QUASITRIVIAL SEMIMODULES III

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Abstract. The paper continues the investigation of semimodules. Main emphasis is laid on minimal (i.e., every proper subsemimodule has just one element), almost minimal and congruence-simple semimodules.

This paper is a continuation of [1] and [2] and we use the same notation. When referring to these two papers, we use e.g. I.4.1 for Proposition 4.1 from [1] and II.2 for section 2 from [2].

1. Almost minimal semimodules (a)

A left semimodule ${}_SM$ will be called *almost minimal* if it has both an additively absorbing element o_M and an additively neutral element 0_M and if $So = o \neq 0 = S0$, $Sx = M$ for every $x \in M \setminus P$, $P = \{o, 0\}$, $|P| = 2$. Throughout this section, let M be almost minimal.

1.1 Lemma. (i) *$\{o\}$, $\{0\}$, P and M are just all subsemimodules of ${}_SM$.*
(ii) *${}_SM$ has either three (iff $|M| = 2$) or four (iff $|M| \geq 3$) different subsemimodules.*

(iii) *$P = P({}_SM) = Q({}_SM)$.*
(iv) *${}_SM$ is quasitrivial if and only if it is minimal and if and only if $|M| = 2$ (then ${}_SM \simeq Q_{1,S}$ - see I.3.2).*

Proof. Easy. □

1.2 Lemma. *$x + y \neq 0$ for all $x, y \in M$, $x \neq 0$.*

Proof. Assume, on the contrary, that $x + y = 0$. Then $x \notin P$, and hence $sx = o$ for some $s \in S$. Now, $o = o + sy = sx + sy = s(x + y) = s0 = 0$, a contradiction. □

1.3 Lemma. *Put $\eta = \eta_0$ (see II.2). Then:*
(i) *η is a congruence of ${}_SM$ and $(x, y) \in \eta$ if and only if $\{ s \mid xs = 0 \} = \{ s \mid sy = 0 \}$.*
(ii) *$(x, 0) \notin \eta$ for every $x \neq 0$.*
(iii) *$(y, o) \notin \eta$ for every $y \neq o$.*
(iv) *$\eta \neq M \times M$.*
(v) *$\eta = \eta_o$.*

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(vi) $(x, 2x) \in \eta$ *for every* $x \in M$.
(vii) η *is the unique (proper) maximal congruence of* ${}_SM$.

Proof. By 1.2 and II.2.2, η is a congruence of ${}_SM$. Moreover, $(0:0) = S$, $(o:0) = \emptyset$ and $\emptyset \neq (x:0) \neq S$ for every $x \notin P$. Now, the assertions (i) – (iv) are clear.

Let $(x, y) \in \eta$. If $s \in (x:o)$ then $(o, sy) = (sx, sy) \in \eta$, $sy = o$ by (iii) and $s \in (y:o)$. We have shown that $(x:o) \subseteq (y:o)$. Symmetrically, $(y:o) \subseteq (x:o)$, so that $(x:o) = (y:o)$ and $(x,y) \in \eta_o$. Thus $\eta \subseteq \eta_o$.

Let $(u, v) \in \eta_o$. If $s \in (u:0)$ then $(0, sv) = (su, sv) \in \eta_o$. That is, $\emptyset = (0:o) = (sv:o)$, and therefore $sv = 0$ and $s \in (v:0)$. We have shown that $(u:0) \subseteq (v:0)$. Symmetrically, $(v:0) \subseteq (u:0)$, so that $(u:0) = (v:0)$ and $(u,v) \in \eta_0 = \eta$. Thus $\eta_o \subseteq \eta$.

Let $x \in M$. If $sx = 0$ then $s2x = 2sx = 0$. Conversely, if $r2x = 0$ then $rx + rx = 0$ and $rx = 0$ by 1.2. Thus $(x, 2x) \in \eta$.

Finally, let σ be a proper congruence of ${}_SM$. If $(o, 0) \in \sigma$ then$(o, x) = (o + x, 0 + x) \in \sigma$ for every $x \in M$, so that $\sigma = M \times M$, a contradiction. It follows that $(o, 0) \notin \sigma$. Similarly, if $(o, x) \in \sigma$ for some $x \neq o$ then $sx = 0$, $s \in S$, and we get $(o, 0) = (so, sx) \in \sigma$, a contradiction. Consequently, if $(x, y) \in \sigma$, $x \neq y$, then $x \neq o \neq y$. Moreover, if $tx = o$ then $(o, ty) \in \sigma$ and $ty = o$. Similarly the other case and we see that $(x, y) \in \eta_o = \eta$ (by (v)). Thus $\sigma \subseteq \eta$. □

1.4 Proposition. ${}_SN = {}_SM/\eta$ *is an (additively) idempotent congruence-simple almost minimal semimodule. If* ${}_SM$ *is not quasitrivial then the same is true for* ${}_SN$.

Proof. Combine 1.3 and 1.1(i). □

1.5 Corollary. *The following conditions are equivalent:*

(i) ${}_SM$ *is congruence-simple.*
(ii) $\eta = \mathrm{id}_M$.
(iii) *If* $x, y \in M \setminus P$ *are such that* $x \neq y$ *then* $0 \in \{sx, sy\}$ *and* $sx \neq sy$ *for at least one* $s \in S$. □

1.6 Lemma. *If* $(x, y) \in \eta$ *then* $\{\, u \mid x + u = o \,\} = \{\, v \mid y + v = o \}$.

Proof. If $x + u = o$ then $(o, y + u) = (x + u, y + u) \in \eta$, and hence $y + u = o$. □

1.7 Lemma. *Either* $M(+)$ *is idempotent or* $\mathrm{Id}(M(+)) = P$.

Proof. $\mathrm{Id}(M(+))$ is a subsemimodule of ${}_SM$ and $P \subseteq \mathrm{Id}(M(+))$. □

1.8 Lemma. $\eta_w \nsubseteq \eta$ *for every* $w \in M \setminus P$.

Proof. If $w \notin P$ then $(0:w) = \emptyset = (o:w)$, and hence $(0, o) \in \eta_w$. □

2. Almost minimal semimodules (b)

This section is an immediate continuation of the preceding one.

2.1 Lemma. (i) *The set* $(x:0)$ *is a left ideal of the semiring* S *for every* $x \in M \setminus \{o\}$.
(ii) $(x:0)y$ *is a subsemimodule of* ${}_SM$ *for all* $x, y \in M$, $x \neq o$.
(iii) $(x:0) \cap (y:0) = (x + y:0)$ *for all* $x, y \in M$.
(iv) $(x:0)y = \{o\}$ *if and only if* $x \neq o = x + y$.

Proof. (i) and (ii) are checked easily, while (iii) follows from 1.2. As concerns (iv), assume first that $(x:0)y = o$. Then $(x:0) \neq \emptyset$, and so $x \neq o$. Moreover, by (iii), $\emptyset = (x:0) \cap (y:0) = (x+y:0)$, and therefore $x+y = o$. Conversely, if $x \neq o = x+y$ then $(x:0) \cap (y:0) = (o:0) = \emptyset$ by (iii), and hence $0 \notin (x:0)y$. By (ii), $(x:0)y$ is a subsemimodule of ${}_SM$ and $(x:0)y = o$ now follows from 1.1(i). $\square$

2.2 Lemma. *The following conditions are equivalent for $x, y \in M$:*

(i) $(x:0)y \subseteq \{0\}$.
(ii) $(x:0) \subseteq (y:0)$.
(iii) $(x, x+y) \in \eta$.

Moreover, if ${}_SM$ is congruence-simple then these conditions are equivalent to:

(iv) $x + y = x$.

Proof. (i) implies (ii) trivially.
(ii) implies (iii). By 2.1(iii), $(x+y:0) = (x:0)$, so that $(x+y, x) \in \eta$.
(iii) implies (i). We have $(x:0) = (x+y:0) = (x:0) \cap (y:0)$, and hence $(x:0) \subseteq (y:0)$ and $(x:0)y \subseteq \{0\}$.
Assume, finally, that ${}_SM$ is congruence-simple. Then $\eta = \mathrm{id}_M$ by 1.5, and therefore the conditions (iii) and (iv) coincide in this case. $\square$

2.3 Lemma. *The following conditions are equivalent for $x, y \in M$:*

(i) $(x:0)y = \{0\}$.
(ii) $x \neq o$ *and* $(x:0) \subseteq (y:0)$.
(iii) $x \neq o$ *and* $(x, x+y) \in \eta$.

Moreover, if ${}_SM$ is congruence-simple then these conditions are euiqvalent to:

(iv) $x + y = x \neq o$.

Proof. We have $(x:0) \neq \emptyset$ for $x \neq o$ and the rest is clear from 2.2. $\square$

2.4 Lemma. *Assume that ${}_SM$ is congruence-simple. If $x, y \in M$ are such that $x + y \neq x$ then there is at least one $t \in S$ with $tx = 0$ and $ty = o$.*

Proof. Since $x + y \neq x$, we have $x \neq o$ and $(x:0) \neq \emptyset$. Now, it follows from 2.1(ii) and 2.2 that $o \in (x:0)y$ and our result is clear. $\square$

2.5 Lemma. (i) *The set $(x:o)$ is a left ideal of the semiring S for every $x \in M \setminus \{0\}$.*
(ii) $(x:o) + S \subseteq (x:o)$ *for every $x \in M \setminus \{0\}$.*
(iii) $(x:o)y$ *is a subsemimodule of ${}_SM$ for all $x, y \in M$, $x \neq 0$.*
(iv) $(x:o)y + M \subseteq (x:o)y$ *for all $x, y \in M$, $x \neq 0 \neq y$.*

Proof. (i), (ii) and (iii). Since $x \neq 0$, we have $(x:o) \neq \emptyset$ and the remaining assertions are easy to check.
(iv) If $y = o$ then $(x:o)y = \{o\}$. If $y \neq o$, $s \in (x:o)$ and $z \in M$ then $z = ry$ for some $r \in S$ and $sy + z = sy + ry = (s+r)y \in (x:o)y$, since $s + r \in (x:o)$ by (ii). $\square$

2.6 Lemma. (i) $(0:o)y = \emptyset$ *for every $y \in M$.*
(ii) $(o:o)o = \{o\}$.
(iii) $(o:o)0 = \{0\}$.
(iv) $(o:o)y = M$ *for every $y \in M \setminus P$.*

Proof. We have $(0 : o) = \emptyset$, $(o : o) = S$ and the rest is clear. □

2.7 Lemma. *Let $x \in M \setminus P$. Then:*
(i) $(x : o)o = \{o\}$.
(ii) $(x : o)0 = \{0\}$.
(iii) *If $(x : o) \subseteq (y : o)$, $y \in M$, then $(x : o)y = \{o\}$.*

Proof. We have $(x : o) \neq \emptyset$ and the rest is clear. □

2.8 Lemma. *Assume that ${}_SM$ is congruence-simple. If $x, y \in M$, $y \neq 0$, then either $(x : o)y = \emptyset$ or $(x : o)y = \{o\}$ or $(x : o)y = M$.*

Proof. Put $K = (x : o)y$ and $\alpha = (K \times K) \cup \mathrm{id}_M$. By 2.5(iii) and 2.5(iv), we see that α is a congruence of ${}_SM$. If $\alpha = \mathrm{id}_M$ then either $K = \emptyset$ or $K = \{o\}$. If $\alpha - M \times M$ then $K = M$. □

2.9 Lemma. *Assume that ${}_SM$ is congruence-simple. Let $x, y \in M \setminus \{0\}$. If $(x : o) \nsubseteq (y : o)$ then $(x : o)y = M$ (and hence for every $z \in M$ there is at least one $t \in S$ with $tx = o$ and $ty = z$).*

Proof. Since $x \neq 0$, we have $(x : o) \neq \emptyset$. Moreover, $(x : o) \nsubseteq (y : o)$, and hence $(x : o)y \neq \{o\}$. Now, $(x : o)y = M$ by 2.8. □

2.10 Lemma. *Assume that ${}_SM$ is congruence-simple. Let $x, y \in M$ be such that $x + y = x \neq y$. Then:*
(i) *$x \neq 0$, $y \neq o$ and $(x : o) \nsubseteq (y : o)$.*
(ii) *If $y \neq 0$ then for every $z \in M$ there is at least one $t \in S$ with $tx = o$ and $ty = z$.*

Proof. (i) Since $x+y = x \neq y$, we have $x \neq 0$ and $y \neq o$. Moreover, $(y : o) \subseteq (x : o)$. But $\eta = \mathrm{id}_M$ and $x \neq y$. Thus $(x : o) \nsubseteq (y : o)$.
(ii) Combine (i) and 2.9. □

3. Almost minimal semimodules (c)

Throughout this section, let ${}_SM$ be an almost minimal semimodule that is not quasitrivial (see 1.1(iv)).

3.1 Lemma. (i) *The semiring S is not left quasitrivial.*
(ii) *The semiring S contains no left multiplicatively absorbing element.*
(iii) *The homomorphism $\varphi : S \to \mathrm{End}(M(+))$ given by $(\varphi(s))(x) = sx$* (see II.4.1) *is injective, provided that S is congruence-simple.*

Proof. (i) and (ii). Since ${}_SM$ is not quasitrivial, we can find $x \in M \setminus P$ and then ${}_SM = Sx$ is a homomorphic image of ${}_SS$. Now, if $q \in S$ were left multiplicatively absorbing then $qM = qSx = qx$, and so $|qM| = 1$. But $q0 = 0 \neq o = qo$, a contradiction.
(iii) Use II.4.1(v). □

3.2 Lemma. *Assume that M is finite. Then there is at least one $q \in S$ such that:*
(i) *$qx = o$ for every $x \in M \setminus \{0\}$.*
(ii) *$qy = (q + s)y$ for all $s \in S$ and $y \in M$.*
(iii) *$qz = tqz$ for all $t \in S$ and $z \in M$.*

Proof. For every $x \in M \setminus \{0\}$ there is $q_x \in S$ with $q_x x = o$. Put $q = \sum q_x$, $x \in M$, $x \neq 0$. Then $q(M \setminus \{0\}) = o$. Moreover, if $y \neq 0$ then $(q+s)y = qy+sy = o+sy = o$. Of course, $(q + s)0 = 0 = qy$. Similarly, if $z \neq 0$ then $sqz = so = o = qz$. Again, $sq0 = 0 = q0$. □

3.3 Proposition. *Assume that S is congruence-simple and M is finite. Then S contains an additively absorbing element o_S such that o_S is right multiplicatively absorbing. On the other hand, S has no left multiplicatively absorbing element.*

Proof. Combine 3.1(ii), 3.1(iii), 3.2(ii) and 3.2 (iii). □

3.4 Lemma. *Assume that ${}_SM$ is finite and congruence-simple. Then for every $u \in M \setminus \{o\}$ there is at least one $t \in S$ such that $tx = 0$ if $x + u = u$ and $tx = o$ if $x + u \neq u$.*

Proof. Put $L = \{\, x \mid x + u \neq u \,\}$. Then L is a non-empty finite set (we have $o \in L$ and $0 \notin L$) and for every $x \in L$ there is $t_x \in S$ with $t_x x = o$ and $t_x u = 0$. Put $t = \sum t_x$, $x \in L$. Then $tL = o$ and $tu = 0$. Now, if $y + u = u$ then $0 = tu = ty + tu = ty$. □

3.5 Lemma. *Assume that ${}_SM$ is finite and congruence-simple. Then for all $u \in M \setminus P$ and $v \in M$ there is at least one $s \in S$ such that $su = v$, $sx + v = v$ if $x + u = u$ and $sx = o$ if $x + u \neq u$.*

Proof. By 3.4, there is $t \in S$ with $tx = 0$ if $x+u = u$ and $tx = o$ if $x+u \neq u$. Since $u \notin P$, there is $r \in S$ with $ru = v$. Put $s = r + t$. Then $su = ru + tu = v + 0 = v$. If $x + u = u$ then $v = su = sx + su = sx + v$. If $x + u \neq u$ then $sx = rx + tx = rx + o = o$. □

4. A sort of minimal semimodules (a)

In this section, let ${}_SM$ be a minimal semimodule such that $o = o_M \in M$ and $So = o$ (i.e., $o \in P({}_SM)$). If ${}_SM$ is quasitrivial then $|M| = 2$ and ${}_SM$ is isomorphic to one of the semimodules $Q_{1,S}$, $Q_{2,S}$ and $Q_{4,S}$ (see I.4.1). Now, we will assume that ${}_SM$ is not quasitrivial. Then $Q({}_SM) = P({}_SM) = \{o\}$.

4.1 Lemma. (i) *$\{o\}$ and M are just all subsemimodules of ${}_SM$.*
(ii) *For all $x, y \in M$, $x \neq o$, there is at least one $s \in S$ with $sx = y$.*

Proof. It is easy. □

4.2 Lemma. (i) *η_o is an equivalence* (see II.2).
(ii) *If $(x, y) \in \eta_o$ then $(sx, sy) \in \eta_o$ for every $s \in S$.*
(iii) *$(x, o) \notin \eta_o$ for every $x \in M$, $x \neq o$.*

Proof. It is easy. □

4.3 Lemma. *Define a relation λ_o on M by $(x, y) \in \lambda$ if and only if $(x : o) \subseteq (y : o)$. Then:*
(i) *λ_o is a quasiordering (i.e., it is reflexive and transitive).*
(ii) *$\ker(\lambda_o) = \eta_o$.*
(iii) *$(x, o) \in \lambda_o$ for every $x \in M$.*
(iv) *$(o, y) \notin \lambda_o$ for every $y \in M \setminus \{o\}$.*
(v) *$(x, x + y) \in \lambda_o$ for all $x, y \in M$.*

Proof. It is easy. □

4.4 Lemma. *The following conditions are equivalent for $x, y \in M$:*

(i) *$(x, y) \in \lambda_o$.*
(ii) *$(x : o)y = \{o\}$.*
(iii) *$(x : o)y \neq M$.*

Proof. Use the fact that $(x : o)y$ is a subsemimodule of ${}_SM$. □

4.5 Lemma. *Let $x \in M$, $x \neq o$, be such that the set $L = \{ y \in M \mid (y, x) \notin \lambda_o \}$ is finite. Then for every $z \in M$ there is at least one $s \in S$ such that $sx = z$ and $sy = o$ for every $y \in L$.*

Proof. By 4.4, $(y : o)x = M$, and so there is $s_y \in S$ with $s_y y = o$ and $s_y x = z$. Now, we put $s = \sum s_y$, $y \in L$. □

4.6 Lemma. *Assume that M is finite. Then $tM = \{o\}$ for at least one $t \in S$.*

Proof. For every $x \in M$, there is $t_x \in S$ with $t_x x = o$. Now, we put $t = \sum t_x$, $x \in M$. □

4.7 Lemma. *Assume that the semiring S is congruence-simple and M is finite. Then S contains a bi-absorbing element o_S such that $o_S M = \{o\}$.*

Proof. See II.4.3. □

5. Partial summary

5.1 Lemma. *Let ${}_SM$ be a semimodule such that $I = M$ whenever I is a subsemimodule of ${}_SM$ with $I + M \subseteq I$ and $|I| \geq 2$ (e.g., ${}_SM$ congruence-simple). If $w \in P({}_SM)$ (i.e., $Sw = w$) then either $w = 0_M$ or $w = o_M$.*

Proof. Put $I = M + w$. Then $(I + M) \cup SI \subseteq I$ and $w \in I$. If $I = M$ then $w = 0_M$. If $|I| = 1$ then $w = o_M$. □

5.2 Corollary. *Let ${}_SM$ be a semimodule as in 5.1. Then $|P({}_SM)| \leq 2$.* □

5.3 Lemma. *Let S be a bi-ideal-simple semiring (e.g., S congruence-simple). If $q \in S$ is multiplicatively absorbing then either $q = 0_S$ is additively neutral or $q = o_S$ is bi-absorbing.*

Proof. The set $S + q$ is a bi-ideal of S. □

5.4 Proposition. *The following conditions are equivalent for a congruence-simple semiring S:*

- (i) *S is finite, not left quasitrivial and S has the multiplicatively absorbing element q (then either $q = 0_S$ is additively neutral or $q = o_S$ is bi-absorbing* - see 5.3*).*
- (ii) *There is a finite non-quasitrivial minimal semimodule ${}_SM$ with $Q({}_SM) \neq \emptyset$.*
- (iii) *There is a finite non-quasitrivial congruence-simple minimal semimodule ${}_SN$ with $Q({}_SN) \neq \emptyset$.*

Proof. (i) implies (ii). By I.7.5, there exists a finite minimal semimodule ${}_SM$ that is not quasitrivial. Moreover, by I.7.6(ii), we have $P({}_SM) \neq \emptyset$.
(ii) implies (iii). By I.6.3, there is a congruence ϱ of ${}_SM$ such that ${}_SN = {}_SM/\varrho$ is minimal, congruence-simple and not quasitrivial. Obviously, N is finite and $Q({}_SM)/\varrho \subseteq Q({}_SN)$.
(iii) implies (i). By I.5.9, the semiring S is finite and it is not left quasitrivial due to I.5.8(ii). Furthermore, by II.3.1, $Q({}_SN) = P({}_SN) = \{w\}$, $Sw = w$ and, by II.3.4, either $w = 0_M$ or $w = o_M$ (see also II.4.4(ii)). Finally, by II.4.4(iii) and II.4.4(iv), the semiring S contains the multiplicatively absorbing element q and either $q = 0_S$ or $q = o_S$. □

5.5 Proposition. *Let S be a semiring satisfying the equivalent conditions of 5.4 and let ${}_SM$ be a (finite) non-quasitrivial congruence-simple minimal semimodule. Then just one of the following two cases holds:*

1. *S contains the additively neutral and multiplicatively absorbing element 0_S, $\mathrm{Ann}({}_SM) = \{0_S\}$, $Q({}_SM) = P({}_SM) = \{0_M\}$ and $S \cdot 0_M = 0_M = 0_S \cdot M$;*
2. *S contains the bi-absorbing element o_S, $\mathrm{Ann}({}_SM) = \{o_S\}$, $Q({}_SM) = P({}_SM) = \{o_M\}$ and $S \cdot o_M = o_M = o_S \cdot M$.*

Proof. We have $M = Sx$ for any $x \in M \setminus Q({}_SM)$. The rest is clear from 5.4 and II.4.4. □

5.6 Lemma. *Let ${}_SM$ be a finite minimal semimodule such that $Q({}_SM) = \emptyset$.*
(i) *If $M(+)$ is idempotent then $M(+)$ has an absorbing element o_M.*
(ii) *If $o_M \in M$ then $qM = o_M$ for at least one $q \in S$.*
(iii) *If S is congruence-simple then q is uniquely determined, q is both additively and left multiplicatively absorbing in S and q is not right multiplicatively absorbing (consequently, S has no right multiplicatively absorbing element at all).*

Proof. (i) We have $o_M = \sum x$, $x \in M$.
(ii) We have $Sx = M$ for every $x \in M$, and so $q_x x = o_M$ for some $q_x \in S$. If $q = \sum q_x$, $x \in M$, then $qM = o_M$.
(iii) By II.4.3(i) and II.4.3(v), q is both additively and left multiplicatively absorbing in S. In particular, q is uniquely determined. On the other hand, it follows from II.4.5(ii) that S has no right multiplicatively absorbing element. □

5.7 Lemma. *Let S be a congruence-simple semiring. Then at least one of the following two cases holds:*

1. *$Q({}_SM) \neq \emptyset$ for every finite minimal left semimodule ${}_SM$;*
2. *$Q(N_S) \neq \emptyset$ for every finite minimal right semimodule N_S.*

Proof. Let ${}_SM$ be a finite minimal left semimodule with $Q({}_SM) = \emptyset$. Since $M(+)$ is a finite (commutative) semigroup, the set I of idempotent elements of $M(+)$ is non-empty. Moreover, I is a subsemimodule of ${}_SM$. Now, if $I = \{w\}$ is one-element then $Sw = w$ and $w \in Q({}_SM) = \emptyset$, a contradiction. Thus $|I| \geq 2$ and we get $I = M$, since M is minimal. That is, $M(+)$ is idempotent and it follows from 5.6 that S has a left multiplicatively absorbing element but no right one. The rest is clear. □

5.8 Lemma. (i) *If S is a finite semiring then every minimal (left, right) semimodule is finite.*
(ii) *If S is a congruence-simple semiring such that there exists a non-quasitrivial finite (left, right) semimodule then S is finite.*

Proof. See I.5.10 and I.5.9. □

5.9 Classification. Now, (finite congruence-simple) semirings S will be divided into the following four pair-wise disjoint classes:

(A) There exists at least one non-quasitrivial minimal left S-semimodule and at least one non-quasitrivial minimal right S-semimodule.
(B) There exists at least one non-quasitrivial minimal left semimodule and all minimal right semimodules are quasitrivial.

(C) There exists at least one non-quasitrivial minimal right semimodule and all minimal left semimodules are quasitrivial.

(D) All minimal left or right semimodules are quasitrivial.

(Notice that the classes (B) and (C) are dual via forming the opposite semirings.)

5.10 Proposition. *Let S be a finite congruence-simple semiring of type (A). Then:*
(i) *S is neither left nor right quasitrivial.*
(ii) *S contains the multiplicatively absorbing element q such that either $q = 0_S$ is additively neutral or $q = o_S$ is bi-absorbing.*
(iii) *If $q = 0_S$ then either S is additively idempotent or S is a ring.*
(iv) *If $q = o_S$ then either S is additively idempotent or $S + S = \{o_S\}$.*
(v) *If ${}_SM$ (N_S, resp.) is a non-quasitrivial minimal left (right, resp.) semimodule then M (N, resp.) is finite and $Q({}_SM) \neq \emptyset$ ($Q(N_S) \neq \emptyset$, resp.)* (see 5.5 and II.4.4).

Proof. First, it follows from I.5.8(ii) (and its dual) that S is neither left nor right quasitrivial. Now, let ${}_SM$ (N_S, resp.) be a non-quasitrivial minimal left (right, resp.) seminodule. By 5.8(i), M (N, resp.) is finite. Moreover, taking into account 5.7, we can assume that $Q({}_SM) \neq \emptyset$ (the other case being dual). Now, by 5.4, S has the multiplicatively absorbing element q such that either $q = 0_S$ is additively neutral or $q = o_S$ is bi-absorbing.

Assume that $q = 0_S$ and that ${}_SM$ is congruence-simple (see I.6.3). By 5.5(1), we have $0_M \in M$ and $S0_M = 0_M = 0_SM$. Define a relation κ on M by $(x, y) \in \kappa$ iff $x + u = my$ and $y + v = nx$ for some $u, v \in M$ and positive integers m, n. It is easy to check that κ is a congruence of ${}_SM$ and $(z, 2z) \in \kappa$ for every $z \in M$. If $\kappa = \mathrm{id}_M$ then $z = 2z$ and $M(+)$ is idempotent. On the other hand, if $\kappa \neq \mathrm{id}_M$ then $\kappa = M \times M$, $(z, 0_M) \in \kappa$ for every $z \in M$ and this fact easily implies that $M(+)$ is a group, i.e., M is a module. However, by II.4.1(v), the semiring S is isomorphic to a subsemiring of the (finite) semiring $\mathrm{End}(M(+))$ and we conclude that either S is additively idempotent or it is a ring.

Next, assume that $q = o_S$ and that ${}_SM$ is congruence-simple (see I.6.3). By 5.5(2), $So_M = o_M = o_SM$. Consider the congruence κ of ${}_SM$. If $\kappa = \mathrm{id}_M$ then $M(+)$ is idempotent and the same is true for $S(+)$. If $\kappa = M \times M$ then, for every $z \in M$, $(z, 0_M) \in \kappa$, and so $mz = o_M$ for a positive integer m. The set $J = \{\, z \mid 2z = o_M \,\}$ is a subsemimodule of ${}_SM$. If $|J| = 1$ then $J = \{o_M\}$ and $2w \neq o_M$ for every $w \in M \setminus \{o_M\}$. Now, if n is the smallest positive integer with $nw = o_M$ then $w \geq 3$, $(n-1)w \neq o_M$ and $(n-1)w \in J$, a contradiction. Thus $|J| \geq 2$ and we have $J = M$, since M is minimal. We have shown that $2x = o_M$ for every $x \in M$. Further, put $\theta = ((M + M) \times (M + M)) \cup \mathrm{id}_M$. Again, θ is a congruence of ${}_SM$. If $\theta = \mathrm{id}_M$ then $M + M = \{o_M\}$ and $S + S = \{o_S\}$ by II.4.1(v). If $\theta = M \times M$ then $M + M = M$ and $M(+)$ is a non-trivial commutative nil-semigroup of index 2 and without irreducible elements. However, any such semigroup is infinite, a contradiction.

Finally, if $Q(N_S) = \emptyset$ then, proceeding similarly as in the proof of 5.7, we can show that $N(+)$ is idempotent and S has no left multiplicatively absorbing element, a contradiction. □

5.11 Remark. Let S be a finite congruence-simple semiring of type (A) (see 5.10).
(i) If S is a ring then S is a copy of a matrix ring over a (finite) field (use I.5.7 and

the fact that S is not quasitrivial). Non-quasitrivial minimal semimodules are just the usual simple modules.
(ii) If $S + S = \{o_S\}$ then the multiplicative semigroup $S(\cdot)$ is congruence-simple (see e.g. [???]).
(iii) Let S be additively idempotent. Then S has the multiplicatively absorbing element q and either $q = 0_S$ is additively neutral or $q = o_S$ is bi-absorbing.

Assume that $q = 0_S$ (the subtype (A1)). If ${}_SM$ (N_S, resp.) is a non-quasitrivial minimal semimodule then $0_M \in M$ ($0_N \in N$, resp.) and $S \cdot 0_M = \{0_M\} = 0_S \cdot M$ ($0_N \cdot S = \{0_N\} = N \cdot 0_S$, resp.). Moreover, ${}_SM$ (N_S, resp.) is additively idempotent.

Now, assume that $q = o_S$ (the subtype (A2)). If ${}_SM$ (N_S, resp.) is a non-quasitrivial minimal semimodule then $o_M \in M$ ($o_N \in N$, resp.) and $S \cdot o_M = \{o_M\} = o_S \cdot M$ ($o_N \cdot S = \{o_N\} = N \cdot o_S$, resp.). Moreover, ${}_SM$ (N_S, resp.) is additively idempotent.

5.12 Proposition. *Let S be a finite congruence-simple semiring of type (B). Then:*
(i) *S is not left quasitrivial.*
(ii) *If S is right quasitrivial then $S \simeq \mathbb{K}_1^{\mathrm{op}}$.*
(iii) *If $|S| \geq 3$ then S is neither left nor right quasitrivial.*
(iv) *S contains the additively absorbing element q such that q is left multiplicatively absorbing.*
(v) *S has no right multiplicatively absorbing element.*
(vi) *S is additively idempotent.*
(vii) *If ${}_SM$ is a non-quasitrivial minimal left semimodule then M is finite and $Q({}_SM) = \emptyset$.*
(viii) *S^{op} is of type (C).*

Proof. First, it follows from I.5.8(ii) that S is not left quasitrivial. If S is right quasitrivial then S is not commutative and it follows from the right-hand form of I.5.7 that $S \simeq \mathbb{K}_1^{\mathrm{op}}$. Combining this with the right-hand form of I.7.5, we conclude that S has no right multiplicatively absorbing element. Now, let ${}_SM$ be a non-quasitrivial minimal left semimodule. By 5.8(i), M is finite. By I.6.3, there is a congruence ϱ of ${}_SM$ such that ${}_SN = {}_SM/\varrho$ is non-quasitrivial, minimal and congruence-simple. If $Q({}_SM) \neq \emptyset$ then $Q({}_SN) \neq \emptyset$. On the other hand, it follows from II.4.4 that $Q({}_SN) = \emptyset$. Thus $Q({}_SM) = \emptyset$ as well. Moreover, proceeding similarly as in the proof of 5.7, we can show that $M(+)$ and $N(+)$ are idempotent. Then, of course, S is additively idempotent (use II.4.1(v)). We have proved the assertions (i), (ii), (iii), (v), (vi) and (vii). Finally, (iv) follows from 5.6 and (viii) is clear. □

5.13 Remark. Let S be a finite congruence-simple semiring of type (B) (see 5.12). Then S is additively idempotent and S has the additively absorbing element q such that q is left multiplicatively absorbing but not right muliplicatively absorbing. Moreover, there exists a non-quasitrivial congruence-simple minimal left semimodule ${}_SM$ with $Q({}_SM) = \emptyset$; we have $Sx = M$ for every $x \in M$ (i.e., S acts transitively on M). Further, if S is not isomorphic to $\mathbb{K}_1^{\mathrm{op}}$ then, according to I.7.3 (and 1.4), there exists a non-quasitrivial congruence-simple almost minimal right semimodule N_S. Both semimodules ${}_SM$ and N_S are additively idempotent.

5.14 Proposition. *Let S be a finite congruence-simple semiring of type (D). Then S is commutative, quasitrivial and either S is isomorphic to one of $\mathbb{K}_2$, $\mathbb{K}_3$, $\mathbb{K}_4$ or*

S is a zero multiplication ring of prime order (see I.5.7).

Proof. Assume that S is not left quasitrivial. Let ${}_SM$ be a non-quasitrivial finite semimodule with minimal $|M|$ (see I.6.8). Since S is of type (D), the semimodule ${}_SM$ is not minimal. Then, by I.6.8(i) and I.6.8(iv), we see that ${}_SM$ is congruence-simple and $P({}_SM) = Q({}_SM) \simeq Q_{1,S}$. Moreover, using I.7.3 and its proof, we conclude that ${}_SM$ is almost minimal. Now, by 3.3, S contains the additively absorbing element 0_S such that 0_S is also right multiplicatively absorbing. Consequently, applying the dual of I.7.5, we see finally that S is right quasitrivial. The rest is clear from I.5.7 and its dual. $\square$

5.15 Remark. Let S be a finite additively idempotent congruence-simple semiring. The element $o_S = \sum x$, $x \in S$, is additively absorbing. If o_S is neither left nor right multiplicatively absorbing then $0_S \in S$ and 0_S is multiplicatively absorbing.

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