COMPLETELY CORETRACTABLE RINGS

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Abstract

A module M is said to be coretractable if there exists a non-zero homomorphism of every non-zero factor of M into M. We prove that all right (left) modules over a ring are coretractable if and only if the ring is Morita equivalent to a finite product of local right and left perfect rings.

Keywords and phrases: endomorphism ring, coretractable module.

1. Introduction

Modules M whose every non-zero submodule contains a non-zero endomorphic image of M are known under the term retractable modules. The notion has been studied by many authors from various points of view. Significance of the notion has appeared in connection with study of Baer modules [6, 7], endomorphism rings of non-singular modules [3, 4, 5] and lattices of modules [9]. The recent work [2] is devoted to rings whose every module is retractable.

The dual concept to retractable modules was introduced in the paper [1]; a module M is called *coretractable* if there exists a non-zero homomorphism of M/K to M for every proper submodule $K \subseteq M$. Main results of [1] describe rings whose every right module is coretractable. Such rings are called *right completely coretractable.*

In this short note we give a characterization of completely coretractable rings which generalizes [1, Theorem 3.14] in non-commutative case. Namely, a ring is right (left) completely coretractable iff it is isomorphic to a finite product of matrix rings over right and left perfect rings (Theorem 2.4) and,

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equivalently, every cyclic right and left R-module is coretractable (Proposition 3.2).

Throughout the paper a *ring R* means an associative ring with unit, and a *module* means a right R-module. If M is a module, we denote by $J(M)$ the Jacobson radical of M and by $E(M)$ an injective envelope of M. Recall that a ring R is local provided $J(R)$ is a right (left) maximal ideal and R is *right perfect* iff $J(R)$ is right T-nilpotent and R contains a set of orthogonal idempotents $\{e_i,\ i \leq n\}$ such that $1 = \sum_{i \leq n} e_i$ and $e_i R/J(e_i R)$ is a simple module for every index $i \leq n$; then the set $\{e_i,\ i \leq n\}$ is called a *complete set* of orthogonal primitive idempotents. We say that a ring is *perfect* provided it is both right and left perfect. Finally, recall that a ring R is said to be *right max* provided every nonzero R-module contains a maximal submodule, it is well known that each right perfect ring is right max.

2. Structure of completely coretractable rings

Before we prove a general ring-theoretic criterion of complete coretractability, we investigate the role of idempotents in right completely coretractable rings. First, we prove an elementary observation about products of matrix rings over completely coretractable rings.

LEMMA 2.1. *If R is a local perfect ring and n is a natural number, then a full matrix ring $M_n(R)$ is right and left completely coretractable.*

PROOF. Note that R and so $M_n(R)$ is a right and left semiartinian, hence it contains a simple submodule. As $M_n(R)$ is, moreover, a right and left max ring and there exists up to an isomorphism a unique simple module, the assertion follows from [1, Proposition 3.5] □

The following two lemmas clarify correspondence between ring direct summands and module direct summands over a completely correctable ring.

LEMMA 2.2. *Let R be a right perfect ring and $\{e_i|\ i \leq n\}$ a complete set of orthogonal primitive idempotents of R. If every nonzero factor of $e_j R$ is coretractable and $\mathrm{Hom}(e_i R, e_j R) \neq 0$, then $e_i R \cong e_j R$.*

PROOF. Put $S_k = e_k R/e_k J(R)$ and note that S_k are simple modules for all $k \leq n$. Since $\mathrm{Hom}(e_i R, e_j R) \neq 0$, S_i is isomorphic to some submodule of a suitable factor of $e_j R$, hence the inclusion $S_i \subset E(S_i)$ can be extended to a non-zero homomorphism $\varphi : e_j R \to E(S_i)$. As $\varphi(e_j R)$ is a coretractable module by the hypothesis and $\varphi(e_j R)/J(\varphi(e_j R)) = \varphi(e_j R)/\varphi(e_j J(R)) \cong S_j$, there exists a non-zero homomorphism of S_j into $\varphi(e_j R)$. Moreover, S_i is an essential submodule of $\varphi(e_j R)$, hence we get that $S_i \cong S_j$. Finally, $e_i R \cong e_j R$ because $e_i R$ and $e_j R$ form projective covers of the isomorphic simple modules $S_i \cong S_j$. □

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LEMMA 2.3. *If every right cyclic module of a right perfect ring R is coretractable, then $R \cong \prod_{i \leq k} M_{n_i}(R_i)$ for a system of local right perfect rings R_i, $i \leq k$.*

PROOF. Denote by $\{e_i \mid i \leq n\}$ a complete set of orthogonal primitive idempotents of R. Obviously, $R \cong \operatorname{End}(R_R) = \operatorname{End}(\bigoplus_{i=1}^{n} e_i R)$, and applying Lemma 2.2 we obtain that either $e_i R \cong e_j R$ or $\operatorname{Hom}(e_i R, e_j R) = 0$. Moreover, the relation $\sim$ on $N = \{1, \ldots, n\}$ given by the rule $i \sim j$ iff $e_i R \cong e_j R$ is an equivalence. If we put $k = |N/\sim|$, $N_1, \ldots, N_k$ denote all pairwisely different equivalence classes of $\sim$, $n_i = |N_i|$, $j_i \in N_i$ and $R_i = \operatorname{End}(e_{j_i} R)$, we see that $R \cong \operatorname{End}(R) \cong \prod_{i \leq k} \operatorname{End}(e_{j_i} R^{n_i}) \cong \prod_{i \leq k} M_{n_i}(R_i)$. Finally, note that $R_i \cong e_{j_i} R e_{j_i}$ is a local right perfect ring since $J(e_{j_i} R e_{j_i}) = e_{j_i} J(R) e_{j_i} \subset J(R)$ is a right T-nilpotent ideal and $e_{j_i} R e_{j_i}/J(e_{j_i} R e_{j_i})$ is a simple right ideal. $\square$

Now, we are ready to generalize [1, Theorem 3.14]:

THEOREM 2.4. *Let R be a ring. The following conditions are equivalent:*

(1) R is right completely coretractable;

(2) R is left completely coretractable;

(3) $R \cong \prod_{i \leq k} M_{n_i}(R_i)$ for a system of local perfect rings R_i, $i \leq k$.

PROOF. As the last condition is left-right symmetric, it is enough to prove the equivalence (1)$\Leftrightarrow$(3).

(1)$\Rightarrow$(3) Note that R is a left and right perfect ring by [1, Theorem 3.10]. As $R \cong \prod_{i \leq k} M_{n_i}(R_i)$ for a system of local right perfect rings R_i, $i \leq k$, by Lemma 2.3 and R is left perfect, all the rings R_i are perfect.

(3)$\Rightarrow$(1) Applying Lemma 2.1 we see that all rings $M_{n_i}(R_i)$ are right completely coretractable and a finite product of them is right completely coretractable by [1, Lemma 3.13(b)]. $\square$

Since we have proved that right and left completely coretractable rings coincide, we may define *completely coretractable* rings as right (or left) completely coretractable rings.

We can formulate two easy structural consequences of Theorem 2.4.

COROLLARY 2.5. *The product $\prod_{i \in I} R_i$ of rings R_i is completely coretractable iff I is finite and all R_i are completely coretractable.*

However it could be directly verified that complete coretractability is Morita invariant (cf. e.g. proof of [2, Theorem 2]), we get this fact as an immediate consequence of Theorem 2.4. Moreover, we can find for every completely coretractable ring $\prod_{i \leq k} M_{n_i}(R_i)$ with local perfect rings R_i a "minimal" Morita equivalent ring $\prod_{i \leq k} R_i$.

COROLLARY 2.6. *Every completely coretractable ring is Morita equivalent to a finite product of local perfect rings.*

3. Kasch rings and complete coretractability

Recall that a ring is right Kasch if it contains an isomorphic copy of every right simple module. Note that right Kasch rings are precisely those rings R such that R_R is a coretractable module by [1, Theorem 2.14].

LEMMA 3.1. *If R is a right perfect ring such that every factor-ring of R is right Kasch, then R is left perfect and every cyclic right module is coretractable.*

PROOF. Let C be a cyclic right R-module and M a maximal submodule of C. Then $R/r(C)$ is right Kasch and both the modules C and C/M have a natural structure of a right $R/r(C)$-modules, where $r(C)$ denotes the right annihilator of C. Since $R/r(C)$ contains a simple submodule isomorphic to C/M by the hypothesis and C is faithful over $R/r(C)$, C contains a simple submodule isomorphic to C/M as well. We have proved that every cyclic module has a non-zero socle, i.e. R is right semiartinian and $\mathrm{Hom}(C/N, C) \neq 0$ for every submodule $N \subseteq M$, which implies that C is coretractable. Moreover $R/J(R)$ is semisimple, hence R is left perfect by [8, Theorem VIII.5.1]. □

Applying the previous observation we are able to generalize [1, Theorem 3.14] using the notion of a Kasch ring:

PROPOSITION 3.2. *The following conditions are equivalent for any ring R:*

(1) R is completely coretractable;

(2) every cyclic right and left R-module is coretractable;

(3) R is right perfect and every factor-ring of R is right Kasch;

(4) R is left perfect and every factor-ring of R is left Kasch.

PROOF. (1)⇒(2) It follows immediately from Theorem 2.4.

(2)⇒(3) R is right and left perfect by [1, Proposition 3.8]. If I is an ideal of R, then every cyclic right R/I-module is coretractable, hence R/I is right Kasch by [1, Theorem 2.14].

(3)⇒(1) Every cyclic right module is coretractable and R is left perfect by Lemma 3.1. Applying Lemma 2.3 we get right perfect rings R_i, $i \leq k$ such that $R \cong \prod_{i \leq k} M_{n_i}(R_i)$. As R_i is left perfect as well for each $i \leq k$, R is completely coretractable by Theorem 2.4.

The proof of equivalence (1)⇔(4) is symmetric using the left-hand version of Lemma 3.1. □

As [1, Example 3.15] shows, there exists a perfect right Kasch ring R which is not right completely coretractable. Hence Theorem 2.4 implies that R is not left completely coretractable and there exists a factor of R which is not right Kasch by Proposition 3.2.

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