# ADDITIVELY DIVISIBLE COMMUTATIVE SEMIRINGS

TOMÁŠ KEPKA AND MIROSLAV KORBELÁŘ

Abstract. Commutative semirings with divisible additive semigroup are studied.

## 1. Preliminaries

Throughout the paper, all semigroup, groups, semirings, rings and fields are assumed to be commutative (but, possibly, without additively and/or multiplicatively neutral elements). Furthermore, the following notation will be used in the sequel:

$\mathbb{N}$ ... the semiring of positive integers;
$\mathbb{N}_0$ ... the semiring of non-negative integers;
$\mathbb{Z}$ ... the ring of integers;
$\mathbb{Q}^+$ ... the parasemifield of positive rationals.

## 2. Congruences of $\mathbb{N}$

Define a relation $\rho(k,t)$ on $\mathbb{N}$ for all $k,t \in N$ by $(m,n) \in \rho(k,t)$ iff $m-n \in \mathbb{Z}t$ and either $m=n$ or $m \geq k$ and $n \geq k$.

**Lemma 2.1.** *$\rho(k,y)$ is a congruence of the semiring $\mathbb{N}$.*

*Proof.* It is easy to check that the relation $\rho(k,t)$ is an equivalence and that it is stable under addition and multiplication. □

**Lemma 2.2.** *The congruence $\rho(k,t)$ has exactly $k+t-1$ blocks and these are just the following subsets of $\mathbb{N}$: $\{1\},\{2\},\dots,\{k-1\},\{m+lt|\ l \in \mathbb{N}_0\}$, $m \in \mathbb{N}$, $k \leq m \leq k+t-1$.*

*Proof.* The assertion follows easily from the definition of the congruence $\rho(k,t)$. □

**Lemma 2.3.** *$(k,k+t) \in \rho(k,t)$.*

*Proof.* The assertion follows directly from the definition of the congruence $\rho(k,t)$. □

**Lemma 2.4.** *Let $m \in \mathbb{N}$. Then $(m,2m) \in \rho(k,t)$ iff $k \leq m$ and $t$ divides $m$.*

*Proof.* The assertion follows immediately from the definition of the congruence $\rho(k,t)$. □

**Lemma 2.5.** (i) *$(kt,2kt) \in \rho(k,t)$.*
(ii) *$(l,2l) \in \rho(k,t)$, where $l = \mathrm{lcm}(k,t)$.*
(iii) *$(k+t-s, 2(k+t-s)) \in \rho(k,t)$, where $r,s \in \mathbb{N}_0$ are such that $k = rt+s$ and $s < t$.*

---

2000 *Mathematics Subject Classification.* 16Y60.
*Key words and phrases.* commutative semiring, divisible semigroup.
This work is a part of the research project MSM00210839 financed by MŠMT.

*Proof.* Use 2.4. □

**Lemma 2.6.** *Let $m_1, m_2 \in \mathbb{N}$ be such that $(m_1, 2m_1) \in \rho(k,t)$ and $(m_2, 2m_2) \in \rho(k,t)$. Then $(m_1, m_2) \in \rho(k,t)$.*

*Proof.* Combine 2.4 and the definition of $\rho(k,t)$. □

**Lemma 2.7.** *Let $w \in \mathbb{N}$ be such that $(w, 2w) \in \rho(k,t)$ (see 2.4, 2.5 and 2.6). Then $(m, m+w) \in \rho(k,t)$ for every $m \in \mathbb{N}$, $m \geq k$.*

*Proof.* By 2.4, $w \geq k$ and $t$ divides $w$. The rest is clear. □

**Lemma 2.8.** *Let $w \in \mathbb{N}$ be such that $(w, 2w) \in \rho(k,t)$ (see 2.4, 2.5, 2.6 and 2.7). Then:*

- (i) *For every $m \in \mathbb{N}$, $m \geq k$, there exists at least one $n \in \mathbb{N}$ such that $n \geq k$ and $(m+n, w) \in \rho(k,t)$.*
- (ii) *If $m, n_1, n_2 \in \mathbb{N}$ are such that $m \geq k$, $n_1 \geq k$, $n_2 \geq k$, $(m+n_1, w) \in \rho(k,t)$ and $(m+n_2, w) \in \rho(k,t)$, then $(n_1, n_2) \in \rho(k,t)$.*

*Proof.* (i) Choose $l \in \mathbb{N}$ such that $m+k \leq lw$ and put $n = lw - m$. Then $n \geq k$ and $m+n = lw$. But $(w, 2w) \in \rho(k,t)$, $(2w, 3w) \in \rho(k,l), \dots, ((l-1)w, lw) \in \rho(k,t)$ (for $l \geq 2$) and we have $(w, lw) \in \rho(k,t)$. Thus $(m+n, w) \in \rho(k,t)$.

(ii) We have $(m+n_1, w) \in \rho(k,t)$, $(m+n_2, w) \in \rho(k,t)$, and hence $(m+n_1, m+n_2) \in \rho(k,t)$. Furthermore, $(m+2n_1, w+n_1) \in \rho(k,t)$, $(m+n_1+n_2, w+n_2) \in \rho(k,t)$ and $(m+2n_1, m+n_1+n_2) \in \rho(k,t)$. Consequently, $(w+n_1, w+n_2) \in \rho(k,t)$. Now, $(n_1, n_2) \in \rho(k,t)$ follows from 2.7. □

**Lemma 2.9.** *$\rho(k,t)$, as a congruence of the additive semigroup $\mathbb{N}(+)$ is generated by the single pair $(k, k+t)$.*

*Proof.* Denote by $\rho$ the congruence of $\mathbb{N}(+)$ generated by the ordered pair $(k, k+t)$. Since $(k, k+t) \in \rho(k,t)$, we have $\rho \subseteq \rho(k+t)$. Conversely, we have to show that $(m,n) \in \rho(k,t)$ implies $(m,n) \in \rho$; we can assume that $m < n$. Then $m = k+l$ and $n = k+l+rt$ for some $l \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Of course, $(m, m+t) = (k+l, k+l+t) = (k, k+t) + (l,l) \in \rho$, $(m+t, m+2t) = (k+l+t, k+l+2t) = (k, k+t) + (l+t, l+t) \in \rho, \dots, (m+(r-1)t, m+rt) \in \rho$. Using transitivity, we get $(m,n) = (m, m+rt) \in \rho$. □

## 3. Cyclic semigroups

It is well known, that every congruence of $\mathbb{N}(+)$ is either identity or $\rho(k,t)$ for some $k, t \in \mathbb{N}$.

**Proposition 3.1.** *The congruences $id_{\mathbb{N}}$ and $\rho(k,t)$, $k, t \in \mathbb{N}$, are just all congruences of the semiring $\mathbb{N}(+,\cdot)$ of positive integers.*

*Proof.* Easy to verify. □

**Lemma 3.2.** *Let $t(S)$ denote the set elements of finite order of a semigroup $S$. If $t(S) \neq 0$, then $t(S)$ is a subsemigroup. of $S$*

*Proof.* It is easy. □

A semigroup $S$ will be called torsion if every element of $S$ has finite order.

**Lemma 3.3.** *Let $A$ be a non-empty subset of a semigroup $S$ such that there exists $m \in \mathbb{N}$ with $\mathrm{ord}_S(a) \leq m$ for every $a \in A$. Then there exists $n \in \mathbb{N}$ such that $2nb = nb$ for every $b \in \langle A \rangle_S$.*

*Proof.* For every $a \in A$ there are $k_a, t_a \in \mathbb{N}$ with $\langle A\rangle_S \cong C(k_a, t_a) \cong \mathbb{N}(+)/\rho(k_a, t_a)$. Of course, $k_a + t_a \le m+1$. By 2.5(ii), $2m_a a = m_a a$ for some $m_a \in \mathbb{N}$, $m_a \le m+1$. Now, it suffices to put $n = (m+1)!$. □

**Lemma 3.4.** *Let $A$ be a non-empty subset of a semigroup $S$ such that there exists $m \in \mathbb{N}$ with $\mathrm{ord}_S(a) \le m$ for every $a \in A$. Then there exists $l \in \mathbb{N}$ with $\mathrm{ord}_S(b) \le l$ for every $b \in \langle A\rangle_S$.*

*Proof.* By 4.6, $2nb = nb$ for some $n \in \mathbb{N}$ and all $b \in \langle A\rangle_S$. We have $\langle b\rangle_S \cong C(k_b, t_b)$ and $\mathrm{ord}_S(b) = k_b + t_b - 1$. Since $nb = 2nb$, $n \ge k_b$ and $t_b$ divides $2n - n = n$. Consequently, $k_b + t_b - 1 \le 2n - 1$. □

**Lemma 3.5.** *Let $S$ be a semigroup and let $a, b \in S$ be such that $ka = la + b$ for some $k, l \in \mathbb{N}$, $k \ne l$. If $\mathrm{ord}_S(b)$ is finite, then $\mathrm{ord}_S(a)$ is so.*

*Proof.* We have $mb = nb$ for some $m, n \in \mathbb{N}$, $m < n$. Now, $nka = nla + nb = nla + mb = (n-m)la + m(la+b) = (n-m)la + mka = ((n-m)l + mk)a$. Since $k \ne l$, we have $(n-m)k \ne (n-m)l$ and $nk \ne (n-m)l + mk$. Consequently, $\langle a\rangle_S$ is finite. □

## 4. Divisible semigroups

A (commutative) semigroup $S(= S(+))$ is called divisible if $S = mS$ for every $m \in \mathbb{N}$.

**Proposition 4.1.** (i) *The class of divisible semigroups is closed under homomorphic images and cartesian products.*

(ii) *The class of divisible semigroups contains all semilattices (i.e., idempotent commutative semigroups) and all divisible abelian groups.*

(iii) *The additive semigroup $\mathbb{Q}^+(+)$ ($\mathbb{Q}_0^+(+)$, resp.) of positive (non-negative, resp.) rational numbers are divisible.*

*Proof.* It is easy to see. □

**Proposition 4.2.** *A finite semigroup is divisible if and only if it is idempotent (i.e., it is a semilattice).*

*Proof.* All semilattices are divisible. On the other hand, if $S$ is a finite semigroup, then for every $a \in S$ there is $m_a \in \mathbb{N}$ with $2m_a a = m_a a$ (4.6). If $m = \prod m_a$, $a \in S$, then $2ma = ma$ for every $a \in S$. Finally, if $S$ is divisible, then $mS = S$ and $S$ is idempotent. □

**Lemma 4.3.** *Let $S$ be a semigroup and $a \in S$. Define a relation $\rho_a$ on $S$ by $(u, v) \in \rho_a$ iff $u + ka = v + la$ for some $k, l \in \mathbb{N}$. Then $\rho_a$ is a congruence of $S$ and $(a, 2a) \in \rho_a$.*

*Proof.* Clearly, $\rho_a$ is reflexive, symmetric and stable under the addition of the semigroup $S$. It remains to show that $\rho_a$ is transitive. If $u + ka = v + la$ and $v + ra = w + sa$, $k, l, r, s \in \mathbb{N}$, then $u + (k+r)a = v + la + ra = w + (l+s)a$. □

**Proposition 4.4.** *Let $S$ be a semigroup. Then $S$ is finitely generated and divisible if and only if $S$ is a finite semilattice.*

*Proof.* Assume that $S$ is divisible and generated by a finite set $A$. Let $m$ be the set number of non-idempotent element of $A$. We proceed by induction on $m$.

If $m = 0$, then $S$ is generated by a set of idempotents and if follows easily that $S$ is idempotent itself. Of course, a finitely generated semilattice is finite. Now, assume that $m \ge 1$. If $a \in A$ is such that $a \ne 2a$, then $S/\rho_a$ is a (finite) semilattice by induction (see 5.3). Since $S$ is divisible, we have $a = 2b$ for some $b \in S$ and

$(a,b) = (2a,b) \in \rho_a$ (since $S/\rho_a$ is idempotent). Then $ka = b + la$ for some $k, l \in \mathbb{N}$ and we get $2ka = 2b + 2la = (2l+1)a$. Since $2k \neq 2l+1$, we conclude that the cyclic subsemigroup $\langle a \rangle_S$ generated by $\{a\}$ is finite.

We have proved that $\langle a \rangle_S$ is finite for every $a \in A$. Since $A$ is finite and $S$ is generated by $A$, one checks easily that $S$ is finite, too. By 5.2, $S$ is a finite semilattice. □

**Lemma 4.5.** *Let $S$ be a semigroup. Define a relation $\sigma(S)$ on $S$ by $(u,v) \in \sigma(S)$ iff $mu = mv$ for some $m \in \mathbb{N}$. Then $\sigma(S)$ is a congruence of $S$ and $\sigma(S/\sigma(S)) = id$.*

*Proof.* Clearly, $\sigma(S)$ is reflexive, symmetric and stable under the addition. If $mu = mv$ and $nv = nw$, then $mnu = mnw$, and hence $\sigma(S)$ is transitive as well. Thus $\sigma(S)$ is a congruence of the semigroup. Finally, if $(mu, mv) \in \sigma(S)$, then $kmu = kmv$ and $(u,v) \in \sigma(S)$. □

**Corollary 4.6.** *If $S$ is a divisible semigroup, then $S/\sigma(S)$ is a uniquely divisible semigroup.*

**Lemma 4.7.** *Let $S$ be a semigroup. Define a relation $\tau(S)$ on $S$ by $(u,v) \in \tau(S)$ iff $mu = nv$ for some $m, n \in \mathbb{N}$. Then $\tau(S)$ is a congruence of $S$, $\sigma(S) \subseteq \tau(S)$ and $\tau(S/\tau(S)) = id$.*

*Proof.* Similar to 5.5. □

**Corollary 4.8.** *If $S$ is a divisible semigroup, then $S/\tau(S)$ is a uniquely divisible semigroup.*

**Lemma 4.9.** *Let $S$ be a semigroup such that the factor-semigroup $S/\sigma(S)$ is torsion. Then $S$ is torsion.*

*Proof.* For every $a \in S$ there are $k, l \in \mathbb{N}$ such that $(ka, la) \in \sigma(S)$ and $k < l$. Furthermore, there is $m \in \mathbb{N}$ with $mka = mla$. Clearly, $mk < ml$, and hence $\mathrm{ord}_S(a)$ is finite. □

**Proposition 4.10.** *Let $S$ be a divisible semigroup such that there exists $m \in \mathbb{N}$ with $\mathrm{ord}_S(a) \leq m$ for every $a \in S$. Then $S$ is a semilattice.*

*Proof.* By 4.6, there is $n \in \mathbb{N}$ such that $2na = na$ for every $a \in S$. Now, $a = nb$, and hence $2a = 2nb = nb = a$. □

**Lemma 4.11.** *Let $S$ be a uniquely divisible semigroup. If $a \in S$ is such that $\mathrm{ord}_S(a)$ is finite, then $2a = a$.*

*Proof.* There is $m \in \mathbb{N}$ with $2ma = ma$. Then $2a = a$, since $S$ is uniquely divisible. □

## 5. Additively divisible semirings

**Lemma 5.1.** *Let $S$ be a semiring. Then:*

(i) *$\sigma(S)$ is a congruence of $S$ and $\sigma(S/\sigma(S)) = id$.*
(ii) *$\tau(S)$ is a congruence of $S$ and $\tau(S/\tau(S)) = id$.*

*Proof.* Clearly, both $\sigma(S)$ and $\tau(S)$ are stable under the multiplication of the semiring $S$ and the rest follows from 5.5 and 5.7. □

**Corollary 5.2.** *Let $S$ be an additively divisible semiring. Then both $S/\sigma(S)$ and $S/\tau(S)$ are additively uniquely divisible semirings.*

*Remark* 5.3. Let $S$ be an additively uniquely divisible semiring.

(i) For all $m, n \in \mathbb{N}$ and $a \in S$, there is a uniquely determined $b \in S$ with $ma = nb$ and we put $(m/n)a = b$. If $m_1, n_1 \in \mathbb{N}$ and $b_1 \in S$ are such that $m/n = m_1/n_1$ and $m_1 a = n_1 b_1$, then $k = mn_1 = m_1 n$ and $kb_1 = mm_1 a = kb$ and $b_1 = b$. Consequently, we get a (scalar) multiplication $\mathbb{Q}^+ \times S \to S$ (one checks easily that $q(a_1 + a_2) = qa_1 + qa_2$, $(q_1 + q_2)a = q_1 a + q_2 a$, $q_1(q_2 a) = (q_1 q_2)a$ and $1a = a$ for all $q_1, q_2 \in \mathbb{Q}^+$ and $a_1, a_2, a \in S$) and $S$ becomes a unitary $\mathbb{Q}^+$-semimodule. Furthermore, $qa_1 \cdot a_2 = a_1 \cdot qa_2$ for all $q \in \mathbb{Q}^+$ and $a_1, a_2 \in S$, and therefore $S$ is a unitary $\mathbb{Q}^+$-algebra.

(ii) Let $a \in S$ be multiplicatively but not additively idempotent (i.e., $a^2 = a \neq 2a$). Put $Q = \mathbb{Q}^+ a$. Then $Q$ is a subalgebra of the $\mathbb{Q}^+$-algebra $S$ and the mapping $\varphi : q \mapsto qa$ ia a homomorphism of the $\mathbb{Q}^+$-algebras and, of course, of the semirings as well. Since $a \neq 2a$, we have $\ker(\varphi) \neq \mathbb{Q}^+ \times \mathbb{Q}^+$. But $\mathbb{Q}^+$ is a congruence-simple semiring and it follows that $\ker(\varphi) = id$. Consequently, $Q \cong \mathbb{Q}^+$.

Put $T = Sa$. Then $T$ is an ideal of the $\mathbb{Q}^+$-algebra $S$, $Q \subseteq T$ (we have $qa = a \cdot qa \in T$) and $a = 1_Q = 1_T$ is a multiplicatively neutral element of $T$. The mapping $s \mapsto sa$ is a homomorphism of the $\mathbb{Q}^+$-algebras. Consequently, $T$ is additively uniquely divisible. Furthermore, $T$ is a finitely generated semiring, provided that $S$ is so.

**Proposition 5.4.** *Let $S$ be an additively divisible semiring with $1_S \in S$. Then:*

(i) *$S$ is additively uniquely divisible.*
(ii) *Either $S$ is additively idempotent or $S$ contains a subsemiring $Q$ such that $Q \cong \mathbb{Q}^+$ and $1_S = 1_Q$.*
(iii) *If $\mathrm{ord}_{S(+)}(1_S)$ is finite, then $S$ is additively idempotent.*

*Proof.* For every $m \in \mathbb{N}$, there is $w_m \in S$ such that $1_S = mw_m$. That is, $w_m = (m1_S)^{-1}$. If $ma = mb$, then $a = w_m ma = w_m nb = b$ and we see that $S$ is additively uniquely divisible. The rest is clear from 6.3. □

**Lemma 5.5.** *Let $S$ be a semiring such that $t(S) = t(S(+)) \neq \emptyset$. Then $t(S)$ is an ideal of $S$. Moreover, if $S$ is additively divisible, then $t(S)$ is so.*

*Proof.* By 4.5, $t(S)$ is a subsemigroup of $S(+)$. Furthermore, if $a \in t(S)$, then $ka = la$ for some $k, l \in \mathbb{N}$, $k < l$, and then $kab = lab$ for every $b \in S$. It means that $ab \in t(S)$ and $t(S)$ becomes an ideal of the semiring $S$. Finally, if $a = mc$, $m \in \mathbb{N}$, $c \in S$, then $kmc = ka = la = lmc$ and $km < lm$. Thus $c \in t(S)$. □

**Proposition 5.6.** *Let a semiring $S$ be generated as a (left) $S$-semimodule by a subset $A$ such that $\mathrm{ord}_{S(+)}(a) \leq m$ for some $m \in \mathbb{N}$ and all $a \in A$. If $S$ is additively divisible, then $S$ is additively idempotent.*

*Proof.* Put $B = \{b \in S \mid \mathrm{ord}_{S(+)}(b) \leq m\}$. Then $A \subseteq B$ and $b \in B$ for all $s \in S$ and $b \in B$. Furthermore, $\langle B \rangle_{S(+)} = S$, and hence there is $l \in \mathbb{N}$ with $\mathrm{ord}_{S(+)}(r) \leq l$ for every $r \in S$ (by 4.7). Now, it remains to use 5.10. □

**Corollary 5.7.** *Let an additively divisible semiring $S$ be generated as an $S$-semimodule by a finite set of elements of finite additive orders. Then $S$ is additively idempotent.*

**Corollary 5.8.** *Every additively divisible and torsion finitely generated semiring is additively idempotent.*

*Remark* 5.9. The zero multiplicative ring defined on $\mathbb{Z}_{p^\infty}$ is both additively divisible and additively torsion. Of course, the ring is neither additively idempotent nor finitely generated. The (semi)group $\mathbb{Z}_{p^\infty}(+)$ is not uniquely divisible.

*Remark* 5.10. Let $R$ be a (non-zero) finitely generated ring. Then $R$ has at least one maximal ideal $I$ and the factor $R/I$ is a finitely generated simple ring. However, any such a ring is finite and consequently, $R$ is not additively divisible.

**Proposition 5.11.** *Let $S$ be a non-trivial additively cancellative and divisible semiring. Then $S$ is not finitely generated.*

*Proof.* Consider the difference ring $R = S - S$ of $S$. It is easy to check that $R$ is additively divisible. According to 6.10, $R$ is not finitely generated. Then $S$ is not finitely generated either. □

## 6. One-generated additively divisible semirings

**Lemma 6.1.** *Let $S$ be a semiring such that $1_s \in S$ ($1_S$ being multiplicatively neutral). Let $w \in S$, $a, b, c \in \langle w\rangle_S$ and $m \in \mathbb{N}$ be such that $ma = nb$ and $mc = w$. Then $a = b$.*

*Proof.* We have $\langle w\rangle_S = \langle w, w^2, w^3, \dots\rangle_{S(+)}$ and it follows easily that for every $d \in \langle w\rangle_S$ there is $d' \in S$ with $d = wd'$. Now, $a = wa' = mca' = mwc'a' = mac' = mbc' = mwc'b' = mcb' = wb' = b$. □

**Proposition 6.2.** *Every additively divisible one-generated semiring is uniquely divisible.*

*Proof.* Let $S$ be an additively divisible semiring generated by a single element $w$. First, put $T = S \cup \{0_T\}$, where $0_T$ is additively neutral and multiplicatively absorbing. Then $T$ becomes an additively divisible semiring and $S$ a subsemiring of $T$. Next, let $R = T \times \mathbb{N}_0$ be the Dorroh extension. That is, $(r, m) + (s, n) = (r + s, m + n)$ and $(r, m)(s, n) = (rs + nr + ms, mn)$. Clearly, $T(= T \times \{0\})$ is a subsemiring of $R$, $0_R = (0_T, 0)$ is additively neutral and multiplicatively absorbing in $R$ and $1_R = (0_T, 1)$ is multiplicatively neutral in $R$. Now, if $a, b \in S$ and $m \in \mathbb{N}$ are such that $ma = mb$, then $w = mc$ for some $c \in S$ and we get $a = b$ by 7.1. □

**Lemma 6.3.** *Let $S$ be an additively divisible semiring generated by an element $w$. If $\mathrm{ord}_{S(+)}(w^m)$ is finite for some $m \in \mathbb{N}$, then $S$ is additively idempotent.*

*Proof.* If $\mathrm{ord}_{S(+)}(w)$ is finite, then $S$ is additively idempotent by 6.7. Consequently, assume that $n \geq 2$, where $n \in \mathbb{N}$ is the smallest number with $\mathrm{ord}_{S(+)}(w^n)$ finite. Since $S(+)$ is divisible, we have $w = 2v$ for some $v \in S$. Moreover, there are $1 \leq i_1 < i_2 < \cdots < i_k$, $k \in \mathbb{N}$, such that $v = n_{i_1}w^{i_1} + n_{i_2}w^{i_2} + \cdots + n_{i_k}w^{i_k}$ for some $n_{i_j} \in \mathbb{N}$. From this we see that $w^{n-1} = 2n_{i_1}w^{i_1+n-2} + 2n_{i_2}w^{i_2+n-2} + \cdots + 2n_{i_k}w^{i_k+n-2}$. If $k = 1$, then $w^{n-1} = 2n_{i_1}w^{i_1+n-2}$, where $i_1 + n - 2 \geq n - 1$. If $i_1 = 1$, then $w^{n-1} = 2n_{i_1}w^{n-1}$ and $\mathrm{ord}_{S(+)}(w^{n-1})$ is finite, a contradiction. If $i_1 \geq 2$, then $w^{n-1} = 2n_{i_1}w^n w^{i_1-2}$, and $\mathrm{ord}_{S(+)}(w^{n-1})$ is finite, too. If $k \geq 2$, then $3 \leq i_2$ and $\mathrm{ord}_{S(+)}(u)$ is finite, where $u = 2n_{i_2}w^{i_2+n-2} + \cdots + 2n_{i_k}w^{i_k+n-2}$. If $i_1 \geq 2$, then $\mathrm{ord}_{S(+)}(2n_{i_1}w^{i_1+n-2})$ is finite, and so the same is true for $\mathrm{ord}_{S(+)}(w^{n-1})$. Finally, if $i_1 = 1$, then $w^{n-1} = 2n_{i_1}w^{n-1} + u$ and $\mathrm{ord}_{S(+)}(w^{n-1})$ is finite by 4.8, the final contradiction. □

*Remark* 6.4. Let $S$ be an additively divisible semiring generated by an element $w$ (see 7.2 and 7.3). There exists $v \in S$ with $2v = w$ and $v = n_1w^{i_1} + \cdots + n_kw^{i_k}$ for some $n_1, \dots, n_k, k \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_k$. Then $w = 2n_1w^{i_1} + \cdots + 2n_kw^{i_k}$.

(i) Assume that $i_1 \geq 2$. Then $w = we$, where $e = 2n_1w^{i_1-1} + \cdots + 2n_kw^{i_k-1}$ and we conclude easily that $e = 1_S$ is the multiplicatively neutral element of $S$. Furthermore, $1_S = e = wf$, $f = 2n_1w^{i_1-2} + \cdots + 2n_kw^{i_k-2}$ (here, $w^0 = 1_S$), and hence $f = w^{-1}$. Thus $w^{S^*}$, where $S^*$ denotes the group of multiplicatively invertible elements of $S$.

(ii) Assume that $i_1 = 1$. Then $k \geq 2$ by 7.3 and $w = w + u$, $u = (2n_1 - 1)w + 2n_2w^{i_2} + \cdots + 2n_kw^{i_k}$. Now, it is easy to see that for every $r \in S$ there is at least one $s \in S$ with $r + s = r$.

*Remark* 6.5. Let $S$ be a non-trivial additively divisible semiring generated by an element $w$ (see 7.2,7.3 and 7.4). Consider a maximal congruence $\rho$ of $S$. Then $T = S/\rho$ is a congruence-simple semiring and, of course, $T$ is additively divisible and one-generated. According to [1, 10.1], $T$ is additively idempotent and either $T \cong Z_3, Z_4$ or $T \cong V(G)$ for a finite cyclic group $G$.

(i) If $T \cong \mathbb{Z}_3$, then the congruence $\rho$ has just two blocks $A$ and $B$, where $A$ is a bi-ideal of $S$, $SS \subseteq A$ and $w \in B$, $B + B \subseteq B$. Then $w^2, w^3, \dots \in A$ and it follows that $B = \langle w \rangle_{S(+)} = \{w, 2w, 3w, \dots\}$. On the other hand, $w = 2v$ for some $v \in S$ and $v \in B$. This means that $\mathrm{ord}_{S(+)}(w)$ is finite and it follows from 7.3 that $S$ is additively idempotent.

(ii) If $T \cong Z_4$, then the congruence $\rho$ has just two blocks $A$ and $B$, where $A + S \subseteq A$, $w \in A$, $AA \subseteq B$, $B$ is an ideal of $S$ and $SS \subseteq B$. Then $w^2, w^3, \dots \in B$ and, in fact, $A = \{n_1w + n_2w^2 + \cdots + n_kw^k |\ k \in \mathbb{N}, n_i \in \mathbb{N}_0, n_1 \neq 0\}$. Notice that $B$, as a semiring, is generated by the set $\{w^2, w^3\}$. Finally, notice that $S$ possesses no multiplicatively neutral element.

(iii) Finally, assume that $T \cong V(\mathbb{Z}_m(+))$ for some $m \in \mathbb{N}$ and denote by $\varphi$ a projection of $S$ onto $V(\mathbb{Z}_m)$. We have $V(\mathbb{Z}_m(+)) = \mathbb{Z}_m \cup \{o\} = \{o, 0, 1, \dots, m-1\}$. Put $A = \varphi^{-1}(o)$ and $B_k = \varphi^{-1}(k)$ for every $k = 0, 1, \dots, m-1$. Then $A$ is a bi-ideal of $S$, $B_0$ is a subsemiring of $S$ and $B_1, \dots, B_{m-1}$ are subsemigroups of $S(+)$. Furthermore, $B_kB_l \subseteq B_t$, $t = k + l \pmod{(m)}$ for all $0 \leq k, l \leq m-1$, and $B_k + B_l \subseteq A$ for $k \neq l$. Without loss of generality, we can assume that $w \in B_1, w^2 \in B_2, \dots, w^{m-1} \in B_{m-1}$ and $w^m \in B_0$. Now, it is clear that $B_k = \{n_1w^k + n_2w^{k+m} + \cdots + n_jw^{k+(j-1)m} |\ j \in \mathbb{N}, n_i \in \mathbb{N}_0, \sum n_i \neq 0\}$ for every $1 \leq kleqm - 1$ and $B_0 = \{n_1w^k + n_2w^{2m} + \cdots + n_jw^{jm} |\ j \in \mathbb{N}, n_i \in \mathbb{N}_0, \sum n_i \neq 0\}$. Consequently, $B_0$, as a semiring, is generated by $w^m$. Of course, all $B_0, \dots, B_{m-1}$ are additively divisible.

Since $B_0$ is a subsemiring of $S$, we have $w \notin B_0$ and $m \geq 2$.

(iv) Consider the situation from (iii) and assume that $1_S \in S$. Then $1_S \in B_0$ and $1_S = a_1w^m + a_2w^{2m} + \cdots + a_jw^{jm}$, where $j \in \mathbb{N}$, $a_i \in \mathbb{N}_0$ and $\sum a_i \neq 0$. Consequently, $1_S = w(a_1w^{m-1} + a_2w^{2m-1} + \cdots + a_jw^{jm-1})$ and it follows that $w^{-1} = a_1w^{m-1} + \cdots + a_jw^{jm-1} \in S$, $w^{-1} \in B_{m-1}$, $w^{-m} = a_11_S + a_2w^m + \cdots + a_jw^{(j-1)m} \in B_0$. notice also that $S$ is additively idempotent if and only if $B_0$ is so (i.e., iff $1_S = 2_S$). If $w^t = 1_S$ for some $t \in \mathbb{N}$, then $m$ divides $t$.

(v) Consider that situation from (iv) and assume that $S$ is not additively idempotent. We are going to show that $w^t \neq 1_S$ for every $t \in \mathbb{N}$.

Let, on the contrary, $w^t = 1_S$ for some $t \in \mathbb{N}$. We will proceed by induction on $t$. As we know from (iv), $t = t_1m$, $t_1 \in \mathbb{N}$. If $t_1 = 1$, that $w^m = 1_S$ and then $B_0 = \{n1_S |\ n \in \mathbb{N}\}$ is not additively divisible, a contradiction.

Thus $2 \leq t_1 < t$. But $B_0$ is additively divisible, it is not additively idempotent, and $B_0$ is generated by $w^m$. Since $(w^m)^{t_1} = 1_{B_0}$, we get a contradiction (see (i),(ii) and (iii)).

**Lemma 6.6.** *Let $S$ be a semiring generated by an element $w$. Then $S$ is additively divisible if and only if for every prime integer $p$ there is $v_p \in S$ with $w = pv_p$.*

*Proof.* The direct implication is trivial. Conversely, if $w = pv_p$, then $w \in pS$, and so $pS = S$, since $pS$ is an ideal of the semiring $S$. Furthermore, given $m \in \mathbb{N}$, $m \geq 2$, we have $m = p_1^{k_1} \dots p_n^{k_n}$, and hence $mS = S$ as well. □

## 7. A few conjectures

Consider the following assertion:

(A) A finitely generated semiring is additively idempotent, provided that it is additively divisible.

(A1) A finitely generated semiring is additively idempotent, provided that it is additively uniquely divisible.

(B) A finitely generated senmiring contains no subsemiring isomorphic to $\mathbb{Q}^+$.

(B1) A finitely generated semiring with a unit element contains no subsemiring having the unit and isomorphic to $\mathbb{Q}^+$.

(C) A finitely generated semiring is additively idempotent, provided that it is ideal-simple and infinite.

(C1) A finitely generated semiring is additively idempotent, provided that it is a parasemifield.

**Proposition 7.1.** (A) $\Leftrightarrow$ (A1) $\Rightarrow$ (B) $\Leftrightarrow$ (B1) $\Rightarrow$ (C) $\Leftrightarrow$ (C1).

*Proof.* First, it is clear that (A) $\Rightarrow$ (A1) and (B) $\Rightarrow$ (B1). Furthermore, (C) $\Leftrightarrow$ (C1) by [2, 3.5]. Now, assume that (A1) is true and let $S$ be a finitely generated additively divisible semiring. By 6.2, $S/\sigma(S)$ is additively uniquely divisible and, of course, this semiring inherits the property of being finitely generated. By (A1), the semiring $S/\sigma(S)$ is additively idempotent, and hence the semiring $S$ is additively torsion by 5.9. Finally, $S$ is additively idempotent by 6.8. We have shown that (A1) $\Rightarrow$ (A) and consequently, (A) $\Leftrightarrow$ (A1).

Next, let (B1) be true and let $S$ be a finitely generated semiring containing a subsemiring $Q \cong \mathbb{Q}^+$. Put $T = S1_Q$. Then $T$ is an ideal of $S$, $1_Q = 1_T$, $Q \subseteq T$ and the map $s \mapsto s1_Q$ is a homomorphism of $S$ onto $T$. Thus $T$ is a finitely generated semiring and this is a contradiction with (B1). We have shown that (B1) $\Rightarrow$ (B) and consequently, (B) $\Leftrightarrow$ (B1).

Now, we are going to show that (A) $\Rightarrow$ (B1). Indeed, let $S$ be a finitely generated semiring such that $1_S \in S$ and $S$ contains a subsemiring $Q$ with $1_S \in Q$ and $Q \cong \mathbb{Q}^+$. If $a \in S$ and $m \in \mathbb{N}$, then $b = (m1_S)^{-1}a \in S$ and $mb = a$. It follows that $S$ is additively divisible, and hence additively idempotent by (A). But $Q$ is not so, a contradiction. We have shown that (A) $\Rightarrow$ (B1).

It remains to show that (B1) $\Rightarrow$ (C1). Let $S$ be a parasemifield that is not additively idempotent and let $Q$ denote the subparasemifield generated by $1_S$. Then $Q \cong \mathbb{Q}^+$, $1_Q = 1_S$ and $S$ is not finitely generated due to (B1). □

## References

[1] R. El Bashir, J. Hurt, A. Jančařík and T. Kepka, *Simple commutative semirings*, J. Algebra **236**(2001), 277-306.

[2] V. Kala, T. Kepka and M. Korbelář, *Notes on commutative parasemifields* (preprint)

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Prague 8, Czech Republic

*E-mail address*: kepka@karlin.mff.cuni.cz

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Prague 8, Czech Republic

*E-mail address*: miroslav.korbelar@gmail.com