ON SEPARATING SETS OF WORDS III

V. FLAŠKA, T. KEPKA AND J. KORTELAINEN

ABSTRACT. Transitive closures of special replacement relations in free monoids are studied.

1. Introduction

This article is an immediate continuation of [1] and [2]. References like I.3.3 (II.3.3, resp.) lead to the corresponding section and result of [1] ([2], resp.) and all definitions and preliminaries are taken from the same sources.

2. The transitive closure of the replacement relation

Let Z be a set of words and $\psi : Z \to A^*$ a mapping. Put ($\rho_{Z,\psi} =$) $\rho = \{(uzv, u\psi(z)v) \mid z \in Z, u, v \in A^*\}$, ($\lambda_{Z,\psi} =$) $\lambda = \rho \cup \mathrm{id}_{A^*}$, denote by ($\tau_{Z,\psi} =$) τ the smallest transitive relation defined on A^* and containing ρ (i. e., the transitive closure of ρ) and put ($\xi_{Z,\psi} =$) $\xi = \tau \cup \mathrm{id}_{A^*}$.

A sequence $w_0, w_1, \dots, w_m$ of words from A^*, $m \geq 1$, will be called a ρ-sequence (λ-sequence, resp.) if $(w_i, w_{i+1}) \in \rho$ ($(w_i, w_{i+1}) \in \lambda$, resp.) for every i, $0 \leq i < m$. The positive integer m is the length of the sequence and the sequence is said to lead from w_0 to w_m.

Proposition 2.1.

(i) *$(u, v) \in \tau$ if and only if there exists at least one ρ-sequence leading from u to v.*

(ii) *$(u, v) \in \xi$ if and only if there exists at least one λ-sequence leading from u to v (and hence ξ is the transitive closure of λ).*

Proof. Obvious from the definition of the relations τ and ξ. □

Proposition 2.2.

(i) *τ is stable and transitive.*

(ii) *ξ is stable, reflexive and transitive (and hence ξ is a stable quasiordering of the monoid A^*).*

Proof. Easy (use 2.1). □

Put ($\nu_{Z,\psi} =$) $\nu = \ker(\tau)$ (i. e., $(u, v) \in \nu$ iff $(u, v) \in \tau$ and $(v, u) \in \tau$) and ($\mu_{Z,\psi} =$) $\mu = \nu \cup \mathrm{id}_{A^*}$.

The work is a part of the research project MSM0021620839 financed by MŠMT and the second author was supported by the Grant Agency of Czech Republic, No. 201/09/0296.

Proposition 2.3.

(i) *ν is stable, symmetric and transitive.*
(ii) *If $(u,v) \in \nu$, then $(u,u) \in \nu$, $(v,v) \in \nu$, $(u,u) \in \tau$ and $(v,v) \in \tau$.*
(iii) *μ is a congruence of the monoid A^*.*
(iv) $\mu = \ker(\xi)$.

Proof. Easy. □

Proposition 2.4. *The following conditions are equivalent:*

(i) *ν is reflexive.*
(ii) $\nu = \mu$.
(iii) *ν is a congruence of the monoid A^*.*
(iv) *τ is reflexive.*
(v) *For every $u \in A^*$ there is at least one ρ-sequence leading from u to u.*

Proof. Easy. □

3. On when the closure is antisymmetric

Proposition 3.1. *The following conditions are equivalent:*

(i) *τ is a stable near-ordering on A^*.*
(ii) *τ is antisymmetric.*
(iii) *$w_0 \neq w_m$ whenever $w_0, w_1, \dots, w_m$ is a ρ-sequence of length $m \geq 2$ such that $w_i \neq w_0$ for at least one i, $1 \leq i < m$.*
(iv) $\nu \subseteq \mathrm{id}_{A^*}$.
(v) *ξ is a stable (reflexive) ordering on A^*.*
(vi) *ξ is antisymmetric.*
(vii) $\mu = \mathrm{id}_{A^*}$.

Proof. Easy (use 2.2,2.3 and 2.4). □

Remark 3.2. The equivalent conditions of 3.1 are satisfied if $u = v$ whenever $(u,v) \in \tau$ and $(v,u) \in \rho$.

Indeed, assume that the latter condition is true. Let $w_0, w_1, \dots, w_m$ is a ρ-sequence of length $m \geq 2$ such that $w_i \neq w_0$ for at least one i, $1 \leq i < m$. Let j be the largest number with $1 \leq j \leq m$ and $w_j \neq w_0$. If $j < m$, then $w_{j+1} = w_0$, $(w_0, w_j) \in \tau$, $(w_j, w_0) \in \rho$, a contradiction. Thus $j = m$ and $w_m \neq w_0$.

Proposition 3.3. *The following conditions are equivalent:*

(i) *τ is a stable sharp ordering on A^*.*
(ii) *τ is irreflexive.*
(iii) *τ is irreflexive and antisymmetric.*
(iv) *$w_0 \neq w_m$ whenever $w_0, w_1, \dots, w_m$ is a ρ-sequence.*
(v) $\nu = \emptyset$.

Proof. Easy (use 2.2,2.3 and 2.4). □

Proposition 3.4. *Assume that $|\psi(z)| < |z|$ ($|z| < |\psi(z)|$, resp.) for every $z \in Z$. Then:*

(i) $|v| < |u|$ *($|v| < |u|$, resp.) for every $(u, v) \in \tau$.*
(ii) *τ is a stable sharp ordering.*
(iii) *ξ is a stable ordering.*

Proof. Easy (use 3.1 and 3.3). □

Lemma 3.5. *Let $Z \subseteq A^+$ be a strongly separating set and let $w_0, \dots, w_m$ be a ρ-sequence. Then:*

(i) $\mathrm{tr}(w_0) \leq \mathrm{tr}(w_m) + m$.
(ii) *If, for every $z \in Z$, either $|\psi(z)| \leq 2$ or $\psi(z)$ is reduced, then $\mathrm{tr}(w_m) \leq \mathrm{tr}(w_0) + m$.*
(iii) *If $|\psi(z)| \leq 1$ for every $z \in Z$, then $\mathrm{tr}(w_m) \leq \mathrm{tr}(w_0)$*

Proof. The result follows by induction from I.7.6. □

Proposition 3.6. *Assume that $|\psi(z)| \leq 1$ for every $z \in Z$. If $w \in A^*$ is a meagre word and $(w, v) \in \xi$ then v is meagre.*

Proof. The result follows immediately from 3.5 (iii). □

4. Reduced and pseudoreduced words

Proposition 4.1. *The following conditions are equivalent for a word w:*

(i) *w is reduced.*
(ii) *$(w, x) \notin \rho$ for every $x \in A^*$.*
(iii) *$(w, x) \notin \tau$ for every $x \in A^*$.*

Proof. Obvious. □

A word w will be called *strongly ((Z, ψ)-) pseudoreduced* (or *almost ((Z, ψ)-) reduced*) if $x = w$ for all $(w, x) \in \rho$.

Proposition 4.2. *The following conditions are equivalent for a word w:*

(i) *w is strongly pseudoreduced.*
(ii) *$x = w$ for all $(w, x) \in \lambda$.*
(iii) *$x = w$ for all $(w, x) \in \tau$.*
(iv) *$x = w$ for all $(w, x) \in \xi$.*
(v) *$\psi(z) = z$ for every $z \in Z$ that is a factor of w.*

Proof. Easy. □

Corollary 4.3. *If $\psi(z) \neq z$ for every $z \in Z$, then every strongly pseudoreduced word is reduced.*

A word w will be called *(weakly) ((Z, ψ)-) pseudoreduced* if $(w, x) \in \rho$ implies $(x, w) \in \rho$ (i. e., $(w, x) \in \ker(\rho)$).

Proposition 4.4. *Assume that* $\ker(\rho) \subseteq \mathrm{id}_{A^*}$ *(e. g.,* $\nu \subseteq \mathrm{id}_{A^*}$ *– see 3.1). Then a word* w *is pseudoreduced iff it is strongly pseudoreduced.*

Proof. Clearly, every strongly pseudoreduced word is pseudoreduced. On the other hand, if w is pseudoreduced and $(w, x) \in \rho$, then $(x, w) \in \rho$, $(w, x) \in \ker(\rho)$ and $w = x$. □

A word w will be called *((Z, ψ)-) quasireduced* if $(w, x) \in \tau$ implies $(x, w) \in \tau$ (then $(w, x) \in \nu$).

Proposition 4.5. *A word* w *is quasireduced iff* $(w, x) \in \xi$ *implies* $(x, w) \in \xi$

Proof. Obvious. □

Proposition 4.6.

(i) *Every strongly pseudoreduced word is quasireduced.*
(ii) *If* $\nu \subseteq \mathrm{id}_{A^*}$ *(see 3.1), then every quasireduced word is strongly pseudoreduced.*

Proof. Obvious. □

Proposition 4.7. *Assume that* $\nu \subseteq \mathrm{id}_{A^*}$ *(e. g., if* ψ *is strictly length decreasing or strictly length increasing – see 3.4). Then the following conditions are equivalent for a word* w*:*

(i) w *is pseudoreduced.*
(ii) w *is strongly pseudoreduced.*
(iii) w *is quasireduced.*

Moreover, if $\psi(z) \neq z$ *for every* $z \in Z$*, then these conditions are equivalent to:*

(iv) w *is reduced.*

Proof. Combine 4.2, 4.3, 4.4 and 4.6. □

Proposition 4.8. *Assume that the mapping* ψ *is strictly length decreasing. Then for every word* $w \in A^*$ *there exists at least one reduced word* $r \in A^*$ *such that* $(w, r) \in \xi$*.*

Proof. Easy (by induction on $|w|$). □

5. Meagre words

A word w will be called *meagre* if $\mathrm{tr}(w) \leq 1$.

Proposition 5.1. *(II.5.4) Let* Z *be a strongly separating set of words such that* $Z \neq \{\varepsilon\}$ *and, for every* $z \in Z$*, either* $|\psi(z)| \leq 1$ *or* $\psi(z)$ *is reduced. Assume further that there exists no pair* $(z_1, z_2) \in Z \times Z$ *such that either* $\psi(z_1) = z_2$*,* $\psi(z_2) = z_1$ *or* $z_1 = ur$*,* $z_2 = sv$*,* $\psi(z_1) = us$*,* $\psi(z_2) = rv$*,* $u, v, r, s \in A^+$*. Then every pseudoreduced meagre word is reduced.*

A word w will be called *pseudomeagre* if $(w, x) \in \rho$ for at most one $x \in A^*$. Clearly, every meagre word is pseudomeagre.

Proposition 5.2. *(II.6.7) Let Z be a strongly separating set of words such that $Z \neq \{\varepsilon\}$. Assume further that the following two conditions are satisfied:*

(c1) *$\varepsilon \neq \psi(z) \neq z$ and $\psi(z) \neq zxz$ for all $z \in Z$ and $x \in A^*$.*
(c2) *If $z_1, z_2 \in Z$ and $x, y \in A^*$ are such that $\psi(z_1) = yxz_1$ and $\psi(z_2) = z_2xy$, then $\psi(z_1) = \psi(z_2)$.*

Then every pseudomeagre word is meagre.

Proposition 5.3. *(II.6.8) Let Z be a strongly separating set of words such that $Z \neq \{\varepsilon\}$.*

(i) *If $\psi(z) \neq \varepsilon$ and z is neither a prefix nor a suffix of $\psi(z)$ for every $z \in Z$, then every pseudomeagre word is meagre.*
(ii) *If $|\psi(z)| \leq |z|$ for every $z \in Z$, then every pseudomeagre word is meagre if and only if $\varepsilon \neq \psi(z) \neq z$ for every $z \in Z$.*

Proposition 5.4. *(II.7.3) Let Z be a strongly separating set of words as in 5.1. Assume further that there exists no triple $(z_1, z_2, z_3) \in Z \times Z \times Z$ such that $z_1 = uv$, $z_3 = gh$ and $\psi(z_2) = vpg$ for some $u, v, g, h \in A^+$ and $p \in A^*$. If $(w, v) \in \xi$ and w is meagre, then v is meagre.*

Corollary 5.5. *Let Z be a strongly separating set of words such that $\varepsilon \notin Z$ and $\psi(Z) \subseteq A$. Then:*

(i) *A word v is meagre, provided that $(w, v) \in \xi$ for a meagre word w.*
(ii) *If $\psi(z) \neq z$ for every $z \in Z$, then every pseudomeagre word is meagre.*
(iii) *If there exists no pair $(z_1, z_2) \in Z \times Z$ such that $\psi(z_1) = z_2$ and $\psi(z_2) = z_1$, then every pseudoreduced pseudomeagre word is reduced.*

6. Confluency

Proposition 6.1. *Assume that for all $u, v, w \in A^*$ such that $(u, v) \in \rho$, $(u, w) \in \rho$, $(v, w) \notin \rho$, $(w, v) \notin \rho$ and $v \neq w$ there exists at least one $x \in A^*$ with $(v, x) \in \rho$ and $(w, x) \in \rho$ (then $v \neq x \neq w$ and $v \neq u \neq w$). Then the relation ξ is confluent (i. e., for all $(p, q) \in \xi$, $(p, r) \in \xi$ there exists at least one $s \in A^*$ with $(q, s) \in \xi$ and $(r, s) \in \xi$).*

Proof. It follows easily from our assumption that the relation λ is confluent. We have to show that the transitive closure ξ of λ is confluent as well.

Let $u_0, u_1, \ldots, u_m$ and $v_0, v_1, \ldots, v_n$ be a λ-sequences such that $u_0 = v_0$.

Assume first that $m = 1$. Proceeding by induction, we find words $r_1, \ldots, r_n$ in A^* in the following way: Since λ is confluent, we have

$(u_1, r_1) \in \lambda$ and $(v_1, r_1) \in \lambda$ for some $r_1 \in A^*$. Now, if $1 \le j < n$ and $r_1, \dots, r_j$ are found such that $u_1, r_1, r_2, \dots, r_j$ is a λ-sequence and $(v_1, r_1) \in \lambda$, $(v_2, r_2) \in \lambda$, $\dots$, $(v_j, r_j) \in \lambda$, then $(r_j, r_{j+1}) \in \lambda$ and $(v_{j+1}, r_{j+1}) \in \lambda$ for some $r_{j+1} \in A^*$. Consequently, by induction, $(v_n, r_n) \in \lambda$ and $u_1, r_1, \dots, r_n$ is a λ-sequence. Thus $(u_m, r_n) = (u_1, r_n) \in \xi$ and $(v_n, r_n) \in \xi$.

In the general case, we proceed by induction on $m+n$. Due to the preceding part of the proof, we can assume that $m \ge 2$. Then $(u_{m-1}, r) \in \xi$ and $(v_n, r) \in \xi$ for some $r \in A^*$. Furthermore, $(u_{m-1}, u_m) \in \lambda$, and hence $(u_m, s) \in \xi$ and $(r, s) \in \xi$ for at least one $s \in A^*$. Consequently, $(u_m, s) \in \xi$ and $(v_n, s) \in \xi$. □

Remark 6.2. Assume that ξ is confluent (see 6.1). If $(u, v) \in \tau$ and $(u, w) \in \tau$, then $(v, r) \in \xi$ and $(w, r) \in \xi$ for some $r \in A^*$. If $v \ne r \ne w$, then $(v, r) \in \tau$ and $(w, r) \in \tau$. If $v = r \ne w$, then $(w, v) \in \tau$. If $v \ne r = w$, then $(v, w) \in \tau$. The final case is $v = r = w$ (cf. 6.1).

Remark 6.3. Let the assumption of 6.1 be satisfied and let $w \in A^*$ be such that $(x, w) \in \tau$ whenever $(w, x) \in \rho$. We show that w is quasireduced. Indeed, if $w = w_0, w_1, \dots, w_m = x$ is a ρ-sequence, we show by induction on m that $(x, w) \in \tau$. To this purpose, we can assume that $x \ne w$. The case $m = 1$ is clear. Let $m \ge 2$. We have $(w_{m-1}, w_m) \in \tau$ by induction and $(w_{m-1}, x) \in \rho$. Proceeding similarly as in the proof of 6.1, we find a word $r \in A^*$ such that $(w, r) \in \lambda$ and $(x, r) \in \xi$. Then $(r, w) \in \xi$, and hence $(x, w) \in \xi$. Since $x \ne w$, we get $(x, w) \in \tau$.

Proposition 6.4. *Let $Z \subseteq A^+$ be a strongly separating set. Then:*

(i) *The relation ξ is confluent.*
(ii) *If $(u, v) \in \tau$ and $(u, w) \in \tau$, then either $(v, r) \in \tau$ and $(w, r) \in \tau$ for some $r \in A^*$ or $(v, w) \in \tau$ or $(w, v) \in \tau$ or $v = w$.*

Proof. Combine I.7.11, 6.1 and 6.2. □

Proposition 6.5. *Let $Z \subseteq A^+$ be a strongly separating set and let ψ be strictly length-decreasing. Then for every $w \in A^*$ there exists a uniquely determined reduced word r such that $(w, r) \in \xi$.*

Proof. Combine 4.8 and 6.4. □

Lemma 6.6. *Let $Z \subseteq A^+$ be a strongly separating set and let ψ be strictly length-decreasing. If $(u_1 u_2 \cdots u_m, r) \in \xi$, $(u_i, v_i) \in \xi$, $1 \le i \le m$, and r is reduced, then $(v_1 v_2 \cdots v_m, r) \in \xi$.*

Proof. We have $(u_1 u_2 \cdots u_m, v_1 v_2 \cdots v_m) \in \xi$ and the rest follows from 6.4. □

7. Regularity

We will say that the replacement relation ρ (or the pair (Z, ψ)) is *regular* if $m = n$ whenever $w_0, w_1, \dots, w_m$ and $v_0, v_1, \dots, v_n$ are

ρ-sequences with $w_0 = v_0$ and $w_m = v_n$. In such a case, we put $(\mathrm{dist}_{(Z,\psi)}(w_0, w_m) =)\ \mathrm{dist}(w_0, w_m) = m$.

Lemma 7.1. *Assume that ρ is regular. If $(u, v) \in \tau$ and $(v, w) \in \tau$, then* $\mathrm{dist}(u, w) = \mathrm{dist}(u, v) + \mathrm{dist}(v, w)$.

Proof. Easy. □

Remark 7.2. Assume that ρ is regular. Then τ is irreflexive, and hence τ is a stable sharp ordering on A^* by 3.3. Now, setting $\mathrm{dist}(w, w) = 0$, we have $\mathrm{dist}(u, v)$ for all $(u, v) \in \xi$. Clearly, $\mathrm{dist}(u, w) = \mathrm{dist}(u, v) + \mathrm{dist}(v, w)$ for all $(u, v) \in \xi$ and $(v, w) \in \xi$.

Lemma 7.3. *Assume that for all $u, v, w \in A^*$ such that $(u, v) \in \rho$, $(u, w) \in \rho$ and $v \neq w$ there is at least one $r \in A^*$ with $(v, r) \in \rho$ and $(w, r) \in \rho$. If $u_0, u_1, \dots, u_m$ and $v_0, v_1, \dots, v_n$ are ρ-sequences with $u_0 = v_0$, $u_m = v_n$ and u_m is reduced, then $m = n$*

Proof. We will proceed by induction on $m+n$. We have $m+n \geq 2$ and, if $m + n = 2$, then $m = n = 1$. Henceforth, assume that $1 \leq n \leq m$ and $2 \leq m$.

If $u_1 = v_1$, then $n \geq 2$, since v_n is reduced. Now, $u_1, \dots, u_m$ and $v_1, \dots, v_n$ are ρ-sequences of length $m - 1$ and $n - 1$, resp. Then $m - 1 = n - 1$ by induction, and so $m = n$.

It remains to consider the case $u_1 \neq v_1$. According to our assumption, there is $r_1 \in A^*$ with $(u_1, r_1) \in \rho$ and $(v_1, r_1) \in \rho$. Since v_n is reduced, we have $n \geq 2$ and, proceeding similarly, we find an index $1 \leq k < n$ and words $r_1, \dots, r_k$ such that $(r_i, r_{i+1}) \in \rho$ for every $1 \leq i < k$, $(v_j, r_j) \in \rho$ for every $1 \leq j \leq k$ and $r_k = v_{k+1}$ (use again the fact that v_n is reduced). Clearly, $u_1, r_1, r_2 \dots, r_{k-1}, v_{k+1}, \dots, v_n$ and $u_1, \dots, u_m$ are ρ-sequences of length $n - 1$ and $m - 1$, resp. Thus $n - 1 = m - 1$ by induction and we get $m = n$. □

Lemma 7.4. *Let the assumptions of 7.3 be satisfied. If $u_0, u_1, \dots, u_m$ and $v_0, v_1, \dots, v_n$ are ρ-sequences such that $u_0 = v_0$, $u_m = v_n$ and $(u_m, r) \in \xi$ for at least one reduced word $r \in A^*$, then $m = n$.*

Proof. If $u_m = r$, then u_m is reduced and the rest follows from 7.3. If $u_m \neq r$, then $(u_m, r) \in \tau$ and there is a ρ-sequence $w_0, w_1, \dots, w_k$ such that $u_m = w_0$ and $r = w_k$. Now, $u_0, u_1, \dots, u_m, w_1, \dots, w_k$ and $v_0, v_1, \dots, v_n, w_1, \dots, w_k$ are ρ-sequences and we have $m + k = n + k$. Then $m = n$. □

Corollary 7.5. *Let the assumptions of 7.3 be satisfied and let for every $u \in A^*$ there exists at least one reduced word $r \in A^*$ with $(u, r) \in \xi$. Then the relation ρ is regular.*

Proposition 7.6. *Assume that $Z \subseteq A^+$ is strongly separating set and that for every $u \in A^*$ there exists at least one reduced word $r \in A^*$ with $(u, r) \in \xi$. Then the relation ρ is regular.*

Proof. Combine I.7.11 and 7.5. □

Theorem 7.7. *Assume that $Z \subseteq A^+$ is strongly separating set and that the mapping ψ is strictly length decreasing. Then the relation ρ is regular.*

Proof. Combine 4.8 and 7.6. □

References

[1] V. Flaška, T. Kepka and J. Kortelainen, On separating sets of words I, *Acta Univ. Carolinae Math. Phys.*, **49/1**(2008), 33-51.
[2] V. Flaška, T. Kepka and J. Kortelainen, On separating sets of words II, *Acta Univ. Carolinae Math. Phys.*, **50/1**(2009), 15-28.

Department of Algebra, MFF UK, Sokolovská 83, 18600 Praha 8

Department of Information Processing Science, University of Oulu, P. O. BOX 3000 FIN-90014, Oulu

E-mail address: flaska@matfyz.cz
E-mail address: kepka@karlin.mff.cuni.cz
E-mail address: juha.kortelainen@oulu.fi