COMMUTATIVE SUBDIRECTLY IRREDUCIBLE RADICAL RINGS

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ABSTRACT. A ring R is *radical* if there is a ring S (with unit) such that $R = \mathcal{J}(S)$ (the Jacobson radical). We study the commutative subdirectly irreducible radical rings and show that such a ring is noetherian if and only if is finite. We present a reflection of the commutative radical rings into the category of the commutative rings and derive a lot of examples of the subdirectly irreducible radical rings with various properties. At last, we show partial results in the classification of the factors R/M of the subdirectly irreducible radical rings R by their monoliths M.

For a ring R we denote $\mathcal{J}(R) = \bigcap\{\mathrm{Ann}_R(M)|\ M$ *is a simple* R*-module*$\}$ the Jacobson's radical of R. Radical rings are just all Jacobson's radicals of all rings. These rings are important not only for this property, but were also massively used, together with their adjoint groups, by E. Zelmanov in solving of the Burnside's problem for finitely generated groups.

Equivalently, a ring R is radical if and only if for every $a \in R$ there is an adjoint element $\widetilde{a} \in R$ such that $a + \widetilde{a} + a\widetilde{a} = 0$. Thus we can view the class of the radical rings as a universal algebraic variety (with one nullary, two unary and two binary operations). Since every simple radical ring is isomorphic to a zero-multiplication ring $\mathbb{Z}_p$ for a prime p, we proceed naturally by investigating of the structure of this variety to the subdirectly irreducible ones.

A subdirectly irreducible ring is one in which the intersection of all the nonzero ideals is a nonzero ideal. These ring are a kind of building blocks, since, by the Birkhoff's theorem, every radical ring is isomorphic to a subdirect product of subdirectly irreducible radical rings.

In this paper we pay our attention on the commutative rings. Subdirectly irreducible commutative ring were already studied by N.H. McCoy [6] and N. Divinsky [7]. In [6] was shown that these rings are of the following three types:

(α) Fields.
(β) Every element is a zero divisor.
(γ) There exists both non-divisors of zero and nilpotent elements.

The subdirectly irreducible commutative radical rings are of type (β). In addition, by [6], if they satisfy either the descending or the ascending condition, they are nilpotent.

An important property of the class of commutative radical rings is also the existence of a reflection of the category of the commutative rings into the category of the commutative radical rings. In this paper we present such reflection, which will be consecutively a very effective tool for constructing of examples of subdirectly irreducible radical rings with various properties. As we will see, a very helpful class for these constructions will be the class of so called *subradical* rings.

2000 *Mathematics Subject Classification.* 16D90, 16E30 (primary), 03E75, 18G25, 20K40, 16D40, 16E05, 16G99 (secondary).

Finally, we are concerned with the following natural question: "Which (universal) algebras are homomorphic images of subdirectly irreducible (universal) algebras?" T. Kepka asked for a characterisation of those algebras in [3]. For semigroups was the complete answer given in [1]. For algebras with only unary operations was the problem partially solved in [2]. In [5], the aswer was given for quasigroups. In this paper we study this question for (commutative) radical rings. We give some necessery conditions for such factors and make a characterization of the case when the factor is zero-multiplication ring.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with or without unit. Henceforth, the word 'ring' will always mean a commutative one.

The *Dorroh extension* $\mathbb{D}(R)$ for a ring R is a ring $\mathbb{Z} \oplus R$ with the multiplication $(n, a) \cdot (m, b) = (nm, ma + nb + ab)$ for $n, m \in \mathbb{Z}$ and $a, b \in R$. We can therefore assume that $R \subseteq \mathbb{D}(R)$. Nontrivial radical ring cannot contain a unit (otherwise $-1 \in R$ and $0 = (-1) + (\widetilde{-1}) + (-1)(\widetilde{-1}) = -1$, a contradiction). Hence we can write $(1 + a)(1 + \widetilde{a}) = 1$ in $\mathbb{D}(R)$.

Let R be a ring, $X, Y \subseteq R$ subsets. Denote $X \cdot Y$ the subgroup of $(R, +)$ generated by the set $\{xy | x \in X, y \in Y\}$. Further, put $X^1 = \{\sum_i x_i | x_i \in X\}$ and $X^{n+1} = X \cdot X^n$ for $n \in \mathbb{N}$.

Let R be a ring and $A \subseteq R$ a subset. We will say that R is *id-generated* by A iff R is generated by A as a R-module. We say that a radical ring R is *rd-generated* by A iff is generated by A as a radical ring.

For a ring R we denote $\mathrm{Ann}(R)$ the annihilator, $\mathcal{N}(R)$ the nilradical, $\mathcal{T}(R)$ the torsion part and $\mathcal{D}(R)$ the divisible part of R.

A ring R is said to be *subdirectly irreducible* iff it has the least non-zero ideal, called *a monolith* and denoted by $\mathcal{M}(R)$. Let $M \neq 0$ be an ideal of R. Clearly, a ring R is subdirectly irreducible with a monolith $\mathcal{M}(R) = M$ iff $M \subseteq Rx$ for every $x \in R \setminus \mathrm{Ann}(R)$ and $M \subseteq \mathbb{Z}x'$ for every $0 \neq x' \in \mathrm{Ann}(R)$ (i.e. iff $M \subseteq Rx + \mathbb{Z}x$ for every $0 \neq x \in R$).

Let R be a subdirectly irreducible radical ring with a monolith M. By [4] 12.1, $\mathcal{T}(R)$ is a p-group and $\mathbb{Z}_p(+) \cong M(+) \subseteq \mathrm{Ann}(R)(+) \cong \mathbb{Z}_{p^n}(+)$, where $1 \leq n \leq \infty$.

Denote by $\mathcal{S}$ the the class of all subdirectly irreducible radical rings.

Lemma 1.1. *Let R be a ring, $A \subseteq R$ a subset. Let $A^n = 0$ for some $n \in \mathbb{N}$ and suppose R is id-generated by A.*

Then R is generated by A as a ring and $R^n = 0$ (hence R is nilpotent).

Proof. Obviously $R = A^1 + R \cdot A^1$. Now, by induction, if $R = A^1 + \cdots + A^k + R \cdot A^k$, then $R = A^1 + \cdots + A^k + (A^1 + R \cdot A^1) \cdot A^k = A^1 + \cdots + A^k + A^{k+1} + R \cdot A^{k+1}$. Hence $R = A^1 + \cdots + A^n$ and R is nilpotent. □

Lemma 1.2. *Let R be a noetherian ring. Then there is $m \in \mathbb{N}$ such that $m \times \mathcal{T}(R) = 0$.*

Proof. Let $\mathbb{P} = \{p_1, p_2, \ldots\}$ be the set of all prime numbers. Put $I_n = \{a \in R | (\exists\ k \in N)(p_1 \ldots p_n)^k \times a = 0\}$. Then $\{I_n\}_{n \in \mathbb{N}}$ is an increasing sequence of ideals of R and $\mathcal{T}(R) = \bigcup_n I_n$. Hence $\mathcal{T}(R) = I_{n_0}$ for some n_0. Further, put $J_k = \{a \in R | (p_1 \ldots p_{n_0})^k \times a = 0\}$. Then $\{J_k\}_{k \in \mathbb{N}}$ is an increasing sequence of ideals of R and $\mathcal{T}(R) = \bigcup_k J_k$. Hence $\mathcal{T}(R) = J_{k_0}$ for some k_0. Finally, set $m = (p_1 \ldots p_{n_0})^{k_0}$. □

Lemma 1.3. *Let S be a subdirectly irreducible radical ring with a monolith $M \cong \mathbb{Z}_p$.*

(i) *If $\mathcal{T}(S) \neq S$, then for every $n \in \mathbb{N}$ there exists a subgroup $G_n \subseteq S(+)$ such that $M \subseteq G_n \cong \mathbb{Z}_{p^n}(+)$.*
(ii) *If S is noetherian then is torsion and nil.*

Proof. (i) Let a be a torsion-free element in S, $n \in \mathbb{N}$. Then $p^{n-1} \times a$ is also torsion-free and hence $p^{n-1} \times a \notin \mathrm{Ann}(R)$. Thus there is $b \in S$ such that $0 \neq b \cdot (p^{n-1} \times a) \in M$. Therefore ba is of order p^n and we put $G_n = \langle ba \rangle$.

(ii) $\mathcal{T}(S)$ is a p-group, hence there is $n \in \mathbb{N}$ such that $p^n \times \mathcal{T}(S) = 0$, by 1.2. Suppose that $S \neq \mathcal{T}(S)$. Then, by (i), for every $k \in \mathbb{N}$ there is $a \in \mathcal{T}(S)$ of order p^k, a contradiction.

Suppose now $a \in S$ is not nilpotent. Put $J_n := \{x \in S | xa^n \in M\}$. Then $\{J_n\}_{n\in\mathbb{N}}$ is an increasing sequence of ideals. Since $a^{n+1} \notin \mathrm{Ann}(S)$, there is $b_n \in R$ such that $0 \neq b_n a^{n+1} \in M$ and obviously $b_n a^n \notin M \subseteq \mathrm{Ann}(S)$ (otherwise $0 \neq b_n a^{n+1} = (b_n a^n)a = 0$, a contradiction). Hence $b_n \in J_{n+1} \setminus J_n$ for every $n \in \mathbb{N}$, a contradiction. □

Theorem 1.4. *Let S be a subdirectly irreducible radical ring. Then S is noetherian if and only it is finite. Hence, if S is noetherian then is also artinian.*

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) S is a finitely id-generated ring. By 1.3(ii) is nil and hence by 1.1 is finitely generated as a ring. Moreover, $p^k \times R = 0$ for some prime p and $k \in N$ by 1.2 and 1.3(ii). Hence is S finite. □

Remark 1.5. Ring $\mathbb{Z}_{p^\infty}$, $p \in \mathbb{P}$, with a trivial multiplication is an example of a subdirectly irreducible radical ring that is artinian, but not noetherian.

2. Reflection of radical rings and subradical rings

Construction 2.1. Let R be a commutative ring, $\mathbb{D}(R)$ its Dorroh extension with a unit $1 = 1_{\mathbb{D}(R)}$. The set $1 + R = \{(1, r) \in \mathbb{D}(R) | r \in R\}$ is a subsemigroup of the semigroup $\mathbb{D}(R)(\cdot)$. Consider localization $(1 + R)^{-1}\mathbb{D}(R)$ of $\mathbb{D}(R)$ and the subring $\mathcal{A}(R) = (1 + R)^{-1}R$ with map $\varphi : R \to (1 + R)^{-1}R$, $\varphi(r) = r/1$, $r \in R$.

Proposition 2.2. *Let R be a commutative ring.*

(i) $\mathcal{A}(R) = (1 + R)^{-1}R$ is a radical ring, $r/(1 + s) = r/1 + \widetilde{s/1} \cdot r/1$ and $r/1 = r/(1 + s) + s/1 \cdot r/(1 + s)$ for every $r, s \in R$.

(ii) $\varphi : R \to (1+R)^{-1}R$, $\varphi(r) = r/1$ is a reflection of the category of the commutative rings into the category of the commutative radical rings (i.e. for every radical ring T and every ring homomorphism $\psi : R \to T$ there is an unique homomorphism of radical rings $f : (1 + R)^{-1}R \to T$ such that $\psi = f \circ \varphi$.)

(iii) $r/(1 + s) = 0$ iff $r/1 = 0$, $\ker(\varphi) = \{x \in R | (\exists a \in R)\ x = ax\}$.

(iv) $(1 + R)^{-1}R = 0$ iff $\varphi = 0$.

Proof. (i) For $a = r/(1+s) \in (1+R)R^{-1}$ take $\widetilde{a} = -r/(1+r+s)$. Then $a+\widetilde{a}+a\widetilde{a} = 0$. The rest is easy.

(ii) φ is a reflection: First, we show uniqueness. Let there be a homomorphism $f : (1 + R)^{-1}R \to T$ of radical rings such that $\psi = f\varphi$, where $\psi : R \to T$ is a ring homomorphism. Since $r/(1 + s) = r/1 + r/1 \cdot \widetilde{(-s/(1 + s))} = r/1 + r/1 \cdot \widetilde{s/1}$, we have $f(r/(1 + s)) = f(r/1) + f(r/1)\widetilde{f(s/1)} = \psi(r) + \psi(r)\widetilde{\psi(s)}$ for all $r, s \in R$.

To show existence, define f as above. Let $\mathbb{D}(T)$ be the Dorroh extension of T, then we have $f(r/(1 + s)) = \psi(r)(1 + \widetilde{\psi(s)})$ in $\mathbb{D}(T)$.

f is well defined: Let $r/(1+s) = r'/(1+s')$, where $r, r', s, s' \in R$, then $(1+u)w = 0$ for some $u \in R$, where $w = r(1 + s') - r'(1 + s)$. Hence $\psi(w) = \psi(-u)\psi(w)$ and thus $\psi(w) = 0$, since A is a radical ring. Therefore $\psi(r)(1+\psi(s')) = \psi(r')(1+\psi(s))$

and $\psi(r)(1+\widetilde{\psi(s)}) = \psi(r)(1+\psi(s'))(1+\widetilde{\psi(s')})(1+\widetilde{\psi(s)}) = \psi(r')(1+\psi(s))(1+\widetilde{\psi(s)})(1+\widetilde{\psi(s')}) = \psi(r')(1+\widetilde{\psi(s')})$.

It is easy to show, that f is a ring homomorphism. Hence $f(\widetilde{a}) = \widetilde{f(a)}$ for every $a \in (1+R)^{-1}R$ and f is a homomorphism of radical rings.

(iii),(iv) Obvious. □

Lemma 2.3. *Let R be a commutative ring.*

(i) If R is generated by X (as a ring), then $(1+R)^{-1}R$ is rd-generated by $\varphi(X)$.

(ii) Let R be a free commutative ring with a basis X (i.e. $R \cong \sum_{x\in X} x\mathbb{Z}[X]$). Then φ is injective and $(1+R)^{-1}R$ is a free radical ring with a basis $\varphi(X)$.

(iii) Let R be a subdirectly irreducible ring with a monolith M. If $\varphi|_M$ is injective, then φ is injective and $(1+R)^{-1}R$ is a subdirectly irreducible radical ring with a monolith $(1+R)^{-1}M$.

(iv) Let R be id-generated by X, then $(1+R)^{-1}R$ is id-generated by $\varphi(X)$.

Proof. (i) Follows immediately from $r/(1+s) = r/1 + r/1 \cdot \widetilde{s/1}$ for all $s, r \in R$.

(ii) (See also [4]11.1.2.) Let $R = \sum_{x\in X} x\mathbb{Z}[X]$ be a free commutative ring with a basis X. Then φ is injective by 2.2(iii) and $(1+R)^{-1}R$ is rd-generated by $\varphi(X)$ by (i). Let A be a radical ring and $g : \varphi(X) \to A$ a map. Then there is a ring homomorphism $\psi : R \to A$ such that $g \circ (\varphi|_X) \subseteq \psi$. Hence there is $f : (1+R)^{-1}R \to A$ a homomorphism of radical rings such that $f\varphi = \psi$. Thus $g \subseteq f$. Since $\varphi(X)$ rd-generates $(1+R)^{-1}R$, is f uniquely determined.

(iii) If $\ker(\varphi) \neq 0$, then by assumption $M \subseteq \ker(\varphi)$ and $\varphi|_M = 0$, a contradiction. We show that $(1+R)^{-1}M$ is a monolith of $(1+R)^{1}R$. Let $I \neq 0$ be an ideal of $(1+R)^{-1}R$ and $0 \neq r/(1+s) \in I$. Then $0 \neq r$ and thus $M \subseteq Rr + \mathbb{Z}r$. Let be $m \in M$ and $t \in R$. Since $r/1 = r/(1+s) + s/1 \cdot r/(1+s) \in I$, we have $m/1 \in I$ and $m/(1+t) = m/1 + \widetilde{t/1} \cdot m/1 \in I$. Hence $(1+R)^{-1}M \subseteq I$.

(iv) Obvious. □

Let $f : R \to T$ be a ring homomorphism, $\varphi_R : R \to \mathcal{A}(R)$ and $\varphi_T : T \to \mathcal{A}(T)$ reflections. Then there is a unique homomorphism of radical rings $f^* : \mathcal{A}(R) \to \mathcal{A}(T)$ such that $f^*\varphi_R = \varphi_T f$. Hence we have a covariant functor $R \mapsto \mathcal{A}(R)$, $f \mapsto f^*$ from the category of the commutative rings into the category of the commutative radical rings.

Definition 2.4. A commutative ring R will be called *subradical* iff $(\forall x, a \in R)$ $(x = xa \Rightarrow x = 0)$.

Remark 2.5. (i) Every radical ring is subradical (see [4] 7.9).

(ii) The class of subradical rings is closed under subrings and products.

(iii) Let R be a commutative ring, then $xR[x]$ and $xR[[x]]$ are subradical.

Indeed, for $0 \neq f = \sum_i a_i x^i \in R[[x]]$ put $m(f) = \min\{n | a_n \neq 0\} \geq 1$. If $0 \neq f = fg$ for some $f, g \in R[[x]]$ then $m(f) = m(f) + m(g)$, a contradiction.

(iv) The ring $R = x\mathbb{Z}[x]$ is subradical but its non-trivial homomorphic image $R/(1-x)R$ has a unit and hence isn't subradical.

(v) Let T be a domain with unit 1_T and R a subring such that $1_T \notin R$. Then R is subradical. (If $a = ax$, $a, x \in R$, then $(1_T - x)a = 0$. Since T is a domain and $1_T \notin R$, we get $a = 0$.)

(vi) Let R be a commutative ring. Put $R_0 = R$ and $R_{n+1} = \mathbb{D}(R_n)$ for $n \geq 0$. Then $T = \bigcup_n R_n$ is a ring without a unit and at the same time for every $x \in T$ there is $a \in T$ such that $x = ax$. Hence T is not subradical and $\mathcal{A}(T) = 0$.

(vii) Let R be a ring, $I = \{x \in R | (\exists a \in R)\ x = ax\}$. Then R/I is subring of $(1+R)^{-1}R$ by 2.2, hence subradical. It is easy to see that the natural projection

$\pi : R \to R/I$ is a reflection of the category of the commutative rings into the category of the commutative subradical rings.

(viii) Let R be subradical. Then $\mathrm{Ann}\big(\mathcal{A}(R)\big) = \mathrm{Ann}(R)$, $\mathcal{N}\big(\mathcal{A}(R)\big) = (1+R)^{-1}\mathcal{N}(R)$ and $\mathcal{T}\big(\mathcal{A}(R)\big) = (1+R)^{-1}\mathcal{T}(R)$.

Let $r/(1+s) \in \mathrm{Ann}\big((1+R)^{-1}R\big)$. Then $ru/(1+s) = 0$ for every $u \in R$. Hence $ru/1 = 0$ by 2.2(ii) and $ru = 0$, since R is subradical. Thus $r \in \mathrm{Ann}(R)$ and $r/(1+s) = r/1$. The rest is similar.

(ix) If $f : R \to T$ is surjective, then f^* is surjective.

(x) Let R be a ring with a unit, such that $\mathcal{J}(R) \neq 0$. Then the inclusion $i : \mathcal{J}(R) \to R$ is injective, but $i^* : \mathcal{J}(R) \to (1+R)^{-1}R = 0$ is a zero homomorphism.

(xi) Let R be a subradical ring and $\nu : T \to R$ be an injective ring homomorphism. Then ν^* is also injective.

(xii) The sequence of (subradical) rings $0 \to 2x\mathbb{Z}[x] \xrightarrow{i} x\mathbb{Z}[x] \xrightarrow{\pi} x\mathbb{Z}_2[x] \to 0$, where i is inclusion and π natural projection, is exact, but $Im\ i^* \neq \ker \pi^*$.

Indeed, denote $R = x\mathbb{Z}[x]$ and $I = 2R$. Then $Im\ i^* = \{r/(1+s) | r, s \in I\}$ and $\ker \pi^* = \{r/(1+s) | r \in I, s \in R\}$. We show that $2x/(1+x) \in \ker \pi^* \setminus Im\ i^*$. Suppose, on contrary, that $2x/(1+x) = 2xf(x)/(1+2xg(x))$ for some $f(x), g(x) \in \mathbb{Z}[x]$. Then $2x(1+2xg(x)) = 2xf(x)(1+x)$, since R is subradical, and thus $1+2xg(x) = f(x)(1+x)$. Using a natural projection $\sigma : \mathbb{Z}[x] \to \mathbb{Z}_2[x]$ we obtain $1 = f(x)(1+x)$ in $\mathbb{Z}_2[x]$, a contradiction, by comparing the degrees of the polynomials.

Corollary 2.6. *Let S be a subdirectly irreducible radical ring. The following are equivalent:*

(i) *S is finite,*
(ii) *S is finitely rd-generated,*
(iii) *S is noetherian.*

Proof. We only need to prove (ii)⇒(iii). The rest follows from 1.4.

We show that every finitely rd-generated radical ring is noetherian. It is enough to prove it only for a free radical ring T with a finite basis. By 2.2(iv) there is a free commutative ring $T = \sum_{i=1}^{n} x_i\mathbb{Z}[x_1, \ldots, x_n]$ and a reflection $\varphi : T \to (1+T)^{-1}T$ such that $U = (1+T)^{-1}T$. We prove that every ideal I in U is finitely generated as a U-module. Obviously $K = \varphi^{-1}(I)$ is finitely generated T-module, since T is a noetherian ring. Hence $(1+T)^{-1}\varphi(K) = I$ is also finitely generated $(1+T)^{-1}T$-module. □

Every finitely generated commutative ring is noetherian. Infinite fields are easy examples of noetherian rings that are not finitely generated. Following example shows a noetherian radical ring that is not finitely rd-generated.

Example 2.7. Put $R = x\mathbb{Z}_n[[x]]$, where $n = 0$ or $n \geq 2$. Then:

(i) R is id-generated by x.
(ii) R is a noetherian subradical ring.
(iii) $S = (1+R)^{-1}R$ is a noetherian radical ring which is not finitely (not even countably) rd-generated.

Proof. (i) Let $f(x) = \lambda_1 x + \lambda_2 x^2 + \cdots$, then $-\lambda_1 x + f(x) = xg(x) \in xR$ for some $g(x) \in R$.

(ii) The ring $\mathbb{Z}_n$ is noetherian, hence $\mathbb{Z}_n[[x]]$ is noetherian. An ideal I of R is also an ideal of $\mathbb{Z}_n[[x]]$. Hence R is noetherian. By 2.5 (iii) is subradical.

(iii) R is uncountable. The rest follows by localization. □

Following lemmas show that a subradical semigroup allows to get a subradical ring with help of the classical construction of a semigroup algebra (eventually the contracted construction, where the zero element of a semigroup is identified with a zero of the ring). In the first case the semigroup needs to be without a zero element.

Definition 2.8. Let A be a commutative semigroup. We call A *subradical* iff every $x \in A$, such that $x = ax$ for some $a \in A$, is a zero element.

Remark 2.9. Let $K \neq \emptyset$ be a finite set and $\varphi : K \to K$ a map. Choose $x \in K$. Since K is finite, there must be $m, n \in N$, $m < n$ such that $\varphi^m(x) = \varphi^n(x)$. Put $a = \varphi^m(x)$ and $k = n - m$. Then $\varphi^k(a) = a$.

Lemma 2.10. *Let R be a ring, A a semigroup with a zero element o. Let $R[A]$ be a semigroup algebra with an ideal $I = R \cdot o$.*

(i) Let R have a unit. If $R[A]/I$ is subradical, then A is subradical.

(ii) Let A be subradical. Then $R[A]/I$ is subradical.

Proof. (i) Easy.

(ii) Suppose, for contradiction, that $[\sum_{i=1}^{n} \lambda_i a_i] \cdot [\sum_{j=1}^{m} \mu_j b_j] = [\sum_{i=1}^{n} \lambda_i a_i]$ in $R[A]/I$, where $n \geq 1, m \geq 0$, $\lambda_i, \mu_j \in R$, $\lambda_i \neq 0$ and $a_i, b_j \in A$, $a_i \neq 0$ for all i, j. From the multiplication in $R[A]/I$ follows that there are maps $\varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and $\psi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $a_i = a_{\varphi(i)} b_{\psi(i)}$ for every $i = 1, \ldots, n$. By 2.9 there are $i_0 \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$ such that $\varphi^k(i_0) = i_0$. Hence $a_{i_0} = a_{\varphi(i_0)} b_{\psi(i_0)} = a_{\varphi^2(i_0)} b_{\psi^2(i_0)} b_{\psi(i_0)} = \cdots = a_{\varphi^k(i_0)} b_{\psi^k(i_0)} b_{\psi^{k-1}(i_0)} \ldots b_{\psi(i_0)}$. Thus $0 \neq a_{i_0} = a_{i_0}.b$ for some $b \in A$, a contradiction. □

Consequence 2.11. *Let R be a ring, A a semigroup and $R[A]$ a semigroup algebra.*

(i) Let R have a unit. If $R[A]$ is subradical, then A is subradical.

(ii) Let A be a subradical without a zero element. Then $R[A]$ is subradical.

Proof. (i) Easy.

(ii) Put $A' = A \cup \{o\}$, where o is new element and set $ao = oa = oo = o$ for every $a \in A$. Then A' is a subradical semigroup with a zero element o. Then by 2.10(ii) $R[A']/I$ is a subradical ring (where $I = R \cdot o$). Since $R[A] \cong R[A']/I$, it follows that $R[A]$ is also subradical. □

Comparing to the subradical rings, the only way how to obtain a radical ring as a semigroup algebra, is to use the contracted construction and a nil semigroup, as we will see in next lemmas. Radical (contracted) semigroup algebras provide thus only a limited class of examples to choose. The subradical semigroups on the other hand extend a much wider class of rings, which will be very useful for the constructions in the next chapter.

Lemma 2.12. *Let A be a commutatice semigroup with a zero element o (i.e. $ao = o$ for all $a \in A$), R be a commutative ring with a unit. Put $S = R[A]/I$, where $R[A]$ is a semigroup algebra and $I = R \cdot o$ an ideal of $R[A]$.*

S is a radical ring if and only if A is a nil semigroup (i.e. $(\forall a \in A)(\exists n \in \mathbb{N})(a^n = o)$). In this case S is a nil ring.

Proof. ($\Leftarrow$) S is generated by the set $\{\lambda a | \lambda \in R, a \in A\}$ of nilpotent elements and hence is it a nil ring and therefore a radical ring.

($\Rightarrow$) For $0 \neq a \in A$ there is $\widetilde{a} \in S$ such that $a + \widetilde{a} + a\widetilde{a} = 0$ and $\widetilde{a} = \sum_{i=1}^{n} \lambda_i a_i$, where $n \geq 1$, $0 \neq \lambda_i \in R$, $0 \neq a_i \in A$ for all $i = 1, \ldots, n$ and $a_i \neq a_j$ for all $i \neq j$. We show by induction on $k \geq 0$ that:

"If $k \leq n$ then $\widetilde{a} = \sum_{i=1}^{k} (-1)^i a^i + \sum_{i=k+1}^{n} \lambda'_i a'_i$ for some $0 \neq \lambda'_i \in R$ and $0 \neq a'_i \in A$ such that $a'_i \neq a'_j$ for $i \neq j$."

For $k = 0$ is it obvious. Suppose now, that the statement is true for $k \geq 0$. Hence

$$0 = a + \big(\sum_{i=1}^{k}(-1)^i a^i + \sum_{i=k+1}^{n} \lambda'_i a'_i\big) + a \cdot \big(\sum_{i=1}^{k}(-1)^i a^i + \sum_{i=k+1}^{n} \lambda'_i a'_i\big) =$$

$$= \big(a + \sum_{i=1}^{k}(-1)^i a^i + \sum_{i=2}^{k+1}(-1)^{i+1} a^i\big) + \sum_{i=k+1}^{n} \lambda'_i a'_i + \sum_{i=k+1}^{n} \lambda'_i a a'_i =$$

$$= (-1)^k a^{k+1} + \sum_{i=k+1}^{n} \lambda'_i a'_i + \sum_{i=k+1}^{n} \lambda'_i a a'_i.$$

Suppose first that $k < n$. Then must be $a^{k+1} \neq 0$ (otherwise would be $a + a' + aa' = 0$ for $a' = \sum_{i=1}^{k}(-1)^i a^i$ and hence $\widetilde{a} = a'$, a contradiction with the choice of n.) Now, if $a^{k+1} \neq a'_i$ for all $k+1 \leq i \leq n$, then there would be $n-k+1$ pairwise different non-zero elements $a^{k+1}, a'_{k+1}, \dots, a'_n$ and no more than $n-k$ pairwise different elements $aa'_{k+1}, \dots, aa'_n$, which would be in contradiction with the zero combination in the sum. Hence (without lose of generality) $a^{k+1} = a'_{k+1}$.

For contradiction suppose that $\lambda'_{k+1} \neq (-1)^{k+1}$. Then $0 = \mu a^{k+1} + \sum_{i=k+2}^{n} \lambda'_i a'_i + \sum_{i=k+1}^{n} \lambda'_i a a'_i$, where $0 \neq \mu = \lambda'_{k+1} + (-1)^k$. Considering again the numbers of pairwise different element in the sum, it must be $a'_i = aa'_{\pi(i)}$ for all i and some permutation π on the set $\{k+1, \dots, n\}$. Obviously $a'_i = a^m a'_{\pi^m(i)}$ for all $m \in \mathbb{N}$ and $\pi^{m_0} = id$ for some $m_0 \in \mathbb{N}$. Hence $0 \neq a'_{k+1} = a^{m_0} \cdot a'_{k+1}$, a contradiction, supposing S being radical.

Finally, let $k = n$. Then $0 = (-1)^k a^{k+1}$ and thus $\widetilde{a} = \sum_{i=1}^{n}(-1)^i a^i$ and a is nilpotent. □

Consequence 2.13. *Let A be a commutatice semigroup, $R \neq 0$ a commutative ring with a unit. Then the semigroup algebra $R[A]$ is never a radical ring.*

Proof. Suppose $R[A]$ is radical. Put $A' = A \cup \{o\}$, where o is a new element such that $ao = oa = oo = o$ for all $a \in A$. Then A' is a semigroup with a zero element o. Obviously $R[A] \cong R[A']/I$, where $I = R \cdot o$ is an ideal in $R[A']$. Hence $R[A']/I$ is a radical ring and by the previous lemma must A' be nilpotent. Thus for every $a \in A$ there is $n \in \mathbb{N}$ such that $a^n = o$, a contradiction, since $o \notin A$. □

3. Examples on subdirectly irreducible radical ring

In this section we will construct examples of the subdirectly irreducible radical rings to investigate the relations between various properties of these ring and the relations between the nilradical, the torsion part, the divisible part and the annihilator.

Our method will usually be to find a subdirectly irreducible subradical ring R with desired properties and then construct the reflection $\mathcal{A}(R) = (1+R)^{-1}R$ (see 2.2). This ring will be a subdirectly irreducible radical ring by 2.3(iii) and (since it is a localization and the reflection is a monomorphism) many of the properties of R will be preserved in $\mathcal{A}(R)$ (see for example 2.3(i)(iv), 2.5(viii)).

For a ring R we have a following sequence of implications:

R is a zero multiplication ring $\Rightarrow R$ is nilpotent $\Rightarrow R$ is a nil ring ($\Leftrightarrow R$ is a radical and Hilbert ring) $\Rightarrow R$ is a radical ring.

For a radical ring R and a subset $A \subseteq R$ we have this sequence of implications:

R is generated by A (as a ring) $\Rightarrow R$ is rd-generated by A (i.e. generated as a radical ring) $\Rightarrow R$ is id-generated by A (i.e. generated as a R-module).

Remark 3.1.

(1) There is a (finite) nilpotent $S \in \mathcal{S}$ that is not a zero-multiplication ring (see 3.3(ii)).
(2) There is a nil ring $S \in \mathcal{S}$ that is not nilpotent (see 3.17).
(3) There is $S \in \mathcal{S}$ that is not nil (see 3.12, 3.14).
(4) Every radical ring, that is finitely generated (as a ring) is nilpotent (see [4] 10.4). By 1.4 every finitely rd-generated R is finite, hence nilpotent.
(5) There is an one-id-generated $S \in \mathcal{S}$ that is not nil, hence not finitely rd-generated (see 3.12).

Remark 3.2. Let $G = \bigoplus_{i \in I} G_i$ be a direct sum of commutative groups. Then $\mathcal{D}(G) = \bigoplus_{i \in I} \mathcal{D}(G_i)$.

Indeed, put $H = \bigoplus_{i \in I} \mathcal{D}(G_i)$. Since H is divisible, we have $H \subseteq \mathcal{D}(G)$. Now, $G/H \cong \bigoplus_{i \in I} G_i/\mathcal{D}(G_i)$ and $G_i/\mathcal{D}(G_i)$ are reduced for every $i \in I$. Hence $\mathcal{D}(G)/H \subseteq \mathcal{D}(G/H) = 0$ and $\mathcal{D}(G) = H$.

Example 3.3. (i) Consider $S = \mathbb{Z}_{p^n}$, $1 \leq n \leq \infty$ with a trivial multiplication. Since every subdirectly irreducible group is of this form, are these rings the only zero-multiplications subdirectly irreducible radical rings.

(ii) Let $\mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, be a ring with the standard multiplication $\mod p^n$. Put $S(k,n) = p^k\mathbb{Z}_{p^n}$, $1 \leq k < n$. The ring $S(k,n)$ is an ideal of $\mathcal{J}(\mathbb{Z}_{p^n})$, hence a subdirectly irreducible radical ring. We have $\operatorname{Ann}(S(k,n)) = S(n-k,n)$ if $2k < n$ and $\operatorname{Ann}(S(k,n)) = S(k,n)$ otherwise.

Subdirectly radical rings can be obtained in following way (see [4] 12.2):

Let R be a radical ring and $a \in R$, $a \neq 0$. The set $\mathcal{A}$ of such ideals J that $a \notin J$ is a non-empty and upwards-inductive. If $K \in \mathcal{A}$ is maximal in $\mathcal{A}$ then $S = R/K$ is a subdirectly irreducible radical ring with a monolith $M = (K + Ra)/K$ if $Ra \not\subseteq K$ and $M = (K + \mathbb{Z}a)/K$ if $Ra \subseteq K$.

It is easy to see that every subdirectly irreducible radical ring is of this form.

In the next lemma we look what kind of rings arises if we apply this construction on the radical rings with quite simple structure - on the one-generated F-algebras.

Lemma 3.4. *Let F be a field, $n \in \mathbb{N}$, $R = xF[x]/x^{n+1}F[x]$. Then R is a nilpotent ring and:*

(i) *Subset $I \subseteq R$ is an ideal if and only if $I = Fx^n \oplus \cdots \oplus Fx^{k+1} \oplus Hx^k$, where $k \in \mathbb{N}$ and H is a subgroup of F.*
(ii) *Let S be a subdirectly irreducible factor of R. Then $S = Fx^n \oplus \cdots \oplus Fx^k \oplus Gx^{k+1}$, where $0 \leq k < n$ and $G = \mathbb{Z}_p$ if $\operatorname{char}F = p > 0$ and $G = \mathbb{Z}_{p^\infty}$ for some $p \in \mathbb{P}$ if $\operatorname{char}F = 0$. The multiplication is given as follows*

$$\Big(\sum_{i=1}^{k} \lambda_i x^i + g x^{k+1}\Big) \cdot \Big(\sum_{j=1}^{k} \mu_j x^j + h x^{k+1}\Big) = \sum_{l=1}^{k} \big(\sum_{i+j=l} \lambda_i \mu_j\big) x^l + \pi\big(\sum_{i+j=k+1} \lambda_i \mu_j\big) x^{k+1}$$

where $\pi : (F,+) \to (G,+)$ is an epimorphism of groups.

Proof. (i) Clearly, R is a vector space over F with a basis $x, \dots, x^n$. Let $a = \lambda_i x^i + \cdots + \lambda_n x^n$, where $\lambda_j \in F$, $\lambda_i \neq 0$ and let J be an ideal generated by a. We show that $J = Fx^n \oplus \cdots \oplus Fx^{i+1} \oplus \mathbb{Z}\lambda_i x^i$. The inclusion " $\subseteq$ " is clear. For " $\supseteq$ " let $n+1 \geq j \geq i+1$ be the least j such that $Fx^n \oplus \cdots \oplus Fx^j \subseteq J$. Suppose that $j > i+1$. Then $\lambda x^{j-1} = (\lambda\lambda_i^{-1} x^{j-(i+1)})a - (\lambda\lambda_i^{-1}\lambda_{i+1}x^{j+1} + \cdots + \lambda\lambda_i^{-1}\lambda_n x^{n+j-(i+1)}) \in J$ for every $\lambda \in F$. Hence $Fx^n \oplus \cdots \oplus Fx^{j-1} \subseteq J$, a contradiction. Hence $j = i+1$ and our claim is obvious.

(ii) S is a subdirectly irreducible factor of R if and only if $S \cong R/M$, where M is an ideal maximal with respect to the property $a \notin M$ for some $a \in R$. Let $a = \lambda_{k+1}x^{k+1} + \cdots + \lambda_n x^n$, where $\lambda_{k+1} \neq 0$, $0 \leq k < n$. Then, by (i), $M = Hx^{k+1} \oplus Fx^{k+2} \oplus \cdots \oplus Fx^n$, where H is a subgroup of F maximal with the respect to the property $\lambda_{k+1} \notin H$. Hence $R/M = Fx \oplus \cdots \oplus Fx^k \oplus Gx^{k+1}$, where $G = F/H$ and the multiplication is as above. Now, if $\mathrm{char}F = p > 0$, then F is a vector space over $\mathbb{Z}_p$ and, by the property of H, we easily get $F/H \cong \mathbb{Z}_p$. On the other hand, if $\mathrm{char}F = 0$, then F is divisible, since $\mathbb{Q} \leq F$, and so F/H is also divisible. From the classification of the divisible groups and the property of H we get that $F/H \cong \mathbb{Q}$ or $F/H \cong \mathbb{Z}_{p^\infty}$ for some $p \in \mathbb{P}$. But if $F/H \cong \mathbb{Q}$ then $S = R/M$ would be torsionfree, a contradiction, since the monolith of S is torsion. □

This example gives us an idea how to construct other subdirectly irreducible rings.

Definition 3.5. Let A be a commutative semigroup with a zero element 0. Put $\mathrm{Ann}(A) = \{a \in A | (\forall\, x \in A) ax = 0\}$ and $A^* = A \setminus \mathrm{Ann}(A)$.

Construction 3.6. Let A be a commutative semigroup with a zero element 0 and $\mathrm{Ann}(A) = \{0, m\}$, $m \neq 0$.

Let R be a commutative ring (not necessary with a unit), $G(+)$ a commutative group and $\varphi : R(+) \to G(+)$ a group homomorphisms.

Put $\mathcal{R}(R, A, G, \varphi) = (\bigoplus_{a \in A^*} R{\cdot}a) \oplus G{\cdot}m$ and set the multiplication on $\mathcal{R}(R, A, G, \varphi)$ as follows:

$$\Big(\sum_{a\in A^*} \lambda_a \cdot a + g\cdot m\Big) \cdot \Big(\sum_{b \in A^*} \mu_b \cdot b + h \cdot m\Big) = \sum_{c \in A^*} \big(\sum_{ab=c} \lambda_a \mu_b\big)\cdot c + \varphi\big(\sum_{aa'=m} \lambda_a \mu_{a'}\big)\cdot m$$

It is easy to verify that $\mathcal{R}(R, A, G, \varphi)$ is a commutative ring.

Example 3.7. Let A be a commutative semigroup with 0, $\mathrm{Ann}(A) = \{0, m\}$, $m \neq 0$ and such that for every $n \geq 1$, $a_1, \dots, a_n \in A^*$, there is $1 \leq i_0 \leq n$ and $b \in A$ such that $a_{i_0} b = m$ and $a_i b = 0$ for $a_i \neq a_{i_0}$.

(i) Let F be a field and set $G = \mathbb{Z}_p$ if $\mathrm{char}F = p > 0$ and $G = \mathbb{Z}_{p^\infty}$ for some $p \in \mathbb{P}$ if $\mathrm{char}F = 0$. Let $\pi : F \to G$ be a group epimorphism. Then $R = \mathcal{R}(F, A, G, \pi)$ is a subdirectly irreducible ring with a monolith $\mathbb{Z}_p \cdot m$. Further, $\mathrm{Ann}(R) = G \cdot m$, $\mathcal{D}(R) = 0$ if $\mathrm{char}F > 0$, and $\mathcal{D}(R) = R$ if $\mathrm{char}F = 0$.

(ii) Let be $p \in \mathbb{P}$, $i \in \mathbb{N}$ and $i \leq k \leq \infty$. Let $\nu : (\mathbb{Z}_{p^i}, +) \to (\mathbb{Z}_{p^k}, +)$ be inclusion. Then $R = \mathcal{R}(\mathbb{Z}_{p^i}, A, \mathbb{Z}_{p^k}, \nu)$ is a subdirectly irreducible ring with a monolith $\mathbb{Z}_p \cdot m$. Further, $\mathrm{Ann}(R) = \mathbb{Z}_{p^k} \cdot m$, $\mathcal{D}(R) = \mathbb{Z}_{p^\infty} \cdot m$ if $k = \infty$, and $\mathcal{D}(R) = 0$ otherwise.

(iii) Let R be the ring constructed in (i) or (ii). If A is a subradical semigroup, then R is also subradical, $(1+R)^{-1}R$ is radical and $\mathrm{Ann}((1+R)^{-1}R) = \mathrm{Ann}(R)$. Moreover, if A is nil, then they are also nil and, hence, radical.

Proof. (i) By 3.6 is R a ring. We show that $(Rx + \mathbb{Z}x) \cap G \cdot m \neq 0$ for every $0 \neq x \in R$. If $x \in R \setminus G \cdot m$ then $x = \sum_{i=1}^{n} \lambda_i a_i + \lambda m$, where $n \geq 1$, $\lambda \in G$, $0 \neq \lambda_i \in F$ and $a_i \in A^*$ for every i. By assumption there is i_0 and $b \in A$ such that $a_{i_0} b = m$ and $a_i b = 0$ if $a_i \neq a_{i_0}$. There is $\mu \in F$ such that $\pi(\lambda_{i_0}\mu) \neq 0$. Hence $0 \neq x(\mu b) \in G \cdot m$.

Now, since G is a subdirectly irreducible group, we easily get that $\mathbb{Z}_p \cdot m$ is a monolith of R.

Finally, suppose that $x \in \mathrm{Ann}(R)$ and $x = (\sum_{i=1}^{n} \lambda_i \cdot a_i) + g \cdot m$, where $n \in \mathbb{N}$, $0 \neq \lambda_i \in F$, $g \in G$ and $a_i \in A^*$ are pairwise different. By assumption there is $b \in A$ and $i_0 \in \{1, \ldots, n\}$ such that $(1 \cdot b)x = 1 \cdot ba_{i_0} = 1 \cdot m \neq 0$, a contradiction. Thus $\mathrm{Ann}(R) = G \cdot m$.

For divisible part use 3.2. (ii) Similar to (i).

(iii) Similar to the proof of 2.12 and 2.10. □

Remark 3.8. Following semigroups fulfil the conditions of 3.7:

(i) $A = F_0(x_1, \ldots, x_k)/\equiv$, where $F_0(x_1, \ldots, x_k)$ is a free commutative semigroup with a basis $\{x_1, \ldots, x_k\}$ and a zero element 0 and $\equiv$ is a congruence on $F_0(x_1, \ldots, x_k)$ generated by $x_i^{n_i} \equiv 0$, $x_i \in X$, $2 \leq n_i \in \mathbb{N}$, $i = 1, \ldots, k$.

(ii) $A = F(x) \cup \{0, a_0, a_1, \ldots\}$ (a disjoint union), where $F(x)$ is a free commutative semigroup with a basis $\{x\}$, $\{0, a_0, a_1, \ldots\}$ is a zero multiplication semigroup, $x^i 0 = 0x^i = 0$ for every $i \in \mathbb{N}$ and

$$x^i a_j = a_j x^i = \begin{cases} a_{j-i} & , j \geq i \\ 0 & , j < i. \end{cases}$$

(iii) The semigroup constructed in 4.4(iv).

Next construction shows how to glue together the subdirectly irreducible radical rings, with isomorphic monoliths, to get a new one.

Construction 3.9. Let $\{S_i\}_{i \in X}$ be a family of the subdirectly irreducible radical rings and let there for every $i, j \in X$ such that $|\mathrm{Ann}(S_i)| \leq |\mathrm{Ann}(S_j)|$ be a monomorphism $\nu_{i,j} : \mathrm{Ann}(S_i) \to \mathrm{Ann}(S_j)$ such that

(i) $\nu_{i,j} = id$ for every $i \in X$ and

(ii) $\nu_{j,k} \circ \nu_{i,j} = \nu_{i,k}$ if $|\mathrm{Ann}(S_i)| \leq |\mathrm{Ann}(S_j)| \leq |\mathrm{Ann}(S_k)|$.

Let $S = \bigoplus_{i \in X} S_i$ be a direct sum of rings and I an ideal of S generated by the set $\{x - \nu_{i,j}(x) | x \in \mathrm{Ann}(S_i), |\mathrm{Ann}(S_i)| \leq |\mathrm{Ann}(S_j)|, i, j \in X\}$. Then R/I is a subdirectly irreducible radical ring with a monolith $(\bigoplus_{j \in X} M_j + I)/I = (M_i + I)/I$ and an annihilator $(\bigoplus_{j \in X} \mathrm{Ann}(S_j)/I$, where M_j is a monolith of S_j.

Proof. Clearly, $M = (\bigoplus_{j \in X} M_j + I)/I$ is a direct limit of $\{M_i\}_{i \in I}$ and $N = (\bigoplus_{j \in X} \mathrm{Ann}(R_j))/I$ is a direct limit of $\{\mathrm{Ann}(R_i)\}_{i \in I}$. Hence $M = (M_i + I)/I \cong M_i \neq 0$ for every $i \in I$ and $N \cong \mathbb{Z}_{p^n}$, where $1 \leq n \leq \infty$.

Let $0 \neq a = [\sum_i x_i] \in S/I$, $x_i \in S_i$. If $x_{i_0} \in S_{i_0} \setminus \mathrm{Ann}(S_{i_0})$ for some i_0, then there is $r_{i_0} \in S_{i_0}$ such that $0 \neq r_{i_0} x_{i_0} \in M_{i_0}$, hence $0 \neq [r_{i_0}][\sum_i x_i] = [r_{i_0} x_{i_0}] \in M$ and $[\sum_i x_i] \notin \mathrm{Ann}(S/I)$. On the other hand, if $x_i \in \mathrm{Ann}(S_i)$ for every i, then $0 \neq a \in N \cong \mathbb{Z}_{p^n}$, hence $0 \neq p^k \times a \in M \subseteq N$ for some $k \in \mathbb{N}$.

Therefore M is the least nonzero ideal of S/I and $\mathrm{Ann}(S/I) = N$. □

Lemma 3.10. *Let K be a commutative R-algebra. Then $R \oplus K$ with the multiplication given as $(r, x) \cdot (s, y) = (rs, ry + sx + xy)$ is a commutative ring containing R and K as the subrings.*

Proof. Easy to verify. □

Lemma 3.11. *Let N be a R-module.*

(i) If R is a subradical ring and $Fix(r) = \{a \in N | ra = a\} = 0$ for every $r \in R$, then $R \oplus N$ is subradical.

(ii) If N is a faithful (i.e. $\mathrm{Ann}_R(N) = \{r \in R | (\forall a \in N) ra = 0\} = 0$) subdirectly irreducible R-module with a monolith M, then $S = R \oplus N$ is a subdirectly irreducible ring with a monolith M. Moreover, the annihilator of S is equal to $\{a \in N | (\forall r \in R) ra = 0\}$.

Proof. (i) Easy.

(ii) Clearly, M is an ideal of S. We need to show that $M \cap (Sx + \mathbb{Z}x) \neq 0$ for every $0 \neq x \in S$. Then $M = Sm + \mathbb{Z}m \subseteq Sx + \mathbb{Z}x$, where $0 \neq m \in M \cap (Sx + \mathbb{Z}x)$, hence M is the least nonzero ideal in S.

Let $0 \neq x = (r, a) \in S$. We can assume that $r = 0$, since for $r \neq 0$ there is $x \in N$ such that $rx \neq 0$, thus $0 \neq (0, rx) = (r, a)(0, x) \in Sx + \mathbb{Z}x$. Hence $a \neq 0$ and therefore $M \cap (Ra + \mathbb{Z}a) \neq 0$. For that reason $M \cap (Sx + \mathbb{Z}x) \neq 0$. The rest is easy. □

Example 3.12. Let X be a set, $k \in \mathbb{N} \cup \{\infty\}$. Put $G_i = \mathbb{Z}_{p^k}$ for $i \in (X \times \mathbb{N}) \cup \{0\}$ and

$$N = \bigoplus_{i \in (X \times \mathbb{N}) \cup \{0\}} G_i$$

a direct sum of groups and

$$T(X) = \begin{cases} \bigoplus_{x \in X} x\mathbb{Z}_{p^k}[x] & , k \in \mathbb{N} \\ \bigoplus_{x \in X} x\mathbb{Z}[x] & , k = \infty \end{cases}$$

a direct sum of rings.

For $x \in X$ let $\alpha_x \in End(N(+))$ be an endomorphism such that

$$\big(\alpha_x(a)\big)(i) = \begin{cases} a(x, n+1) & , i = (x, n) \\ a(x, 1) & , i = 0 \\ 0 & , \text{otherwise} \end{cases}$$

where $a \in N$.

Since $\alpha_x \circ \alpha_y = 0$ for $x \neq y$ we have a ring endomorphism

$$\alpha : \bigoplus_{x \in X} x\mathbb{Z}[x] \to End(N(+))$$

$$x \mapsto \alpha_x$$

and for $k < \infty$ we have $p^k x\mathbb{Z}[x] \subseteq \ker(\alpha)$ for every $x \in X$. Hence N is a $T(X)$-module (and thus N is a $T(X)$-algebra with $N^2 = 0$.)

(i) $R = T(X) \oplus N$ is a ring.
(ii) R is subradical.
(iii) R is a subdirectly irreducible with a monolith $(\mathbb{Z}_p)_0 \subseteq (\mathbb{Z}_{p^k})_0$.
(iv) $T(X)$ as a subring of R contains non-nilpotent elements.
(v) R is id-generated by X.
(vi) $S = (1 + R)^{-1}R$ is a subdirectly irreducible radical ring id-generated by X and is not a nil ring.
(vii) Let $S = (1 + R)^{-1}R$ be id-generated by Y. Then $|Y| \geq |X|$.
(viii) $\mathrm{Ann}((1 + R)^{-1}R) = \mathrm{Ann}(R) = G_0$.
(ix) $\mathcal{N}((1+R)^{-1}R) = (1+R)^{-1}(pT(X) \oplus N)$ and $\mathcal{D}((1+R)^{-1}R) = 0$ if $k \in \mathbb{N}$, and $\mathcal{N}((1 + R)^{-1}R) = \mathcal{D}((1 + R)^{-1}R) = (1 + R)^{-1}N$ if $k = \infty$.

Proof. (i) Follows from 3.10.

(ii) For $0 \neq a \in N$ denote $D(a) = \{n | (\exists x \in X) a(x,n) \neq 0\}$. Put $m(a) = \max D(a)$ if $D(a) \neq \emptyset$, $m(a) = 0$ otherwise and $m(0) = -1$ for $0 \in N$. Now, clearly $m(f \cdot a) < m(a)$ for every $f \in T(X)$ and $0 \neq a \in N$. Hence R is subradical by 3.11.

(iii) Let $0 \neq f = \sum_{x,n} \lambda_{(x,n)} x^n \in T(X)$ and $n_0 \in \mathbb{N}$ be the least such that $\lambda_{(x_0,n_0)} \neq 0$ for some $x_0 \in X$. Clearly, there is $\mu \in \mathbb{Z}_{p^k}$ such that $\lambda_{x_0,n_0}\mu \neq 0$. Put $a(x_0,n_0) = \mu$ and $a(i) = 0$ for $i \in (X \times \mathbb{N}) \cup \{0\}$, $i \neq (x_0,n_0)$. Then $a \in N$ and $fa \neq 0$. Hence N is a faithful $T(X)$-module.

Let $0 \neq a \in N$. If $m(a) = m \geq 1$ and $a(x_0,m) \neq 0$, $x_0 \in X$ then $0 \neq x_0^m a \in (\mathbb{Z}_p)_0$. And if $m(a) = 0$ then $0 \neq p^j \times a \in (\mathbb{Z}_p)_0$ for some $j \in \mathbb{N}_0$. Hence N is a subdirectly irreducible $T(X)$-module with a monolith $(\mathbb{Z}_p)_0$ and R is subdirectly irreducible by 3.11.

(iv),(v) Easy.

(vi) Follows from (iii),(iv) and 2.2.

(vii) Let R be id-generated by Y. Put $I = pR + N + \sum_{x \in X} xT(X)$. Then I is an ideal of R. Let $\pi : R \to R/I$ be a natural homomorphism. Since $\pi^* : (1+R)^{-1}R \to (1+R/I)^{-1}R/I = Q$ is an epimorphism, is Q id-generated by $\pi^*(Y)$. Hence $Q \cong (\mathbb{Z}_p)^{(X)}$ is generated by $\pi^*(Y)$ as a vector space over $\mathbb{Z}_p$ and thus $|X| = \dim Q \leq |\pi^*(Y)| \leq |Y|$.

(viii) Use 3.11.

(ix) We have $\mathcal{N}(\mathbb{Z}_{p^n}[x]) = p\mathbb{Z}_{p^n}[x]$ for the ring of polynomials $\mathbb{Z}_{p^n}[x]$, $n \in \mathbb{N}$ and further $(f+a)^n = f^n + nf^{n-1}a$ for every $f \in T(X)$, $a \in N$ and $n \in \mathbb{N}$. The rest follows from 3.2. □

Remark 3.13. The ring R from 3.12 is isomorphic to $\mathcal{R}(\mathbb{Z}_{p^k}, A, \mathbb{Z}_{p^k}, id|_{\mathbb{Z}_{p^k}})$, if $k \in \mathbb{N}$ and $A = \{0, a_0, a_1, \ldots\} \cup (\bigcup_{x \in X} F(x))$ (a disjoint union), where $F(x)$ is a free commutative semigroup with a basis $\{x\}$, $\{0, a_0, a_1, \ldots\}$ is a zero multiplication semigroup, $x^i 0 = 0 x^i = x^i y^j = 0$ for every $i, j \in \mathbb{N}$, $x, y \in X$, $x \neq y$ and

$$x^i a_j = a_j x^i = \begin{cases} a_{j-i} & , \ j \geq i \\ 0 & , \ j < i. \end{cases}$$

Example 3.14. Let $T = x\mathbb{Q}[x]$ be a ring. Put $G_0 = \mathbb{Z}_{p^\infty}$, $G_n = \mathbb{Q}$ for $n \in \mathbb{N}$ and $N = \bigoplus_{n \in \mathbb{N}_0} G_n$ a direct sum of groups and $\pi : (\mathbb{Q}, +) \to (\mathbb{Z}_{p^\infty}, +)$ an epimorphism of groups.

For $\lambda \in \mathbb{Q}$ and $k \in \mathbb{N}$ let $\alpha_{(\lambda,k)} \in End(N(+))$ be an endomorphism such that

$$\big(\alpha_{(\lambda,k)}(a)\big)(i) = \begin{cases} \lambda \cdot a(k+i) & , \ i \geq 1 \\ \pi(\lambda \cdot a(k)) & , \ i = 0 \end{cases}$$

where $a \in N$.

Put $\alpha(\sum_k \lambda_k x^k) = \sum_k \alpha_{(\lambda_k,k)}$. Then:

(i) $\alpha : T \to End(N(+))$ is a ring homomorphism. Hence N is a T-module (via α) and thus also a T-algebra with $N^2 = 0$.

(ii) The ring $R = T \oplus N$ is isomorphic to $\mathcal{R}(\mathbb{Q}, A, \mathbb{Z}_{p^\infty}, \pi)$, where A is from 3.8(ii).

(iii) R is a subdirectly irreducible subradical ring with a monolith $\mathcal{M}(R) \cong \mathbb{Z}_p$.

(iv) $\mathcal{T}((1+R)^{-1}R) = \mathrm{Ann}((1+R)^{-1}R) = \mathrm{Ann}(R) \cong \mathbb{Z}_{p^\infty}$, $\mathcal{D}((1+R)^{-1}R) = (1+R)^{-1}R$ and $\mathcal{N}((1+R)^{-1}R) = (1+R)^{-1}M$. Hence $(1+R)^{-1}R$ is divisible, but not nilpotent.

Proof. (i) We show $\alpha_{(\lambda,k)}\alpha_{(\mu,l)} = \alpha_{(\lambda\mu,k+l)}$, where $k, l \in \mathbb{N}$, $\lambda, \mu \in \mathbb{Q}$. We have $\big(\alpha_{(\lambda,k)}\alpha_{(\mu,l)}(a)\big)(i) = \lambda \cdot \alpha_{(\mu,l)}(a)(k+i) = \lambda\mu \cdot a(k+l+i) = \big(\alpha_{(\lambda\mu,k+l)}(a)\big)(i)$ for $i \geq 1$ and $\big(\alpha_{(\lambda,k)}\alpha_{(\mu,l)}(a)\big)(0) = \pi\big(\lambda \cdot \alpha_{(\mu,k)}(a)(k)\big) = \pi\big(\lambda\mu \cdot a(k+l)\big) = \big(\alpha_{(\lambda\mu,k+l)}(a)\big)(0)$. The rest is easy.

(ii) As in Example 1, put $m(a) = \max\{n|a(n) \neq 0\}$ for $0 \neq a \in M$ and $m(0) = -1$. Then, clearly, $m(fa) < m(a)$ for every $f \in S$ and $0 \neq a \in N$. Hence R is subradical by 3.11.

(iii) Let $0 \neq f = \sum_k \lambda_k x^k \in S$ and $k_0 \in \mathbb{N}$ be the least such that $\lambda_{k_0} \neq 0$. There is obviously $\mu \in \mathbb{Q}$ such that $\pi(\mu \cdot \lambda_{k_0}) \neq 0$. Put $a(k_0) = \mu$ and $a(k) = 0$ for $k \neq k_0$. Then $a \in N$ and $fa \neq 0$. Hence N is a faithful T-module.

Let $0 \neq a \in N$. If $m(a) = m \leq 1$ and $a(m) \neq 0$, then there is $\lambda \in \mathbb{Q}$ such that $\pi(\lambda \cdot a(m)) \neq 0$. Hence $0 \neq (\lambda x^m)a \in (\mathbb{Z}_p)_0$. If $m(a) = 0$ then $0 \neq p^k \times a \in (\mathbb{Z}_p)_0$ for some $k \in \mathbb{N}_0$. Hence N is a subdirectly irreducible T-module with a monolith $(\mathbb{Z}_p)_0$ and R is subdirectly irreducible by 3.11.

(iv) Easy. □

Remark 3.15.

(i) There is $S \in \mathcal{S}$ such that $\mathcal{D}(S) \subsetneqq \mathcal{N}(S)$ (see 3.12). There is $S \in \mathcal{S}$ such that $\mathcal{N}(S) \subsetneqq \mathcal{D}(S)$ (see 3.14). There is $S \in \mathcal{S}$ such that $\mathcal{D}(S) = \mathcal{N}(S)$ (see 3.12).

(ii) There is $S \in \mathcal{S}$ such that $\mathcal{D}(S) \subsetneqq \mathcal{T}(S)$ (see 3.7(ii) and 3.8(i)). There is $S \in \mathcal{S}$ such that $\mathcal{T}(S) \subsetneqq \mathcal{D}(S)$ (see 3.14). There is $S \in \mathcal{S}$ such that $\mathcal{D}(S) = \mathcal{T}(S)$ (see 3.3(i)).

(iii) There is $S \in \mathcal{S}$ such that $\mathcal{N}(S) \subsetneqq \mathcal{T}(S)$ (see 3.12). There is $S \in \mathcal{S}$ such that $\mathcal{T}(S) \subsetneqq \mathcal{N}(S)$ (see 3.4). There is $S \in \mathcal{S}$ such that $\mathcal{N}(S) = \mathcal{T}(S)$ (see 3.4, 3.3).

(iv) There is $S \in \mathcal{S}$ such that $\mathcal{D}(S) \cap \mathcal{N}(S) \not\subseteq \mathcal{T}(S)$ (see 3.14). There is $S \in \mathcal{S}$ such that $\mathcal{T}(S) \cap \mathcal{N}(S) \not\subseteq \mathcal{D}(S) \neq 0$ (see 3.7(ii) and 3.8(i)).

Lemma 3.16. *Let S be a subdirectly irreducible radical ring.*

(i) $\mathcal{D}(S) \cap \mathcal{T}(S) \subseteq \mathcal{N}(S)$.

(ii) *If $\mathcal{T}(S) = S$, then either $\mathcal{D}(S) = 0$ or $\mathcal{D}(S) = \mathrm{Ann}(S) \cong \mathbb{Z}_{p^\infty}$, $p \in \mathbb{P}$.*

Proof. (i) By [4] 1.13.(iii) is $\big(\mathcal{D}(S) \cap \mathcal{T}(S)\big)^2 = 0$.

(ii) By [4] 1.13.(iii) is $\mathcal{D}(S) \cdot \mathcal{T}(S) = 0$. Hence, by [4] 12.1.(vi), follows that $Div(S) \subseteq \mathrm{Ann}(S) \cong \mathbb{Z}_{p^n}$ for some p prime and $1 \leq n \leq \infty$. □

In examples 3.3, 3.4, 3.7 and 3.14 we have for the subdirectly irreducible radical ring S that $\mathrm{Ann}(S) \cong \mathbb{Z}_{p^\infty}$ assuming $\mathcal{T}(S) \neq S$. The following example shows that this is not true in common.

Example 3.17. Let $S_k = p\mathbb{Z}_{p^k}$, $k \geq 3$ be ideal of $\mathbb{Z}_{p^k}$. Then S_k is a subdirectly irreducible radical ring and $\mathrm{Ann}(S_k)$ is identical with the monolith $M (\cong \mathbb{Z}_p)$. Consider $T = (\bigoplus_{k=3}^{\infty} S_k)/I$ be the subdirectly irreducible radical ring with identified monoliths of all S_k (as in 3.9). Let $M \cong \mathbb{Z}_p$ be a monolith of T. Put $\varphi : p\mathbb{Z} \to End(T(+))$, $\varphi(pk)(x) = pk \times x$, $x \in T$. Then:

(i) T is a $p\mathbb{Z}$-algebra (via φ) and hence $R = p\mathbb{Z} \oplus T$ is a ring.

(ii) R is subradical.

(iii) R is a subdirectly irreducible with a monolith $M \cong \mathbb{Z}_p$, $\mathcal{T}(R) \neq R$ and $\mathcal{D}(R) = 0$.

(iv) $(1 + R)^{-1}R$ is radical, $\mathcal{T}((1 + R)^{-1}R) \neq R$ and $\mathcal{D}((1 + R)^{-1}R) = 0$.

Proof. (i) Easy.

(ii) Let $(pk, a) = (pk, a)(pl, b) = (p^2kl, pk \times b + pl \times a + ab)$, where $k, l \in \mathbb{Z}$, $a, b \in T$. Then $pk = p^2kl$, thus $k = 0$. Hence we have $a = pl \times a + ba$ and $(1 - pl) \times a = ba$. By induction we get $(1 - pl)^n \times a = b^n a$ for every $n \in \mathbb{N}$. Hence $(1 - pl)^{n_0} \times a = 0$ for some n_0, since T is a nil ring. Since a must be of order p^m, where $m \in \mathbb{N}_0$, we get $a = 0$. Thus R is subradical.

(iii) Let $0 \neq x = (pk, a) \in R$. If $k \neq 0$ then by the construction of A there obviously is $b \in T$ such that $ba = 0$ and the order of b is greater than $p|k|$. Then $(pk, a)(0, b) = (0, pk \times b) \neq 0$. Hence $R(0, a') + \mathbb{Z}(0, a') \subseteq Rx + \mathbb{Z}x$ for some $0 \neq (0, a') \in R$. Since T is a subdirectly irreducible by 3.9 with a monolith M, we have $M \subseteq Ta' + \mathbb{Z}a' \subseteq R(0, a') + \mathbb{Z}(0, a') \subseteq Rx + \mathbb{Z}x$. The ring R is thus subdirectly irreducible with a monolith M.

Since T and $p\mathbb{Z}$ are reduced, R is also reduced. □

In view of 3.17 we can ask whether $\mathcal{D}(\mathcal{T}(S)) \neq 0$ implies $\mathrm{Ann}(S) \cong \mathbb{Z}_{p^\infty}$, $p \in \mathbb{P}$, for a subdirectly irreducible radical ring S. Example 3.18 shows that the answer is again negative.

Example 3.18. Let a_1 be an element of order p in $\mathbb{Z}_{p^\infty}$. Put $U = (\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty})/K$, where K is a subgroup of $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ generated by $(a_1, -a_1)$. Let $T = p\mathbb{Z} \times p\mathbb{Z}$ be a product of rings. Put

$$\varphi : T \to End(U(+))$$
$$\varphi(pk, pl)\Big((a, b) + K\Big) = (pk \times a, pl \times b) + K,$$

$(a, b) \in \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. Then:

(i) U is a T-module and hence $R = T \oplus U$ is a ring.
(ii) R is subradical.
(iii) R is a subdirectly irreducible with a monolith $M = (\mathbb{Z}_p \oplus \mathbb{Z}_p)/K$ and $\mathcal{T}(R) = U$ is a divisible group.
(iv) $(1 + R)^{-1}R$ is a subdirectly irreducible radical ring with a monolith $\cong \mathbb{Z}_p$ and $\mathcal{T}((1 + R)^{-1}R) = (1 + R)^{-1}\mathcal{T}(R)$ is a divisible group.

Proof. (i) Easy.

(ii) Clearly, for $0 \neq a \in U$ is the order of a greater than the order of $(pk, pl)a$ for every $(pk, pl) \in U$. Hence R is subradical by 3.11.

(iii) Let $0 \neq (pk, pl) \in T$. Obviously there are $a, b \in \mathbb{Z}_{p^\infty}$ such that at least one of the elements $pk \times a$, $pl \times b$ is of order at least p^2. Then $(pk, pl) \cdot ((a, b) + K) \neq 0$. Hence U is a faithful T-module.

Let $0 \neq (a, b) + K \in U$. Suppose $(a, b) + K \notin M$. Then at least one of the orders of the elements a, b (say a) must be p^k, where $k \geq 2$. Hence $0 \neq (p^{k-1}, 0) \cdot ((a, b) + K) \in M$.

It follows by 3.11, that R is a subdirectly irreducible with a monolith M.

(iv) Follows from 2.2. □

4. Factors of the subdirectly irreducible radical rings by their monoliths

Corollary 4.1. *Let $R \neq 0$ be an artinian subradical ring. Then* $\mathrm{Ann}(R) \neq 0$.

Proof. Let $\mathrm{Ann}(R) = 0$ and $0 \neq a \in R$. Then there is $0 \neq b \in R$ such that $0 \neq ab$. Hence there is a sequence $a_1, a_2, \dots$ such that $0 \neq a_{n+1} \in Ra_n$ for every $n \in \mathbb{N}$. Put $I_n = Ra_n$. Then $\{I_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of ideals and $a_{n+1} \in I_n \setminus I_{n+1}$, since R is subradical. Hence R is not artinian. □

Corollary 4.2. *Let S be a subdirectly irreducible radical ring with a monolith M. Then:*

(i) *If $S/M \neq 0$ then every element of S/M is a zero divisor.*
(ii) *If $S \setminus \mathrm{Ann}(S) \neq \emptyset$ then $\mathcal{N}(S) \setminus \mathrm{Ann}(S) \neq \emptyset$. Hence if $S/M \neq 0$ then $\mathcal{N}(S/M) \neq 0$.*
(iii) $\mathrm{Ann}(S/M) \subseteq \mathcal{T}(S/M)$.
(iv) *Let S is artinian (e.g. finite) and $S/M \neq 0$. Then* $\mathrm{Ann}(S/M) \neq 0$.
(v) *S/M is noetherian if and only if it is finite.*

Proof. (i) Let $[0] \neq [a] \in S/M$. If $a \in \mathrm{Ann}(S)$, then $[a] \in \mathrm{Ann}(S/M)$ and hence is a zero-divisor. If $a \notin \mathrm{Ann}(S)$, then there is $b \in S$ such that $0 \neq ba \in M$ and hence $[b] \cdot [a] = [0]$ and $[b] \neq [0]$ (otherwise would be $b \in M \subseteq \mathrm{Ann}(S)$ and $ba = 0$, a contradiction).

(ii) Let $a \in S \setminus \mathrm{Ann}(S)$. Suppose that $a \notin \mathcal{N}(S)$. Then $a^2 \notin \mathrm{Ann}(S)$ and hence there is $b \in S$ such that $0 \neq ba^2 \in M \subseteq \mathrm{Ann}(S)$. Thus $ba \in S \setminus \mathrm{Ann}(S)$ (otherwise $ba^2 = (ba)a = 0$) and $(ba)^2 = b(ba^2) = 0$. Therefore $ba \in \mathcal{N}(S) \setminus \mathrm{Ann}(S)$.

(iii) If $[a] \in \mathrm{Ann}(S/M)$, then $ra \in M$ for every $r \in R$. Since $M \cong \mathbb{Z}_p$ we have $r(p \times a) = p \times (ra) \in p \times M = 0$ for every $r \in S$ and hence $p \times a \in \mathrm{Ann}(S)$. The additive group $\mathrm{Ann}(S)$ is a p-group and hence $p^k \times (p \times a) = 0$ for some $k \in \mathbb{N}$ and thus $a \in Tor(S)$.

(iv) Follows immediately from 4.1.

(v) Follows from 1.4. □

It is not difficult to see that for the subdirectly irreducible radical ring S from 3.3, 3.4, 3.12, 3.14, 3.17 and 3.18, such that $S/\mathcal{M}(S) \neq 0$, is $\mathrm{Ann}(S/\mathcal{M}(S)) \neq 0$. Now we construct a subdirectly irreducible radical ring without this property (see 4.5).

Definition 4.3. Let A be a commutative semigroup with a zero element 0. We will say that A *has a basis* $B \subseteq A^*$ *(with respect to* A^**)* iff every element $x \in A^*$ has (up to commutativity) unique form $x = b_1^{i_1} \cdots b_n^{i_n}$, where $b_j \in B$ are pairwise different and $i_k \in \mathbb{N}$ for $k = 1, \dots, n$.

Construction 4.4. Let A be a commutative semigroup with a zero element 0 and a basis $B \subseteq A^*$.

(1) Let F_X be a free commutative semigroup (without a unit) with a basis X. Put $F_X(A) = A \cup F_X \cup (A^* \times F_X)$ (a disjoin union of sets) and set a commutative binary operation $*$ on $F_X(A)$ as follows:

$$a * b = ab \qquad a * w = w * a = \begin{cases} 0 & , a \notin A^* \\ (a, w) & , a \in A^* \end{cases}$$

$$u * w = uw \qquad a * (c, v) = (c, v) * a = \begin{cases} 0 & , ac \notin A^* \\ (ac, v) & , ac \in A^* \end{cases}$$

$$u * (c, v) = (c, v) * u = (c, uv) \qquad (c, v) * (d, t) = \begin{cases} 0 & , cd \notin A^* \\ (cd, vt) & , cd \in A^* \end{cases}$$

for $a, b \in A$, $u, w \in F_X$ and $(c, v), (d, t) \in A^* \times F_X$.

Then $F_X(A)$ is a commutative semigroup with a zero element 0 and a basis $B \cup X$, A is a subsemigroup of $F_X(A)$ and $\mathrm{Ann}(A) = \mathrm{Ann}(F_X(A))$.

Proof. Put $\widetilde{A} = A \cup \{1_A\}$ and $\widetilde{F} = F_X \cup \{1_F\}$, where 1_A and 1_F are new symbols (units), such that $a1_A = 1_A a = a$, $1_A 1_A = 1_A$ and $w1_F = 1_F w = w$, $1_F 1_F = 1_F$ for every $a \in A$, $w \in F_X$. Further denote $S = \widetilde{A} \times \widetilde{F}$ a product of semigroups and $\rho = id|_S \cup \Big((\mathrm{Ann}(A) \times F_X) \cup \{(0, 1_F)\}\Big)^2$ a relation on S. It is easy to see, that ρ is a congruence on S. Set $\varphi : F_X(A) \to S/\rho$, where $a \mapsto (a, 1_F)/\rho$, $w \mapsto (1_A, w)/\rho$ and $(a, w) \mapsto (a, w)/\rho$ with $a \in A$, $w \in F_X$, $(a, w) \in A^* \times F_X$. Now is easy to verify, that φ is a monomorphism and hence $F_X(A)$ is a semigroup.

Let $z = a_1 \dots a_n x_1 \dots x_k = a'_1 \dots a'_m x'_1 \dots x'_l \in A^* \times F_X$ where $n, m, k, l \geq 1$, $a_i, a'_j \in A$, $x_i, x'_j \in F_X$. Then $a_1 \dots a_n = a'_1 \dots a'_m$ and $x_1 \dots x_k = x'_1 \dots x'_l$ hence by assumption z has an unique decomposition (up to commutativity) with respect to $B \cup X$. The rest is easy. □

(2) Choose $0 \neq m \in \mathrm{Ann}(A)$. Then there is a commutative semigroup A' such that:

(i) A is a subsemigroup of A', 0 is a zero element in A' and $\mathrm{Ann}(A') = \mathrm{Ann}(A)$
(ii) A' a has basis B' such that $B \subseteq B'$
(iii) $(\forall\, a \in A^*)(\exists\, b \in A')\; ab = m$.

Proof. Let $F_X(A)$ be as in (1) where $X = \{x_a | a \in A^*\}$. Set $C = \{(a, x_a) | a \in A^*\}$ and $D = F_X(A) \cdot C \setminus \{0\}$. Then $\sigma = Id|_{F_X(A)} \cup (C \times \{m\}) \cup (\{m\} \times C) \cup C^2 \cup (D \times \{0\}) \cup (\{0\} \times D) \cup D^2$ is obviously a congruence on $F_X(A)$.

Put $A' = F_X(A)/\sigma$ and $\varphi : A \to F_X(A)/\sigma$, $a \mapsto [a] = a/\sigma$. Then φ is a monomorphism and A can be identified with a subsemigroup of A'.

$\mathrm{Ann}(A') = \mathrm{Ann}(A)$: For $a \in A^*$ we obviously have $[a] \notin \mathrm{Ann}(A')$ and for $w \in F_X$ is also $[w] \notin \mathrm{Ann}(A')$, since $[w]^2 \neq [0]$. For $(a, w) \in A^* \times F_X$ such that $[(a, w)] \notin \mathrm{Ann}(A)$ suppose that $[(a, w)] \cdot [w] = [0]$. Then by the definition of σ there are $z \in F_X(A)$ and $(b, x_b) \in C$ such that $z(b, x_b) = (a, w^2)$. Hence x_b divides w^2 and therefore, due to the basis of $F_X(A)$, x_b divides w or $x_b = w$. It follows that $(a, w) \in D$ and $[(a, w)] = [0]$, a contradiction. Hence $[(a, w)] \notin \mathrm{Ann}(A')$.

Finally, put $B' = \varphi(B \cup X)$. Obviously $[x_a] \neq [x_b]$ for $a \neq b$. Now, if $[z_1 \dots z_n] = [z'_1 \dots z'_m] \notin \mathrm{Ann}(A'')$ where $z_i, z'_j \in B \cup X$, then, by definition of σ, we have $z_1 \dots z_n = z'_1 \dots z'_m$. Hence the decomposition is unique, since $B \cup X$ is a basis of $F_X(A)$. □

(3) There is a commutative semigroup A'' such that:

(i) A is a subsemigroup of A'', 0 is a zero element in A'' and $\mathrm{Ann}(A'') = \mathrm{Ann}(A)$
(ii) A'' has a basis B'' such that $B \subseteq B''$
(iii) For all $a, a_1, \dots, a_n \in A^*$ such that $a_i \neq a$ and a_i doesn't divide a for any $i = 1, \dots, n$ there exists $b \in A'$ such that $a_i b = 0$ for all $i = 1, \dots, n$ and $ab \in (A'')^*$.

Proof. Let $F_X(A)$ be as in (1) where $X = \{x_K | K \textit{ is finite subset of } A^*\}$. Set $C = \{(a, x_K) \in A^* \times X | K \textit{ is finite subset of } A^*, a \in K\}$. Then $\tau = Id|_{F_X(A)} \cup (C \cup F_X(A) \cdot C)^2$ is obviously a congruence on $F_X(A)$.

Put $A'' = F_X(A)/\tau$ and $\varphi : A \to F_X(A)/\tau$, $a \mapsto [a] = a/\tau$. Then φ is a monomorphism and A can be identified with a subsemigroup of A''.

$\mathrm{Ann}(A'') = \mathrm{Ann}(A)$: For $a \in A^*$ we obviously have $[a] \notin \mathrm{Ann}(A'')$ and for $w \in F_X$ is also $[w] \notin \mathrm{Ann}(A'')$, since $[w]^2 \neq [0]$. For $(a, w) \in A^* \times F_X$ such that $[(a, w)] \notin \mathrm{Ann}(A)$ suppose that $[(a, w)] \cdot [w] = [0]$. Then, by the definition of τ, there are $z \in F_X(A)$, a finite subset K of A^*, $b \in K$ and $(b, x_K) \in C$ such that $z(b, x_K) = (a, w^2)$. Hence x_K divides w^2 and therefore, due to the basis of $F_X(A)$, x_K divides w or $x_K = w$. It follows that $(a, w) \in C \cup F_X(A) \cdot C$ and $[(a, w)] = [0]$, a contradiction. Hence $[(a, w)] \notin \mathrm{Ann}(A'')$.

Let be $a \in A^*$ and $K = \{a_1, \dots, a_n\} \subseteq A^*$ such that $a_i \neq a$ and a_i doesn't divide a for any $i = 1, \dots, n$. Then obviously $[a_i] \cdot [x_K] = [0]$. Suppose that $[(a, x_K)] \in \mathrm{Ann}(A'')$. Then $[(a, x_K^2)] = [0]$. Hence, by the definition of τ, $a_i = a$ or a_i divides a for some i, a contradiction.

Finally, put $B'' = \varphi(B \cup X)$. For the rest see proof of (ii). □

(4) There is a (countable) commutative semigroup D with a zero element 0 such that:

(i) $\mathrm{Ann}(D) = \{0, m\} \subsetneq D$, where $m \neq 0$
(ii) D has an infinite basis $C \subseteq D^*$
(iii) $(\forall\, a \in D^*)(\exists\, b \in D)\; ab = m$

(iv) For all $a, a_1, \dots, a_n \in D^*$ such that $a_i \neq a$ and a_i doesn't divide a for any $i = 1, \dots, n$ there exists $b \in D$ such that $a_i b = 0$ for all $i = 1, \dots, n$ and $ab \in D^*$.

Proof. Let $D_0 = \{0, m\}$, $m \neq 0$ be a zero multiplicative semigroup, $X = \{x\}$. Put $D_1 = F_X(D_0)$. Further, by the induction, set $D_{n+1} = (D_n)'$ (see (2)), if n is odd, and $D_{n+1} = (D_n)''$ (see (3)), if n is even. Now, put $D = \bigcup_n D_n$. The rest is easy. □

Example 4.5. Let D be a semigroup constructed in 4.4 (4) with a zero element o and $\mathrm{Ann}(D) = \{o, m\}$, $m \neq o$. Let p be a prime number. Put $R = \mathbb{Z}_p[D]/I$, where $I = \mathbb{Z}_p \cdot o$ is an ideal in a semigroup algebra $\mathbb{Z}_p[D]$. Then:

(i) D is a subradical semigroup and hence R is a subradical ring, by 2.10(ii).
(ii) R is a subdirectly irreducible with a monolith $M = \mathbb{Z}_p \cdot m = \mathrm{Ann}(R)$ and for every $x \in R \setminus \mathrm{Ann}(R)$ there is $y \in R$ such that $xy \in R \setminus \mathrm{Ann}(R)$.
(iii) $S = (1 + R)^{-1}R$ is a subdirectly irreducible radical ring with a monolith $M \cong \mathbb{Z}_p$. Moreover $\mathrm{Ann}(S) = \mathrm{Ann}(R) = M$ and $\mathrm{Ann}(S/M) = 0$.

Proof. (i) Let $0 \neq a \in D$ such that $ab = a$ for some $b \in D$. Then $a, b \notin \mathrm{Ann}(D)$ and hence there are two different decomposition of a in the basis C, a contradiction.

(ii) Let $x \in R \setminus \mathbb{Z}_p \cdot m$. We show that $xb = \mu c$ for some $0 \neq \mu \in \mathbb{Z}_p$, $b \in D$, $c \in D^*$.

Clearly, $x = \sum_{i=1}^n \lambda_i a_i + \lambda m$, where $n \geq 1$, $\lambda, \lambda_i \in \mathbb{Z}_p$, $a_i \in S^*$, $\lambda_i \neq 0$ for all $i = 1, \dots, n$ and $a_i \neq a_j$ for $i \neq j$.

Suppose first $n = 1$. Since C is infinite, is there $b_0 \in C$ that doesn't divide a_1. Hence we have $c = a_1 b \in D^*$ (and $b_0 b = 0$) for some $b \in D$. Thus $xb = \lambda_1 a_1 b = \lambda_1 c$.

Now, let $n \geq 2$. Then there is i_0 such that a_i doesn't divide a_{i_0} for every $i \neq i_0$. Indeed, suppose on the contrary, that there is a map $\varphi : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $a_i = b_i a_{\varphi(i)}$ for every i, where $b_i \in S$. Then $\varphi^k(i') = i'$ for some $i' \in \{1, \dots, n\}$ and $k \in \mathbb{N}$. Hence $a_{i'} = b_{i'} a_{\varphi(i')} = b_{i'} b_{\varphi(i')} a_{\varphi^2(i')} = \dots = b_{i'} \dots b_{\varphi^{k-1}(i')} a_{\varphi^k(i')}$, a contradiction with the subradicality of D.

Thus there is $b \in D$ such that $a_i b = 0$ for $i \neq i_0$ and $c = a_{i_0} b \in D^*$. Hence $xb = \lambda_{i_0} a_{i_0} b = \lambda_{i_0} c$.

Since $xb \neq 0$, we have $x \in R \setminus \mathrm{Ann}(R)$ and we have proved $\mathrm{Ann}(R) \subseteq \mathbb{Z}_p \cdot m$. The other inclusion is trivial. Hence $\mathrm{Ann}(R) = \mathbb{Z}_p \cdot m$ and $xb \in R \setminus \mathrm{Ann}(R)$.

Finally, there is $b' \in D$ such that $cb' = m$. Hence $0 \neq x(cb') \in \mathbb{Z}_p \cdot m$ and R is a subdirectly irreducible with a monolith $\mathbb{Z}_p \cdot m$.

(iii) Follows from 2.2, 2.5 and (ii). □

Now we classify the subdirectly irreducible radical rings S such that $S/\mathcal{M}(S)$ is a zero-multiplication ring.

Lemma 4.6. *(i) Let S be a subdirecly irreducible radical ring with a monolith $M \cong \mathbb{Z}_p$ such that S/M is a zero multiplication ring. Then there is a bi-additive symmetric form $\mu : S \times S \to \mathbb{Z}_p$ and $0 \neq m \in M$ such that $ab = \mu(a, b)m$ for every $a, b \in S$ and $\mathrm{Ann}(S) = \ker(\mu) = \{x \in S | (\forall a \in S)\ \mu(x, a) = 0\}$.*

(ii) Conversely, let $S(+)$ be a group and $\mu : S \times S \to \mathbb{Z}_p$ a symmetric bi-additive form such that $\ker(\mu) \cong \mathbb{Z}_{p^n}$, where $1 \leq n \leq \infty$. Let $0 \neq m \in \ker(\mu)$ be such that $|m| = p$. Set $a \cdot b = \mu(a, b)m$ for all $a, b \in S$. Then S is a subdirectly irreducible radical ring with a monolith $M = \mathbb{Z}_p \cdot m$ and an annihilator $\ker(\mu)$ such that S/M is a zero multiplication ring.

Proof. (i) For $0 \neq m \in M$ and $a, b \in S$ put $\mu(a, b) = \lambda \in \mathbb{Z}_p$, where $ab = \lambda m$. The rest is easy.

(ii) First we show the associativity of the multiplication. For $a, b, c \in S$ we have $(ab)c = (\mu(a, b)m)c = \mu(\mu(a, b)m, c)m = 0$, since $m \in \ker(\mu)$ and hence

$a(bc) = (bc)a = 0 = (ab)c$. The distributivity is easy to verify. Further put $\widetilde{a} = -a + \mu(a,a)m$ for $a \in S$. Then $a + \widetilde{a} + a\widetilde{a} = a + (-a + \mu(a,a)m) + \mu(a, -a + \mu(a,a)m)m = \mu(a,a)m - \mu(a,a)m + \mu(a, \mu(a,a)m)m = 0$ and hence S is a radical ring.

For $a \in S \setminus \ker(\mu)$ there is $b \in S$ such that $ba = \mu(a,b)m \neq 0$ and for $a \in \ker(\mu)$ there is $k \geq 0$ such that $p^k \times a = m$, thus S is a subdirectly irreducible with a monolith M. The rest is clear. □

Lemma 4.7. *Let G be a commutative group, p a prime number. Then there is a symmetric bi-aditive form $\mu : G \times G \to \mathbb{Z}_p$ such that $\ker \mu \subseteq \mathbb{Z}_{p^\infty}$ if and only if $G \cong (\mathbb{Z}_p)^{(\kappa)} \oplus \mathbb{Z}_{p^n}$, with κ an ordinal number and $1 \leq n \leq \infty$.*

Proof. ($\Rightarrow$) We have $p \times G \subseteq \ker(\mu) \cong \mathbb{Z}_{p^n}$ since $\mu(p \times a, x) = p \times \mu(a,x) = 0$ for every $a, x \in G$. Hence $p \times G \cong \mathbb{Z}_{p^k}$, $0 \leq k \leq \infty$. Now, put $H = p \times G$ if $k = \infty$ and $H = \langle a \rangle$ for some $a \in G$ of order p^{k+1} if $k < \infty$. There exists a subgroup F of G such that $Soc(G) = (H \cap Soc(G)) \oplus F$.

We show that $G = F \oplus H$. Obviously, $H \cap F = H \cap F \cap Soc(G) = 0$. Let $x \in G$. Since $p \times H = p \times G$, there is $b \in H$ such that $p \times x = p \times b$ and hence $x = b + (x - b) \in H + F$.

($\Leftarrow$) Let $\{e_\alpha | \alpha < \kappa\}$ be a basis of $(\mathbb{Z}_p)^\kappa$. Set $\mu(\sum\limits_\alpha \lambda_\alpha e_\alpha + a, \sum\limits_\beta \mu_\beta e_\beta + b) = \sum\limits_\alpha \lambda_\alpha \mu_\alpha$ for $\lambda_\alpha, \mu_\beta \in \mathbb{Z}_p$ and $a, b \in \mathbb{Z}_{p^n}$. The rest is easy. □

The previous classification gives us a hint to find an example of a finite radical ring, that cannot be isomorphic to any factor of a subdirectly irreducible radical ring by its monolith.

Example 4.8. Let $R = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ be a ring with a trivial multiplication. Then R is radical and $N(R) = \mathrm{Ann}(R) = R$, but there is no subdirectly irreducible radical ring S with a monolith M, such that $S/M \cong R$.

Indeed, suppose that $\varphi : S \to R$ is such epimorphism. Then $\psi : S/Soc(S) \to R/Soc(R)$, $\psi(x + Soc(S)) = \varphi(x) + Soc(R)$ is also an epimorphism, where $Soc(G) = \{a \in G | p \times a = 0\}$ for a p-group G. But $S/Soc(S)$ is cyclic by 4.6 and 4.7, while $R/Soc(R) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, a contradiction.

References

[1] S.Bulman-Fleming, E. Hotzel, J. Wang, *Semigroups that are factors of subdirectly irreducible semigroups by their monolith*, Algebra Universalis **51** (2004), 1-7.

[2] J.Jezek, P.Markovic, D.Stanovsky, *Homomorphic images of finite subdirectly irreducible unary algebras*, Czech.Math.J. **57** (2007), 671-677.

[3] T. Kepka, *A note on subdirectly irreducible grupoids*, Acta Univ. Carolinae - Math. and Phys., Vol. 22, No. **1** (1981), 25-28.

[4] T. Kepka, P. Němec, *Commutative radical rings I*, Acta Univ. Carolinae - Math. and Phys., Vol. 48, No. **1** (2007), 11-41.

[5] R. McKenzie, D. Stanovský, *Every quasigroup is siomorphic to a subdirectly irreducible quasigroup modulo its monolith*, Acta Sci. Math. (Szeged) **72** (2006), 59-64.

[6] Neal H. McCoy, *Subdirectly irreducible commutative rings*, Duke Math. J., Vol. 12, No. **2** (1945), 381-387.

[7] N. Divinsky, *Commutative subdirectly irreducible rings*, Proc. Amer. Math. Soc. **8** (1957), 642-648.

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