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ROOTS OF EXT, THEIR ABSTRACT ELEMENTARY CLASSES, DECONSTRUCTION, AND APPLICATIONS

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Abstract. We start by surveying the tools of set–theoretic homological algebra that make it possible to deconstruct roots of the contravariant Ext functor, ie., express them as transfinite extensions of small roots (where small stands for countably or finitely generated).

Then we present several applications to structure theory of modules. In detail, we give a simple proof for the recent result of Bazzoni and Herbera [9] characterizing modules of projective dimension ≤ 1 over semiprime Goldie rings.

We also recall [7] that the roots of Ext yield natural examples of abstract elementary classes in the sense of Shelah, and present their basic model–theoretic properties. The final section consists of open problems.

1. Basic notions and properties

Let R be a ring and $\mathcal{C}$ a class of (right R–) modules. A module M is a *root of Ext* for $\mathcal{C}$ provided $\mathrm{Ext}^1_R(M,C) = 0$ for all $C \in \mathcal{C}$. Given a ring R and a class of modules $\mathcal{C}$, we will denote by ${}^\perp\mathcal{C}$ the class of all roots of Ext for $\mathcal{C}$, that is, ${}^\perp\mathcal{C} = \mathrm{KerExt}^1_R(-,\mathcal{C})$.

We will also deal with the subclass, ${}^{\perp_\infty}\mathcal{C}$, of ${}^\perp\mathcal{C}$ consisting of all higher roots of Ext for $\mathcal{C}$. A module M is a *higher root of Ext* for $\mathcal{C}$ provided that $\mathrm{Ext}^i_R(M,C) = 0$ for all $1 \leq i < \omega$ and all $C \in \mathcal{C}$. Notice that ${}^{\perp_\infty}\mathcal{C} = {}^\perp(\bigcup_{n<\omega}\mathcal{C}_n)$ where $\mathcal{C}_n$ is the class of all nth cosyzygies of the modules in $\mathcal{C}$.

Classes of the form ${}^\perp\mathcal{C}$ are ubiquituos in module and representation theory. For example, for each $n < \omega$, the class $\mathcal{P}_n$ ($\mathcal{F}_n$) of all modules of projective (flat) dimension $\leq n$ is of this form. So are the classes of all torsion–free, Whitehead, and Baer modules; moreover, various classes of roots of Ext naturally arise in (infinite dimensional) tilting theory, see [19].

Any class of the form ${}^\perp\mathcal{C}$ is closed under extensions and arbitrary direct sums. These are particular instances of the more general notion of a transfinite extension.

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For a class of modules $\mathcal{A}$, a module M is a *transfinite extension* of modules in $\mathcal{A}$ provided there is an increasing chain $(M_\alpha \mid \alpha \leq \lambda)$ of submodules of M such that $M_0 = 0$, $M = M_\lambda$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \lambda$, and for each $\alpha < \lambda$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$. We will say that $(M_\alpha \mid \alpha \leq \lambda)$ is a *witnessing chain* for M being a transfinite extension of modules in $\mathcal{A}$.

A basic fact, known as the Eklof lemma, says that the classes of all roots of Ext are closed under transfinite extensions.

Lemma 1.1. (Eklof Lemma) [12, XII.1.5] *Let R be a ring and $\mathcal{C}$ a class of right R–modules. Then the class ${}^\perp\mathcal{C}$ contains all transfinite extensions of modules in ${}^\perp\mathcal{C}$.*

In particular, the following are *necessary* conditions for a class of modules $\mathcal{A}$ to be of the form ${}^\perp\mathcal{C}$: $\mathcal{A}$ is closed under direct summands, transfinite extensions, and $R \in \mathcal{A}$. These conditions are not sufficient in general.

Example 1.2. Let $R = \mathbb{Z}$ and let $\mathcal{D}$ denote the class of all $\aleph_1$*–free* (abelian) groups (ie., the groups M such that each countable subgroup of M is free).

Then $\mathcal{D}$ contains all free groups, $\mathcal{D}$ is closed under direct summands, and by Pontryagin's criterion, $\mathcal{D}$ is closed under transfinite extensions (cf. [12, IV.2.3]). The Baer–Specker theorem says that any direct product of copies of $\mathbb{Z}$ is $\aleph_1$–free (cf. [12, IV.2.8]). Now, by [18, Lemma 1.2], if C is a group such that $\mathrm{Ext}^1_{\mathbb{Z}}(P, C) = 0$ for any direct product P of copies of $\mathbb{Z}$, then C is a cotorsion group, so ${}^\perp\{C\}$ contains all torsion–free groups; in particular $\mathbb{Q} \in {}^\perp\{C\}$, but $\mathbb{Q}$ is not $\aleph_1$–free. It follows that there exists no class of groups $\mathcal{C}$ such that $\mathcal{D} = {}^\perp\mathcal{C}$.

In the example above, the groups in $\mathcal{D}$ behave 'locally' like free groups, and the class of all free groups does form a class of all roots of Ext. So the phenomenon cannot be captured 'locally'.

In order to formulate a *sufficient* condition, we will need more definitions.

Given a cardinal κ, we will use the notation $(\text{Mod-}R)^{<\kappa}$ for the class of all modules that possess a projective resolution consisting of $< \kappa$–generated projective modules. We also define $\text{mod-}R = (\text{Mod-}R)^{<\aleph_0}$. Notice that if the ring R is right coherent, then mod-R is just the class of all finitely presented modules. For an arbitrary class of modules $\mathcal{A}$, we define $\mathcal{A}^{<\kappa} = \mathcal{A} \cap (\text{Mod-}R)^{<\kappa}$.

Definition 1.3. Let R be a ring and $\mathcal{A}$ a class of modules.

Let κ be a cardinal. Then $\mathcal{A}$ is κ*–deconstructible* provided that each module $M \in \mathcal{A}$ is a transfinite extension of modules in $\mathcal{A}^{<\kappa}$.

$\mathcal{A}$ is called *deconstructible* in case there exists a cardinal κ such that $\mathcal{A}$ is κ–deconstructible.

By definition, any member M of a deconstructible class $\mathcal{A}$ is equipped with a chain witnessing that M is a transfinite extension of 'small' modules from $\mathcal{A}$. The surprising fact, known as the Hill Lemma, says that there is actually a large family of such chains available with several remarkable properties.

Lemma 1.4. (Hill Lemma) [21], [19, 4.2.6] *Let R be a ring, κ a regular infinite cardinal, and $\mathcal{C}$ a class of $<\kappa$–presented modules.*

Let M be a transfinite extension of modules in $\mathcal{C}$, and $\mathcal{M} = (M_\alpha \mid \alpha \leq \lambda)$ be a witnessing chain. Then there is a family $\mathcal{H}$ consisting of submodules of M such that

(1) $\mathcal{M} \subseteq \mathcal{H}$,
(2) *$\mathcal{H}$ is closed under arbitrary sums and intersections,*
(3) *P/N is a transfinite extension of modules in $\mathcal{C}$ for all $N, P \in \mathcal{H}$ such that $N \subseteq P$, and*
(4) *If $N \in \mathcal{H}$ and S is a subset of M of cardinality $<\kappa$, then there exists $P \in \mathcal{H}$ such that $N \cup S \subseteq P$ and P/N is $<\kappa$–presented.*

Typical applications of the Hill Lemma employ the family $\mathcal{H}$ in replacing the original witnessing chain $\mathcal{M}$ by a more suitable one, $\mathcal{M}' \subseteq \mathcal{H}$, which respects additional properties of the module M. In this way, one can eg. prove the following lemma (where, for a class of modules $\mathcal{C}$, we define $\mathcal{C}^\perp = \mathrm{KerExt}^1_R(\mathcal{C}, -)$).

Lemma 1.5. [31, §2] *Let R be a ring and $\mathcal{S}$ be a set of modules such that $R \in \mathcal{S}$. Let $\mathcal{A} = {}^\perp(\mathcal{S}^\perp)$. Let κ be a regular uncountable cardinal such that $\mathcal{S} \subseteq (Mod\text{-}R)^{<\kappa}$. Then the following are equivalent:*

(1) $M \in \mathcal{A}$;
(2) *M is a direct summand in a transfinite extension of modules in $\mathcal{S}$;*
(3) *M is a transfinite extension of modules in $\mathcal{A}^{<\kappa}$.*

In particular, $\mathcal{A}$ is κ–deconstructible.

Now, we will see that deconstructibility yields a sufficient condition for being a class of all roots of Ext.

Lemma 1.6. *Let R be a ring and $\mathcal{A}$ a class of modules closed under direct summands, transfinite extensions, and containing R.*

Then $\mathcal{A}$ is deconstructible if and only if there is a subset $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{A}^\perp = \mathcal{S}^\perp$. In this case, there is a class of modules $\mathcal{C}$ such that $\mathcal{A} = {}^\perp\mathcal{C}$.

Proof. The 'only if part' follows by the Eklof Lemma 1.1. For the 'if part', note that $\mathcal{A} \subseteq {}^\perp(\mathcal{S}^\perp)$ by assumption; the latter class is contained in $\mathcal{A}$ by the closure properties of $\mathcal{A}$ and by Lemma 1.5. The final claim thus holds for $\mathcal{C} = \mathcal{S}^\perp$. □

Most classes of roots of Ext are deconstructible in ZFC, but there are some whose deconstructibility is known to be independent of ZFC (However, it is not known whether there is an extension of ZFC in which *each* class of roots of Ext is deconstructible, see Open Problem 1.)

Example 1.7. Let R be a countable Dedekind domain which is not a field, and let $\mathcal{W}_1$ denote the class of all *Whitehead modules* (ie., the modules M

such that $\operatorname{Ext}^1_R(M,R)=0$). Then the deconstructibility of $\mathcal{W}_1$ is independent of ZFC + GCH, cf. [13].

Let R be an IC–valuation domain, ie., a valuation domain whose quotient field Q has projective dimension 1 as R–module, such that R has global dimension 2, and $\operatorname{Ext}^1_R(R/I,Q/R)\neq 0$ for each non–principal ideal I of R. Let $\mathcal{W}_2$ denote the class of all modules M satisfying $\operatorname{Ext}^2_R(M,R)=0$. Then the deconstructibility of $\mathcal{W}_2$ is independent of ZFC + GCH. For more details, we refer to [19, 10.2.14].

Example 1.8. On the positive side, we note that the class of all projective modules, $\mathcal{P}_0$, is $\aleph_1$ deconstructible by a classic result of Kaplansky [12, I.2.4]. Moreover, for each $1\leq n<\aleph_0$, the class $\mathcal{P}_n$ is κ^+–deconstructible where κ is the least infinite cardinal such that each right ideal of R is $\leq\kappa$–generated. For each $n<\aleph_0$, the class $\mathcal{F}_n$ is κ^+–deconstructible where κ is the least infinite cardinal $\geq\operatorname{card}(R)$, see eg. [19, §3.2 and §4.1].

Further examples of deconstructible classes of all roots of Ext and their applications will be presented in the following sections.

2. Deconstructibility and some of its applications

A remarkable property of deconstructible classes of all roots of Ext is their close connection to special approximations of modules.

Definition 2.1. Let R be a ring and $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is a *special approximation class* provided that for each module M there exist short exact sequences $0\to D\to C\xrightarrow{f}M\to 0$ and $0\to M\xrightarrow{g}D'\to C'\to 0$ where $C,C'\in\mathcal{C}$ and $D,D'\in\mathcal{C}^\perp$.

Applying the functor $\operatorname{Hom}_R(-,M)$ to the first exact sequence we see that each homomorphism $\overline{f}:\overline{C}\to M$ with $\overline{C}\in\mathcal{C}$ has a (not necessarily unique) factorization through f; so f is called a $\mathcal{C}$*–precover* of M. Dually, applying $\operatorname{Hom}_R(M,-)$ to the latter sequence, we infer that each homomorphism $\overline{g}: M\to\overline{D}$ with $\overline{D}\in\mathcal{C}^\perp$ has a (not necessarily unique) factorization through g; so g is called a $\mathcal{C}^\perp$*–preenvelope* of M.

Theorem 2.2. [14] *Let R be a ring and $\mathcal{C}$ a class of all roots of Ext. If $\mathcal{C}$ is deconstructible then $\mathcal{C}$ is a special approximation class.*

In particular, in view of Example 1.8, all the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ $(n\geq 0)$ are special approximation classes. We note that the reverse implication in Theorem 2.2 remains open (see Open Problem 3 below).

Deconstructibility of a class of all roots of Ext is often shown by proving the $\aleph_1$–deconstructibility. We will now present three results of this kind, each of which has a rather different proof. First we need a definition.

Definition 2.3. Let $\mathcal{X}$ be the class of all modules N of the following form: N is a submodule of a direct product $\prod_{\alpha<\lambda}I_\alpha$ of injective modules such that there is a regular uncountable cardinal $\kappa_N\leq\lambda$ such that N consists of all sequences $z \in \prod_{\alpha<\lambda} I_\alpha$ which are zero on a 'large' subset $I_z \subseteq \lambda$ (ie., on a subset I_z satisfying $\mathrm{card}(\lambda \setminus I_z) < \kappa_N$).

Notice that N is a directed union of injective modules, hence a pure epimorphic image of a direct sum of injective modules. Moreover, N is injective in case each right ideal of R is countably generated.

Lemma 2.4. [30, Theorem 15] (see also [19, 5.2.9]) *Let R be a ring, $\mathcal{T}$ be a class of modules closed under direct sums, and $\mathcal{C} = {}^{\perp_\infty}\mathcal{T}$. Assume that all modules in $\mathcal{C}$ have bounded projective dimension (ie., $\mathcal{C} \subseteq \mathcal{P}_n$ for some $n < \omega$), and that $\mathcal{C} \subseteq {}^{\perp_\infty}\mathcal{X}$. Then $\mathcal{C}$ is $\aleph_1$–deconstructible.*

In particular, $\mathcal{C}$ is $\aleph_1$–deconstructible whenever $\mathcal{C}$ is the class of all higher roots of Ext for a class $\mathcal{T}$ closed under direct sums, such that $\mathcal{C}$ consists of modules of bounded projective dimension, and R is right noetherian.

A dual result also holds true:

Lemma 2.5. [28, Theorem 1.9] *Let R be a right noetherian ring, $\mathcal{T}$ be a class of modules closed under direct sums, and $\mathcal{C} = {}^{\perp_\infty}\mathcal{T}$. Assume that all modules in $\mathcal{T}$ have bounded injective dimension. Then $\mathcal{C}$ is $\aleph_1$–deconstructible.*

A different argument yields the following (where a class of modules $\mathcal{T}$ is called *cogenerating* provided that each module embeds into a product of modules from $\mathcal{T}$):

Lemma 2.6. [27, Theorem and Remark 3.5] *Let R be a ring and $\mathcal{T}$ be a cogenerating class of modules closed under direct products and unions of well–ordered chains. Let $\mathcal{C} = {}^{\perp_\infty}\mathcal{T}$. Then $\mathcal{C}$ is $\aleph_1$–deconstructible.*

In order to understand the structure of the countably presented roots of Ext, the next result going back to [1] and [8] is very useful.

Lemma 2.7. [27, Proposition 2.7] *Let R be a ring and $\mathcal{B}$ be a class of modules closed under countable direct sums. Let $M \in {}^{\perp}\mathcal{B}$ be a countably presented module, and N be a pure submodule of a direct product of modules from $\mathcal{B}$. Then $\mathrm{Ext}^1_R(M, N) = 0$.*

As a sample application of the deconstruction tools above, we consider a class of modules $\mathcal{T}$ which is a *torsion class* (ie., $\mathcal{T}$ is closed under direct sums, homomorphic images and extensions). The roots of Ext for $\mathcal{T}$ are then called the *relative Baer modules* for $\mathcal{T}$.

If R is an integral domain and $\mathcal{T}$ is the class of all torsion modules, then the relative Baer modules for $\mathcal{T}$ are simply called the *Baer modules*. Kaplansky showed already in 1962 that Baer modules have projective dimension ≤ 1, but only recently Angeleri, Bazzoni, and Herbera have proved that Baer modules are actually projective [1]. A short direct proof, by an application of Lemmas 2.4 and 2.7, appears in [33, §3].

In general, relative Baer modules may have a rather complex structure. This is the case of Baer modules for artin algebras, for example: let R

be an arbitrary artin algebra, $\mathcal{S}$ a class of finitely generated modules of projective dimension ≤ 1 containing R and closed under extensions and direct summands, and $\mathcal{L} = (\text{mod-}R) \cap \mathcal{S}^\perp$. Let $\mathcal{T}_\mathcal{S}$ denote the torsion class of all homomorphic images of direct sums of copies of elements of $\mathcal{L}$. The deconstruction tools above then yield

Theorem 2.8. [4] *Let R be an artin algebra. Then the relative Baer modules for $\mathcal{T}_\mathcal{S}$ coincide with the transfinite extensions of modules in $\mathcal{S}$. In other words, the class of all relative Baer modules for $\mathcal{T}_\mathcal{S}$ is $\aleph_0$–deconstructible.*

In particular, when R is an indecomposable hereditary artin algebra, $\mathfrak{p}$ its preprojective component, and $\mathcal{S} = \text{add}(\mathfrak{p})$, then $\mathcal{B} = \mathcal{T}_\mathcal{S}$ is the class of all Baer modules in the sense of Ringel, Okoh and Lukas (see eg. [25], [26]). If R is moreover tame, then one can describe all countably generated modules in $\mathcal{B}$ more precisely, see [26, §4] and [3, §3]. But even in this particular case, the characterization in Theorem 2.8 (saying that $M \in \mathcal{B}$ is equivalent to M being a transfinite extension of modules in $\mathfrak{p}$) is the only one available for all uncountably generated modules in $\mathcal{B}$.

Now we turn to applications to (infinite dimensional) tilting theory.

Definition 2.9. Let R be a ring. A module T is called *tilting* privided that

- (T1) T has finite projective dimension;
- (T2) $\text{Ext}^i_R(T, T^{(I)}) = 0$ for each set I and each $1 \leq i < \omega$;
- (T3) There exists a finite exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that T_i is a direct summand in a direct sum of copies of T for each $i \leq r$.

If T is tilting then the class $\mathcal{T} = T^{\perp_\infty}$ is the *tilting class* induced by T.

Lemmas 2.4 and 2.6 are instrumental in proving

Theorem 2.10. [30], [10] *Let $\mathcal{T}$ be a tilting class. Then the class $\mathcal{A} = {}^\perp\mathcal{T}$ is $\aleph_1$–deconstructible.*

Moreover, there is a tilting module T such that T is a transfinite extension of modules in $\mathcal{A} \cap$ mod-R, and T induces $\mathcal{T}$.

The second assertion of Theorem 2.10 implies that tilting classes can be classified by *resolving subcategories* of mod-R (ie., the subclasses $\mathcal{S} \subseteq$ mod-R of bounded projective dimension, containing R, closed under extensions and direct summands, and containing the first term of any exact sequence once they contain the second and third). This classification takes a tilting class $\mathcal{T}$ to the resolving subcategory $\mathcal{S} = {}^\perp\mathcal{T} \cap \text{mod-}R$, cf. [19, §5.2].

The main applications of tilting modules and classes in the representation theory of artin algebras concern finitely generated modules (see eg. [5, Chap.VI]). The picture changes completely when considering applications to commutative rings. The point is the following result by Bazzoni, Colpi and Menini (cf. [11]) which we present here with a short alternative proof based on an idea from [32].

Lemma 2.11. *Let R be a commutative ring and $T \in$ mod-R.*

1. *Let T be of finite projective dimension $n \geq 1$. Then $Ext^n_R(T,T) \neq 0$.*
2. *If T is a tilting module then T is projective.*

Proof. (1) Let $\mathcal{O}$ be a projective resolution of T such that $\mathcal{O}$ consists of finitely generated modules. Denote by M the $(n-1)$th syzygy of T in $\mathcal{O}$. Then M is a finitely presented module of projective dimension 1, so there is a maximal ideal m of R such that M_m has projective dimension 1 as R_m–module. Moreover, M_m is the $(n-1)$th syzygy of T_m in $\mathcal{O}_m$ where $\mathcal{O}_m$ is the free resolution of the R_m–module T_m obtained by applying the exact (localization) functor $- \otimes_R R_m$ to $\mathcal{O}$.

Assume that $\mathrm{Ext}^n_R(T,T) = 0$. Then $\mathrm{Ext}^1_R(M,T) \cong \mathrm{Ext}^n_R(T,T) = 0$, so $\mathrm{Ext}^1_{R_m}(M_m, T_m) = 0$ by [16, 3.2.5]. Since $0 \neq T_m$ is finitely generated, T_m has a maximal R_m–submodule, and because M_m has projective dimension 1, we infer that $\mathrm{Ext}^1_{R_m}(M_m, R_m/m_m) = 0$.

Since R_m is a local ring, the finitely presented R_m–module M_m has a projective (= free) cover, so there is an exact sequence $0 \to K \subseteq F \to M_m \to 0$ where $0 \neq K$ is a finitely generated superfluos submodule of the finitely generated free module F. In particular, $K \subseteq \mathrm{Rad}(F)$, and there is an R_m–epimorphism $f : K \to R_m/m_m$. As $\mathrm{Ext}^1_{R_m}(M_m, R_m/m_m) = 0$, f can be extended to an R_m–epimorphism $g : F \to R_m/m_m$. Then $\mathrm{Ker}g$ is a maximal submodule of F, so $K \subseteq \mathrm{Rad}(F) \subseteq \mathrm{Ker}g$. This implies that $f = g \restriction K = 0$, a contradiction.

(2) By part (1), each module $T \in$ mod-R satisfying the conditions (T1) and (T2) of Definition 2.9 is projective. □

So applications of tilting theory to the commutative setting necessarily rest in the infinitely generated case. We finish by one such application, concerning the structure of localizations of commutative rings. It clarifies the case when the classical localization of a commutative ring R modulo R decomposes into a direct sum of countably presented R–modules. Its proof uses Theorem 2.10 and appears in [2]. (Recall that an element $r \in R$ is a *non–zero–divisor* if for each $s \in R$, $s = 0$ whenever $s.r = 0$ or $r.s = 0$.)

Theorem 2.12. *Let R be a commutative ring, and Σ be some multiplicative subset of R consisting of non–zero–divisors. Denote by R_Σ the localization of R at Σ. Then the following are equivalent:*

1. *R_Σ has projective dimension ≤ 1 as R–module.*
2. *$R_\Sigma \oplus R_\Sigma/R$ is a tilting R–module of projective dimension ≤ 1.*
3. *The R–module R_Σ/R decomposes into a direct sum of countably presented submodules.*

3. Modules of projective dimension ≤ 1

In this section, we apply the tools of Section 2 to a short alternative proof of the following recent result by Bazzoni and Herbera, whose particular instances for Prüfer and Matlis domains were proved in [24].

Theorem 3.1. [9, Corollary 8.1] *Let R be a semiprime Goldie ring.*

1. *Let M be a module. Then M has projective dimension ≤ 1 if and only if $Ext^1_R(M, D) = 0$ for each divisible module D.*
2. *$\mathcal{P}_1$ coincides with the class of all direct summands of transfinite extensions of cyclically presented modules.*

Before proving the result, we explain the notation and recall several well–known facts.

Let R be any ring. If r is a non–zero–divisor or $r = 0$, then the module R/rR is called *cyclically presented*. Clearly, cyclically presented modules are finitely presented and have projective dimension ≤ 1. The set of all cyclically presented modules will be denoted by $\mathcal{R}$.

Extending the standard notation from commutative domains, we call a module M *divisible* provided that $M.r = M$ for each non–zero–divisor $r \in R$. Equivalently, M is divisible if and only if $M \in \mathcal{R}^\perp$. We will denote by $\mathcal{D}$ the class of all divisible modules, and define $\mathcal{S} = {}^\perp\mathcal{D}$. Clearly, $\mathcal{R} \subseteq \mathcal{S} \subseteq \mathcal{P}_1$. (The modules in $\mathcal{S}$ were called *semi–Baer* in [24]; so Theorem 3.1 just says that if R is a semiprime Goldie ring then semi–Baer = of projective dimension ≤ 1, ie., $\mathcal{S} = \mathcal{P}_1$.)

We will also need the dual notion of a torsion–free module: a left R–module M is *torsion–free* provided that $r.m = 0$ implies $m = 0$ for each non–zero–divisor $r \in R$ and each $m \in M$. Applying the functor $- \otimes_R M$ to the exact sequence $0 \to R \to R \to R/rR \to 0$ for a non–zero–divisor $r \in R$, we see that $\{m \in M \mid r.m = 0\} \cong \mathrm{Tor}^1_R(R/rR, M)$, so M is torsion–free if and only if $\mathrm{Tor}^1_R(\mathcal{R}, M) = 0$.

For a right (left) R–module M, we denote by $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ the *character module* of M, so M^* is a left (right) R–module. The well–known Ext–Tor relations give that a (right R–) module D is divisible if and only if the left R–module D^* is torsion–free, and D is flat iff D^* is injective.

A ring R is called *right Goldie* provided that R contains no infinite direct sum of right ideals and has ACC on right annihilators. Similarly, *left Goldie* and *Goldie* (= left and right Goldie) rings are defined. By a classical result of Goldie, a ring R is semiprime right Goldie if and only if R has a classical right quotient ring Q_r which is semisimple artinian, and similarly on the left. Moreover, the classical left and right quotient rings of a semiprime Goldie ring R coincide: $Q = Q_r = Q_l$, and Q is flat both as a left and as a right R–module, see e.g. [20, 6.20 and Ex. 7G].

Let R be a semiprime right Goldie ring. Then any direct sum of copies of Q_r is injective, and these direct sums are exactly the divisible torsion–free modules, see [20, 7.12 and 7.13].

A module M is called *h–divisible* provided that M is a homomorphic image of an injective module. Any h–divisible module is divisible, but the converse fails in general (for integral domains R, h–divisible = divisible if and only if R is a Matlis domain, see [17, VII.2.8]). We denote by $\mathcal{H}$ the class of all h–divisible modules.

Since $\mathcal{H}$ is just the class of all first cosyzygies of all modules, we have $\mathcal{P}_1 = {}^{\perp}\mathcal{H} = {}^{\perp_\infty}\mathcal{H}$ for any ring R. So the question of whether $\mathcal{S} = \mathcal{P}_1$ just asks whether the (in general distinct) classes of all divisible and all h–divisible modules have the same classes of all roots of Ext.

Now, we can present the proof of Theorem 3.1.

Proof. Since $\mathcal{D} = \mathcal{R}^{\perp} = \mathcal{S}^{\perp}$, by Lemma 1.5, it suffices to show that $\mathcal{P}_1 \subseteq \mathcal{S}$.

First, we only assume that R is a semiprime *right* Goldie ring. Then the class $\mathcal{X}$ from Definition 2.3 satisfies $\mathcal{X} \subseteq \mathcal{H}$, because in this case h–divisible modules coincide with homomorphic images of direct sums of copies of Q. So Lemma 2.4 applies to $\mathcal{T} = \mathcal{H}$, and yields the $\aleph_1$–deconstructibility of $\mathcal{P}_1$. Since $\mathcal{S} = {}^{\perp}(\mathcal{R}^{\perp})$ is $\aleph_1$–deconstructible by Lemma 1.5, our claim is equivalent to proving that $\mathcal{P}_1^{<\aleph_1} \subseteq \mathcal{S}^{<\aleph_1}$.

Now, we will also assume that R is *left* Goldie. Let $M \in \mathcal{P}_1^{<\aleph_1}$. It remains to show that $\operatorname{Ext}^1_R(M, D) = 0$ for each divisible module D.

First we prove that D^{**} is h–divisible: since D^* is a torsion–free left R–module, the injective envelope $E = E(D^*)$ is an injective torsion–free left R–module, hence $E \cong Q^{(X)}$ for a set X. Then E is a flat left R-module, and D^{**} is a homomorphic image of the injective module E^*, so $D^{**} \in \mathcal{H}$.

Finally, D is a pure submodule in D^{**} and $\operatorname{Ext}^1_R(M, D^{**}) = 0$ because $M \in \mathcal{P}_1^{<\aleph_1}$, so $\operatorname{Ext}^1_R(M, D) = 0$ by Lemma 2.7. $\square$

4. Abstract elementary classes

Abstract elementary classes were designed by Shelah in [29] in order to extend classical model theory of first order structures to a much more general abstract setting.

Definition 4.1. A pair $(K, \prec_K)$ is said to be an *abstract elementary class (AEC)* if K is a class of τ–structures (for some fixed vocabulary τ), $\prec_K$ is a binary relation on K, both K and $\prec_K$ are closed under isomorphism and satisfy the following axioms:

Ax 1. If $M \prec_K N$ then M is a substructure of N (or $M \subseteq N$, for short).
Ax 2. $\prec_K$ is a partial order on K.
Ax 3. If $(A_i \mid i < \delta)$ is a continuous $\prec_K$–increasing chain then
 (1) $\bigcup_{i<\delta} A_i \in K$;
 (2) For each $j < \delta$, $A_j \prec_K \bigcup_{i<\delta} A_i$;
 (3) If each $A_i \prec_K M \in K$ then $\bigcup_{i<\delta} A_i \prec_K M$.
Ax 4. If $A, B, C \in K$, $A \prec_K C$, $B \prec_K C$ and $A \subseteq B$ then $A \prec_K B$.
Ax 5. There is a cardinal $\operatorname{LS}(K)$ such that if $A \subseteq B \in K$ there is $A' \in K$ with $A \subseteq A' \prec_K B$ and $\operatorname{card}(A') \leq \operatorname{card}(A) + \operatorname{LS}(K)$.

$\operatorname{LS}(K)$ is called the *Löwenheim–Skolem number* of K, cf. [6].

Recall that $(A_i \mid i < \delta)$ is a *continuous $\prec_K$–increasing chain* provided

- $A_i \in K$,
- $A_i \prec_K A_{i+1}$ for all $i < \delta$, and

- $A_i = \bigcup_{j<i} A_j$ for all limit ordinals $i < \delta$.

Also, the closure of $\prec_K$ under isomorphism means that if $M \prec_K N$ and $f : N \to N'$ is an isomorphism taking M to M' then $M' \prec_K N'$.

If $M \prec_K N$ we will say that M is a *strong substructure* of N. An embedding $f : M \to N$ is called *strong* provided that $\operatorname{Im}(f) \prec_K N$.

Example 4.2. The following are examples of AECs:

(1) The class of all models of a first–order theory with the relation of being an elementary submodel.
(2) The class of all modules over a ring with the relation of being a pure submodule.

In view of Example 4.2, it may come as a surprise that there exist AECs of a rather different sort, consisting of all higher roots of Ext. They were introduced in [7].

Definition 4.3. Let R be a ring and $\mathcal{C}$ a class of modules. Consider the pair $({}^{\perp_\infty}\mathcal{C}, \prec_{\mathcal{C}})$ where $A \prec_{\mathcal{C}} B$ denotes that A is a submodule of B such that $A, B, B/A \in {}^{\perp_\infty}\mathcal{C}$.

In general $({}^{\perp_\infty}\mathcal{C}, \prec_{\mathcal{C}})$ will not be an AEC, but the following result from [7] clarifies when this is the case. (Notice that the class ${}^{\perp_\infty}\mathcal{C}$ always has the *resolving property*, that is, $A \in {}^{\perp_\infty}\mathcal{C}$ whenever there is an exact sequence of modules $0 \to A \to B \to C \to 0$ such that $B, C \in {}^{\perp_\infty}\mathcal{C}$. This easily follows from the long exact sequence for Ext.)

Theorem 4.4. *Let R be a ring, κ an infinite cardinal $\geq \operatorname{card}(R)$, $\mathcal{C}$ a class of modules, and $\mathcal{A} = {}^{\perp_\infty}\mathcal{C}$. Then the following conditions are equivalent:*

(1) *$(\mathcal{A}, \prec_{\mathcal{C}})$ is an AEC with $LS(\mathcal{A}) = \kappa$;*
(2) *$\mathcal{A}$ is κ^+–deconstructible and closed under direct limits.*

Proof. We follow the proof in [7, §1].

(1) implies (2): First we show that $\mathcal{A}$ is closed under direct limits. It suffices to show that if $(M_\alpha, f_{\beta\alpha} \mid \alpha \leq \beta < \sigma)$ is a well–ordered direct system of modules such that $M_\alpha \in \mathcal{A}$ for all $\alpha < \sigma$, then $\varinjlim M_\alpha \in \mathcal{A}$. This is clear when σ is a non–limit ordinal, since then $\varinjlim M_\alpha = M_{\sigma-1} \in \mathcal{A}$ by assumption.

Assume σ is a limit ordinal. Let $0 \to K \hookrightarrow A \to M \to 0$ be the canonical presentation of $M = \varinjlim M_\alpha$, so $A = \bigoplus_{\alpha<\sigma} M_\alpha \in \mathcal{A}$ and K is the submodule of A generated by all elements of the form $x_{\beta\alpha} = m - f_{\beta\alpha}(m)$ where $\alpha \leq \beta < \sigma$ and $m \in M_\alpha$. Let K_γ denote the submodule of K generated only by the $x_{\beta\alpha}$'s with $\beta < \gamma$. Then $(K_\gamma \mid 1 \leq \gamma \leq \sigma)$ is a continuous chain of submodules of A, and $K_\sigma = K$.

By induction on $\gamma \leq \sigma$, we prove that $K_\gamma, A/K_\gamma \in {}^{\perp_\infty}\mathcal{C}$. This is clear for $K_1 = 0$. If γ is non–limit, then $K_\gamma + M_{\gamma-1} \supseteq \bigoplus_{\alpha<\gamma} M_\alpha$, hence $A = K_\gamma \oplus L_\gamma$ where L_γ denotes the direct summand of A generated by all M_α's with $\gamma - 1 \leq \alpha < \sigma$. Since $A/K_\gamma \cong L_\gamma$ is a direct sum of elements of $\mathcal{A}$, we have $A/K_\gamma \in \mathcal{A}$. Then $K_\gamma \in \mathcal{A}$ by the resolving property of the class $\mathcal{A}$.

Let γ be a limit ordinal. By the inductive premise, $K_\delta, A/K_\delta \in \mathcal{A}$ for each $\delta < \gamma$. Then $K_{\delta+1}/K_\delta \in \mathcal{A}$ by the resolving property, so $K_\gamma = \bigcup_{\delta<\gamma} K_\delta \in \mathcal{A}$ by the Eklof Lemma 1.1, and $A/K_\gamma = A/\bigcup_{\delta<\gamma} K_\delta \in \mathcal{A}$ by Ax 3.(3).

In particular, for $\gamma = \sigma$, we conclude that $M \cong A/K_\sigma \in \mathcal{A}$.

Next, we show that $\mathcal{A}$ is κ^+–deconstructible. Let $A \in \mathcal{A}$ and let X be a set of R–generators of A. By induction, we will construct a chain $(A_\alpha \mid \alpha \leq \sigma)$ consisting of strong submodules of A that will witness that A is a transfinite extension of modules in $\mathcal{A}^{<\kappa^+}$.

$A_0 = 0$, and if A_α is defined and there is $x \in A \setminus A_\alpha$ then applying Ax 5 to A/A_α gives a submodule $A_{\alpha+1}$ containing $A_\alpha + xR$ and such that $A_{\alpha+1}/A_\alpha, A/A_{\alpha+1} \in \mathcal{A}$ and $\operatorname{card}(A_{\alpha+1}/A_\alpha) \leq \operatorname{card}((A_\alpha + xR)/A_\alpha) + \kappa$. Since $(A_\alpha + xR)/A_\alpha \cong xR/(A_\alpha \cap xR)$, also $\operatorname{card}((A_\alpha + xR)/A_\alpha) \leq \kappa$, hence $\operatorname{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$. The resolving property of $\mathcal{A}$ gives $A_{\alpha+1} \in \mathcal{A}$.

For a limit ordinal α we define $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, so $A_\alpha \in \mathcal{A}$ by the Eklof Lemma 1.1, and $A/A_\alpha \in \mathcal{A}$ by Ax 3.(3).

(2) implies (1): Ax 1 is clear from the definition, and Ax 2 from $\mathcal{A}$ being closed under extensions. Ax 3.(1) and Ax 3.(2) follow from the Eklof Lemma 1.1.

Ax 3.(3) is where closure under direct limits is needed: for $\bigcup_{i<\delta} A_i \prec_K M$, we have to prove that $M/\bigcup_{i<\delta} A_i \in \mathcal{A}$, but $M/\bigcup_{i<\delta} A_i \cong \varinjlim_i M/A_i$ and $M/A_i \in \mathcal{A}$ by assumption. Ax 4 follows by the resolving property.

In order to prove Ax 5 we take an infinite cardinal $\kappa \geq \operatorname{card}(R)$ such that $\mathcal{A}$ is κ^+–deconstructible, and consider a chain $(B_i \mid i \leq \delta)$ witnessing that B is a transfinite extension of modules in $\mathcal{A}^{<\kappa^+}$. Let $\mathcal{H}$ denote the family corresponding to this witnessing chain by the Hill Lemma 1.4. Let $\{a_\alpha \mid \alpha < \lambda\}$ be a minimal generating subset of A.

By induction, we select from $\mathcal{H}$ an increasing chain $(A_\alpha \mid \alpha < \lambda)$ as follows: $A_0 = 0$; $A_{\alpha+1} = P$ where P is the module from Condition (4) of Lemma 1.4 for $N = A_\alpha$ and $S = \{a_\alpha\}$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ when $\alpha < \lambda$ is a limit ordinal (this is possible by Condition (2) of Lemma 1.4). Let $A' = \bigcup_{\alpha<\lambda} A_\alpha$. By Condition (3) of Lemma 1.4 and by Lemma 1.1, we have $A' \in \mathcal{H}$, and clearly $A \subseteq A'$. Since $\operatorname{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$ by construction, we have $\operatorname{card}(A') \leq \lambda \times \kappa \leq \operatorname{card}(A) + \kappa$, so we can take $\operatorname{LS}(\mathcal{A}) = \kappa$. □

The AEC $(\mathcal{A}, \prec_\mathcal{C})$ from Theorem 4.4 is called the *AEC of the roots of Ext* for $\mathcal{C}$.

Of course, a question remains of how to express the properties of ${}^{\perp_\infty}\mathcal{C}$ being deconstructible and closed under direct limits in terms of properties of the class $\mathcal{C}$. A property that guarantess this to hold is given by the following theorem.

Theorem 4.5. [15] *Let R be a ring and $\kappa = \max(\operatorname{card}(R), \aleph_0)$. Let $\mathcal{C}$ be any class of pure–injective modules. Then ${}^{\perp_\infty}\mathcal{C}$ is κ^+–deconstructible and closed under direct limits, ie., $({}^{\perp_\infty}\mathcal{C}, \prec_\mathcal{C})$ is an AEC with $LS(\mathcal{A}) = \kappa$.*

Remark 4.6. Assume $\mathcal{A} = {}^{\perp_\infty}\mathcal{C}$ is closed under direct limits. Then $\mathcal{A}$ contains all flat modules, and hence $\mathcal{C}$ consists of cotorsion modules (in the sense of Enochs [16]).

If moreover R is right noetherian and right hereditary, then the modules in $\mathcal{C}$ can be replaced by their pure–injective envelopes by [7, Lemma 1.10], so $\mathcal{A} = {}^{\perp_\infty}\mathcal{E}$ for a class of pure–injective modules $\mathcal{E}$, and Theorem 4.5 applies. Whether this extends to an arbitrary ring R remains open (see Open Problem 4 below).

Next, we consider model–theoretic properties of the AECs of the roots of Ext. First we recall the relevant definitions in the general setting of arbitrary AECs.

Definition 4.7. Let $\mathfrak{A} = (K, \prec_K)$ be an AEC.

(1) $\mathfrak{A}$ has a *prime model* provided there is $A \in K$ such that for each $B \in K$ there is a strong embedding $A \to B$.
(2) $\mathfrak{A}$ has *arbitrary large models* provided there is a cardinal κ such that for each $\lambda \geq \kappa$ there exists $A_\lambda \in K$ of cardinality λ.
(3) $\mathfrak{A}$ has *disjoint amalgamation* if the following holds true: if $A, B, C \in K$ satisfy $A \prec_K B$, $A \prec_K C$ and $A = B \cap C$ then there are $D \in K$ and $f : B \cup C \to D$ such that $f \restriction B$ and $f \restriction C$ are strong embeddings, and $f(A) = f(B) \cap f(C)$.
(4) $\mathfrak{A}$ is said to *admit closures* if $\mathrm{cl}(X) = \bigcap\{B \mid X \subseteq B \prec_K A\} \prec_K A$ whenever X is a subset of $A \in K$.

Lemma 4.8. *Let R be a ring, $\mathcal{C}$ be a class of modules such that $\mathfrak{A} = ({}^{\perp_\infty}\mathcal{C}, \prec_{\mathcal{C}})$ is an AEC. Then*

(1) *$\mathfrak{A}$ has a prime model, and has arbitrary large models.*
(2) *$\mathfrak{A}$ has disjoint amalgamation.*
(3) *Assume $\mathcal{C}$ consists of modules of injective dimension ≤ 1. Then $\mathfrak{A}$ admits closures if and only if ${}^{\perp_\infty}\mathcal{C}$ is closed under direct products.*

Proof. (1) is clear, because always $0 \in {}^{\perp_\infty}\mathcal{C}$ and ${}^{\perp_\infty}\mathcal{C}$ contains all free modules, and (2) holds because D can be constructed by a pushout of the strong embeddings $A \to C$ and $B \to C$. (3) is proved in [7, Lemma 2.8]. $\square$

As an example, we examine the case of Dedekind domains in more detail.

Example 4.9. [7] Let R be a Dedekind domain. Then the AECs of the roots of Ext correspod 1–1 to sets of maximal ideals:

If P is a set of maximal ideals, we let $\mathcal{A}_P$ denote the class of all modules M that are p–torsion–free for all $p \in P$ (ie., $\mathrm{Tor}_R^1(R/p, M) = 0$ for each $p \in P$). Then the AEC of the roots of Ext correspondig to P is $(\mathcal{A}_P, \prec)$ where $A \prec B$ if and only if $A, B, B/A \in \mathcal{A}_P$.

By Lemma 4.8, these AECs of the roots of Ext have all the properties (1)-(4) defined in 4.7.

An important model–theoretic property of general AECs is the finite character property introduced by Hyttinen and Kesälä in [22].

Definition 4.10. An AEC $(K, \prec_K)$ has *finite character* provided that for each $A \subseteq B$ we have $A \prec_K B$ whenever for each finite subset F of A there exists $C \in K$ and strong embeddings $f : A \to C$ and $g : B \to C$ such that $f \restriction F = g \restriction F$.

There is a handy test for finite character due to Kueker.

Lemma 4.11. [23] *If $(K, \prec_K)$ has amalgamation, then $(K, \prec_K)$ has finite character if and only if for each $A \subseteq B$ we have $A \prec_K B$ whenever for each finite subset F of A there is a strong embedding $f : A \to B$ such that $f \restriction F = id_F$.*

Finite character is often studied jointly with further properties: an AEC $\mathfrak{A} = (K, \prec_K)$ is called *finitary* provided that $\mathfrak{A}$ has a prime model and arbitrary large models, $\mathfrak{A}$ has has disjoint amalgamation and finite character, and both the vocabulary τ and the Löwenheim–Skolem number $\mathrm{LS}(K)$ are countable, cf. [22, §§2-6] and [23, §§2-6].

We finish by stating a recent result by the author extending the main result from [34] (cf. Theorem 4.5). Notice that here there are no assumptions on the size of the vocabulary or the Löwenheim–Skolem number (ie., no restriction on the size of the ring R).

Theorem 4.12. *Let R be a ring and $\mathcal{C}$ be a class of pure–injective modules. The the AEC $({}^{\perp_\infty}\mathcal{C}, \prec_{\mathcal{C}})$ has finite character.*

In view of Remark 4.6, Theorems 4.5 and 4.12 imply that *all* AECs of the roots of Ext over any right noetherian and right hereditary ring R have finite character (in particular, this holds when R is a Dedekind domain, cf. Example 4.9). It follows that if R is a countable right noetherian and right hereditary ring then all AECs of the roots of Ext are finitary. Whether this extends to non–noetherian or non–hereditary rings remains open (see Open Problem 5).

5. Open problems

1. *Is there an extension of ZFC in which all classes of the roots of Ext are deconstructible? In particular, does this happen assuming Gödel's Axiom of Constructibility ($V = L$)?*

In [28, Theorem 1.7] it is proved assuming V = L that the class of all higher roots of Ext for $\mathcal{C}$, where $\mathcal{C}$ is any *set* of modules of bounded injective dimension, is deconstructible.

2. *Assume that R is a right perfect ring (ie., R contains no infinite strictly decreasing chain of principal left ideals). Are all classes of the roots of Ext deconstructible?*

3. *Let R be any ring and $\mathcal{A}$ be a special approximation class of all roots of Ext. Is $\mathcal{A}$ deconstructible?*

4. *Does each AEC of the roots of Ext have the form* $({}^{\perp_\infty}\mathcal{C}, \prec_\mathcal{C})$ *for a class of pure–injective modules* $\mathcal{C}$*?*

The answer is positive when R is right noetherian and right hereditary (see Remark 4.6).

5. *(i) Does each AEC of the roots of Ext have finite character?*
(ii) Is each AEC of the roots of Ext over a countable ring R finitary?

By Theorem 4.12 a positive answer to 4. implies a positive answer to 5.(i), and the latter clearly implies a positive answer to 5.(ii).

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