, $a, b \in G$.

Firstly, let $u + v \neq o$. If $R \cap a^{-1}bR \neq \emptyset$, then $cw \dashv_S u$ and $cw \dashv_S v$ for some $c \in G$ by 5.19(ii). Consequently, $\mathrm{Ann}_S(cw) \subseteq \mathrm{Ann}_S(u) \cap \mathrm{Ann}_S(v)$, $cw \in \mathrm{Ann}_S(cw)$ implies $cw + u = o$, $u \in \mathrm{Ann}_S(cw)$ and, finally, $u \in \mathrm{Ann}_S(v)$, $u + v = o$, a contradiction. Thus $R \cap a^{-1}bR = \emptyset$, $Ra^{-1} + Rb^{-1} = Ra^{-1} \cap Rb^{-1} \neq \emptyset$ in $\mathbf{Q}(+)$ and we get $\psi(u+v) = \psi(u) \cap \psi(v) = Ra^{-1} \cap Rb^{-1} = Ra^{-1} + Rb^{-1} = \psi(u) + \psi(v)$.

Next, let $u + v = o$. Then $R \cap a^{-1}bR \neq \emptyset$ by 5.20, and therefore $\psi(u+v) = \psi(o) = \emptyset = Ra^{-1} + Rb^{-1} = \psi(u) + \psi(v)$, too. □

Lemma 6.2.3. *The condition* (f2) *is satisfied.*

Proof. By 6.2.2, $\mathbf{Q}(+)$ is a homomorphic image of $S(+)$. Consequently, $\mathbf{Q}(+)$ is a semigroup and (f2) follows by 4.1.3. □

Lemma 6.2.4. $\mathbf{Q}$ *is a non-trivial transitive zs-semimodule and* $\psi : S \to \mathbf{Q}$ *is a projective homomorphism of the semimodules.*

Proof. We have $\pi_S \neq S \times S$, and hence $\mathbf{Q}$ is non-trivial. The rest is clear from 4.1, 6.2.2 and 6.2.3. □

Lemma 6.2.5. *The conditions* (f3) *and* (f5) *are satisfied.*

Proof. By 6.2.4, $\mathbf{Q}$ is a non-trivial zs-semimodule and (f3) follows from 4.1.4. Now, consider the condition (f5). According to 4.1.9 and 4.1.5(v), it suffices to show that $Rb \subseteq Ra$ whenever $a, b \in G$ are such that $Ra \dashv_{\mathbf{Q}} Rb$. We have $Ra = \psi(u)$ and $Rb = \psi(v)$, $u = a^{-1}w$, $v = b^{-1}w$. If $z \in \mathrm{Ann}_S(u)$, then $\psi(z) \in \mathrm{Ann}_{\mathbf{Q}}(Ra)$, and so $\psi(z) \in \mathrm{Ann}_{\mathbf{Q}}(Rb)$ and $\psi(z+v) = \psi(z) + \psi(v) = \emptyset$ $(= o_{\mathbf{Q}})$. Thus $(z+v, o_S) \in \pi_S$, $z + v \in \mathrm{Ann}_S(S) = \{o_S\}$, $z + v = o_S$ and $z \in \mathrm{Ann}_S(v)$. It follows that $u \dashv_S v$ and $Rb = \psi(v) \subseteq \psi(u) = Ra$ by 5.12. □

Theorem 6.2.6. *The conditions* (f1), (f2), (f3) *and* (f5) *are satisfied and there exists a projection of the semimodule S onto the semimodule $\mathbf{Q}(G, R)$. This projection is an isomorphism if and only if S is separable.*

Proof. See 6.2.1, ..., 6.2.5. □

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