COMMUTATIVE SEMIGROUPS THAT ARE SIMPLE OVER THEIR ENDOMORPHISM SEMIRINGS

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Abstract. In the paper, commutative semigroups simple over their endomorphism semirings are investigated. In particular, commutative semigroups having just two fully invariant congruences are classified into five basic types and each of these types is characterized.

Congruence-simple (universal) algebras (i.e., those possessing just two congruence relations) appear as keystone in many algebraic structure theories, but not always. Sometimes, the simple algebras are too many and difficult to handle and sometimes they are quite few. The latter apllies to commutative semigroups (abelian groups included). Congruence-simple commutative semigroups are just two-element semilattices, two-element constant semigroups and p-element cyclic groups, p being a prime. All of these semigroups are finite and tame in many situations. Now, the scenery becomes much wilder if we consider commutative semigroups that are simple over their automorphism groups (i.e., non-trivial commutative semigroups without non-trivial invariant congruences). A few pieces of information concerning these semigroups (let us call them *amc-simple*) can be found in [6] (but see also [7], [9], [10], [11], [12], [14], [15] and [16]).

In the finite case, we do not get much more. Namely, if A is a finite amc-simple commutative semigroup then either A is simple (and then it is one of the semigroups mentioned in the preceding paragraph) or A is a semilattice with at least three elements such that $a+b=c+d$ for all $a,b,c,d \in A$, $a \neq b$, $c \neq d$. In the infinite case, we get many others. Just one example: Put $A = \mathbb{Z} \times \mathbb{Z}$ (the set of ordered pairs of integers) and $(k,m) \oplus (l,n) = (\min(k,l), \min(m,n))$ then $A(\oplus)$ becomes an amc-simple semilattice (notice that $A(\oplus)$ is not a chain).

The aim of the present note is to investigate commutative semigroups that are simple over their endomorphisms semirings (i.e., non-trivial commutative semigroups without non-trivial fully invariant congruences). Some related results are available in [4] and [5].

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1. Basic notions

Throughout the paper, let $A = A(+)$ be a commutative semigroup and $E = \mathrm{End}(A(+))$ be the full endomorphism semiring of A (clearly, E is a unitary semiring and A is a left E-semimodule). Further, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0$ is the set of non-negative integers, $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ is the set of positive real numbers. As usual, 0_A (o_A, resp.) will denote the neutral (absorbing, resp.) element of A and $0_A \notin A$ means that A has no neutral element. If $z \notin A$ then $A_0 = A \cup \{z\}$ is a semigroup such that A is a subsemigroup of A_0 and $x+z = x$ for all $x \in A_0$ (thus $z = 0_{A_0}$). Further, for every $X \subseteq A$ and $a \in A$, we put $X_0 = X \cup \{z\}$ and $a + X_0 = \{\, a+u \mid u \in X_0 \,\} = (a+X) \cup \{a\}$. For $a \in A$, we denote $E(a) = \{\, f(a) \mid f \in E \,\}$.

A subset I of A is an *ideal* if $I \neq \emptyset$ and $A + I \subseteq I$ (then $\beta_I = (I \times I) \cup \mathrm{id}_A$ is a congruence of A and the corresponding factor will be denoted by A/I). A congruence r of A is said to be *fully invariant* if $f(r) = \{\, (f(a), f(b)) \mid (a,b) \in r \,\} \subseteq r$ for every $f \in E$. Similarly, a subsemigroup B (an ideal I, resp.) of A is fully invariant if $f(B) \subseteq B$ ($f(I) \subseteq I$, resp.) for every $f \in E$. Now, we shall say that A is

- *emc-simple* if A has just two fully invariant congruences (then $|A| \geq 2$ and $\mathrm{id}_A, A \times A$ are the congruences);
- *ems-simple* if $|A| \geq 2$ and $|B| = 1$ whenever B is a fully invariant subsemigroup with $B \neq A$;
- *emi-simple* if $|A| \geq 2$ and $|I| = 1$ whenever I is a fully invariant ideal of A with $I \neq A$ (if such an ideal I exists then $I = \{o_A\}$, i.e., A has just two fully invariant ideals);
- a *semilattice* if it satisfies the identity $2x = x$ (i.e., A is idempotent);
- *constant* if $|A + A| = 1$ (i.e., $x + y = x + z$);
- *zeropotent* (or *nil of class* 2) if it satisfies the identity $2x = 3y$ (then $2a = o_A$ for every $a \in A$;
- a *nil-semigroup of class* $m \in \mathbb{N}$, $m \geq 2$, if $mx = (m+1)y$ (then $ma = o_A$ for every $a \in A$;
- a *nil-semigroup* if $o_A \in A$ and for every $a \in A$ there is $m \in \mathbb{N}$ with $ma = o_A$;
- *cancellative* if $a + b = a + c$ implies $b = c$;
- *archimedean* if $A/(A + a)$ is a nil-semigroup for every $a \in A$ (then A/I is a nil-semigroup for every ideal I);
- *strongly archimedean* if it is archimedean, cancellative and not a group.

For every $n \in \mathbb{N}$, the transform $\varphi_n = \varphi_{n,A}$ defined by $\varphi_n(a) = na$ for every $a \in A$, is an endomorphism of A. Hence $\Phi_n : \mathbb{N} \to E$, $\Phi(n) = \varphi_n$, is a unitary homomorphism of semirings.

Now, let r be a fully invariant congruence of A, $\pi : A \to A/r$ be the natural projection and denote $\pi(a)$ by a/r. For every $f \in E$ and $a \in A$ put $\Psi(f)(a/r) = f(a)/r$. Then $\Psi(f) \in \mathrm{End}(A/r)$ and $\Psi : E \to \mathrm{E}nd(A/r)$ is a unitary homomorphism of semirings. Clearly, $\Psi(\varphi_{n,A}) = \varphi_{n,A/r}$ for every $n \in \mathbb{N}$. If s is a fully invariant congruence of A/r then $\pi^{-1}(s) = \{\, (a,b) \in A \times A \mid (\pi(a), \pi(b)) \in s \,\}$ is a fully invariant congruence of A and $r \subseteq \pi^{-1}(s)$. Clearly, $\pi^{-1}(s) = r$ iff $s = \mathrm{id}_{A/r}$, and $\pi^{-1}(s) = A \times A$ iff $s = A/r \times A/r$. Thus A/r is emc-simple, provided that r is a maximal proper fully invariant congruence of A.

1.1 Observation. Consider the following relations on A:

$$\begin{aligned}
\varrho_A &= \{\, (a,b) \mid mb \in A+a \text{ and } na \in A+b \text{ for some } m,n \in \mathbb{N} \,\},\\
\sigma_A &= \{\, (a,b) \mid a+c = b+c \text{ for some } c \in A \,\},\\
\tau_A &= \{\, (a,b) \mid a \in b+A_0 \text{ and } b \in a+A_0 \,\},\\
\alpha_n &= \ker(\varphi_n) \text{ (for } n \in \mathbb{N}),\\
\beta_I &= (I \times I) \cup \mathrm{id}_A \text{ (for a fully invariant ideal } I),\\
\gamma_B &= \{\, (a,b) \mid (a+B) \cap (b+B) \neq \emptyset \,\} \text{ (for a fully invariant subsemigroup } B),\\
\delta_n &= \{\, (a,b) \mid a+nu = b+nv \text{ for some } u,v \in A \,\} \text{ (for } n \in \mathbb{N}),\\
s_{r,n} &= \{\, (a,b) \mid (na,nb) \in r \,\} \text{ (for a fully invariant congruence } r \text{ and } n \in \mathbb{N}),\\
\varepsilon_{u,v} &= \{\, (a,b) \mid a+f(u)+g(v) = b+g(u)+f(v) \text{ for some } f,g \in E \,\} \ (u,v \in A),\\
\lambda_w &= \{\, (a,b) \mid \text{for all } f \in E, x \in A, f(a)+x = w \text{ iff } f(b)+x = w \,\} \text{ (for } w \in A).
\end{aligned}$$

The following observations are straightforward, fairly basic and folklore to much extent. Henceforth, we shall not attribute them to any particular source.
(i) All the relations defined above are fully invariant congruences of A.
(ii) ϱ_A is the smallest conguence such that the corresponding factor is a semilattice. Clearly, $\varrho_A = \mathrm{id}_A$ iff A is a semilattice and $\varrho_A = A \times A$ iff A is archimedean.
(iii) $\tau_A \subseteq \varrho_A$ and $\tau_A = A \times A$ iff A is a group.
(iv) σ_A is the smallest congruence such that the corresponding factor is cancellative.
(v) Assume that $\alpha_n = A \times A$, i.e., $na = nb = o$ for all $a,b \in A$. Then $2o = o$ and $o+a = (n+1)a = \varphi_{n+1}(a)$ for every $a \in A$. If $\alpha_{n+1} = A \times A$ then $o = o_A$ is absorbing and A is a nil-semigroup. If $\alpha_{n+1} = \mathrm{id}_A$ then $o+a = a$ for every $a \in A$, hence $o = 0_A$ and A is a group.
(vi) If B is a fully invariant subsemigroup of A then $B \times B \subseteq \gamma_B$. If $\gamma_B = \mathrm{id}_A$ then $B = \{o\}$, $2o = o$ and $(a+o,a) \in \gamma_B$ for every $a \in A$. Thus $o = 0_A$ and $f(0_A) = 0_A$ for every $f \in E$.
(vii) The sets $I_1 = \{\, 2a+u \mid a \in A, u \in A_0 \,\}$, $I_2 = \{\, 2a+b \mid a,b \in A \,\}$ and $I_3 = A+A$ are fully invariant ideals of A, $\beta_{I_2} \subseteq \beta_{I_1} \subseteq \beta_{I_3}$, β_{I_1} is the smallest congruence such that the corresponding fcctor is zeropotent and β_{I_3} is the smallest congruence such that the corresponding factor is constant.
(viii) Clearly, $\delta_1 = A \times A$ and δ_n is cancellative (i.e., $(a+c,b+c) \in \delta_n$ implies $(a,b) \in \delta_n$) and $(na,nb) \in \delta_n$, $(a,a+nb) \in \delta_n$ for all $a,b \in A$ (in particular, if $\delta_n = \mathrm{id}_A$ then $na = nb = 0_A$ for all $a,b \in A$). Further, δ_n is the smallest congruence such that the corresponding factor is a torsion group of exponent n. If A is cancellative then, clearly, $\delta_n = A \times A$ for every $n \in \mathbb{N}$ iff $\delta_p = A \times A$ for every prime p iff the difference group $G = A - A$ is divisible.
(ix) Let r be a cancellative fully invariant congruence of A and denote $B = A/r$. Then $s_{r,n}$ is cancellative for every $n \in \mathbb{N}$. If $s_{r,n} = A \times A$ for some $n \in \mathbb{N}$ then B is a torsion group of exponent n. If $s_{r,n} = r$ for all $n \in \mathbb{N}$ then the difference group $B - B$ is a torsionfree group.
(x) For all $u,v \in A$, $(u,v) \in \varepsilon_{u,v}$, since $u+u+2v = v+2u+v$. In particular, $\varepsilon_{u,v} \neq \mathrm{id}_A$ for $u \neq v$. Obviously, if A is cancellative then $\varepsilon_{u,v}$ is cancellative. Let r be a fully invariant congruence with $(u,v) \in r$. Then $(f(u)+g(v), g(u)+f(v)) \in r$ for all $f,g \in E$, and consequently $\varepsilon_{u,v} \subseteq r$ if r is cancellative. Hence, if A is cancellative then $\varepsilon_{u,v}$ is the fully invariant cancellative congruence of A generated by the pair (u,v).

(xi) Let G be a groupoid and Φ be the set of all homomorphisms $\varphi : A \to G$. Then $r = \cap \ker\varphi$, $\varphi \in \Phi$, is a fully invariant congruence of A. In particular, if A is emc-simple and $|\varphi(A)| \geq 2$ for at least one $\varphi \in \Phi$ then $r = \mathrm{id}_A$, and hence A imbeds into a cartesian power of G.
(xii) Let A be archimedean. Then A has at most one idempotent element. Obviously, if $0_A \in A$ then A is a group. If A has no idempotents then $(a, 2a) \notin \sigma_A$ for every $a \in A$, i.e., A/σ_A is strongly archimedean (indeed, if $(a, 2a) \in \sigma_A$ then $a + b = 2a + b$ for some $b \in A$, however $b + c = ma$ for some $m \in \mathbb{N}$ and $c \in A$, and hence $e = (m + 1)a = a + b + c = 2a + b + c = (m + 2)a$ is idempotent). Consequently, if $\sigma_A = A \times A$ then A has exactly one idempotent element.

2. Emc-simple semigroups - basic classification

2.1 Lemma. *Assume that $\varrho_A = A \times A$ and $\alpha_k = \mathrm{id}_A$ for some $k \geq 2$. If $a, b, c \in A$ are such that $a + c = b + c$ then $2a = 2b$.*

Proof. Since $(a, c) \in A \times A = \varrho_A$, there are $i \geq 2$ and $d \in A$ with $k^i a = c + d$. Now, $a + k^i a = a + c + d = b + c + d = b + k^i a$, and hence $k(a + k^{i-1}a) = ka + k^i a = (k-1)a + a + k^i a = (k-1)a + b + k^i a = (k-2)a + b + a + k^i a = (k-2)a + 2b + k^i a = \cdots = kb + k^i a = k(b + k^{i-1}a)$. Thus $(a + k^{i-1}a, b + k^{i-1}a) \in \alpha_k = \mathrm{id}_A$ and $a + k^{i-1}a = b + k^{i-1}a$. Proceeding by induction, we obtain $2b = b + a$, and hence $2a = 2b$ by symmetry. □

2.2 Corollary. *If $\varrho_A = A \times A$ and $\alpha_k = \mathrm{id}_A$ for at least one even k then A is cancellative (i.e., $\sigma_A = \mathrm{id}_A$).* □

2.3 Lemma. *Assume that $\alpha_2 = A \times A$. If $a, b, c \in A$ are such that $a + c = b + c$ then $3a = 3b$.*

Proof. Since $(a, c) \in \alpha_2$ and $(b, c) \in \alpha_2$, we have $2a = 2c = 2b$. Now, $3a = a + 2a = a + 2c = a + c + c = b + c + c = b + 2c = b + 2b = 3b$. □

2.4 Corollary. *If $\alpha_2 = A \times A$ and $\alpha_k = \mathrm{id}_A$ for some k divisible by 3 then A is cancellative.* □

2.5 Lemma. *Assume that $\alpha_2 = A \times A = \alpha_3$. Then $2a = 3b$ for all $a, b \in A$ (i.e., A is zeropotent).*

Proof. Since $(2a, b) \in \alpha_3$, we have $6a = 3b$. Further, $(3a, a) \in \alpha_2$, and hence $6a = 2a$. □

2.6 Theorem. *Let A be an emc-simple commutative semigroup. Then just one of the following five cases takes place:*

(1) *A is idempotent (i.e., a semilattice);*
(2) *A is an abelian group;*
(3) *A is cancellative and $a + b \neq a$ for all $a, b \in A$ (and then A is infinite);*
(4) *A is constant;*
(5) *A is zeropotent and $A + A = A$ (and then A is not finitely generated).*

Proof. If $\varrho_A = \mathrm{id}_A$ then A is idempotent. In the opposite case, $\varrho_A = A \times A$. Now, if $\alpha_2 = \mathrm{id}_A$ then A is cancellative by 2.2. On the other hand, if $\alpha_2 \neq \mathrm{id}_A$ and A is not cancellative then $\alpha_2 = A \times A = \alpha_3$ by 2.4, and hence A is zeropotent by 2.5 (we will show in 5.7 that A is infinite if $A + A = A$, and then A is not finitely generated - see 2.12). If $A + A \neq A$ then $\beta_{I_3} = \mathrm{id}_A$ and A is constant. Finally, assume that A is cancellative. If $0_A \notin A$ then $a+b \neq a$ for all $a,b \in A$ and A is infinite. In the opposite case, either $\tau_A = A \times A$ and A is a group or $\tau_A = \mathrm{id}_A$ and it follows that $B = A \setminus \{0_A\}$ is a subsemigroup of A. Now, define a relation r on A by $r = \{\,(a,b) \mid \text{for every } f \in E, f(a) = 0_A \text{ iff } f(b) = 0_A\,\}$. One checks readily that r is a fully invariant congruence of A. Clearly, $(0_A, a) \notin A$ for every $a \neq 0_A$. Thus $r = \mathrm{id}_A$, however $(a, 2a) \in r$, a contradiction. □

2.7 Remark. If A is emc-simple and not a semilattice then $\varrho_A = A \times A$, and hence A is archimedean. Further, if A is emc-simple and I is a proper fully invariant ideal of A then $\beta_I = \mathrm{id}_A$. Thus $I = \{o\}$, $o = o_A$ is absorbing (i.e., A has just two fully invariant ideals, namely A and $\{o_A\}$) and $f(o) = o$ for every $f \in E$. Finally, an abelian group is emc-simple iff it is ems-simple.

Just for the sake of completeness, we include the following well-known result:

2.8 Theorem. *A non-trivial abelian group A is emc-simple (and ems-simple) if and only if it is either p-elementary for some prime p (i.e., A is a direct sum of copies of the p-element cyclic group $\mathbb{Z}_p(+)$) or torsionfree divisible (i.e., A is a direct sum of copies of the additive group $\mathbb{Q}(+)$ of rational numbers).* □

The following lemma is quite familiar:

2.9 Lemma. *Let A be a semilattice and $a,b \in A$ be such that $a+b \neq a \neq b$. If $x,y \in A$ are arbitrary then there is an endomorphism f of A such that $f(a) = x$, $f(a+b) = x+y$ and $f(A) = \{x, x+y\}$.* □

2.10 Theorem. *Every non-trivial semilattice is emc-simple and ems-simple.*

Proof. Let A be a semilattice and $r \neq \mathrm{id}_A$ be a fully invariant congruence of A and take $(a,b) \in r$, $a \neq b$. Then $(a, a+b) \in r$ and we can assume that $a+b \neq a$. Let $x,y \in A$ and f be an endomorphism from 2.9. Then $(x, x+y) \in r$ and, symetrically, $(y, x+y) \in r$, hence $(x,y) \in r$ and $r = A \times A$. Finally, every constant transformation is an endomorphism, and hence A has no proper fully invariant subsemigroups. □

The following result is very easy:

2.11 Theorem. *Every non-trivial constant semigroup A is emc-simple and ems-simple.* □

2.12 Remark. Let A be a finitely generated emc-simple commutative semigroup. Then there is a congruence r of A such that the factor A/r is a (congruence-)simple commutative semigroup. Now, just one of the following three cases takes place:
(1) A/r is a two-element semilattice and A is a finite semilattice;
(2) A/r is a two-element constant semigroup and A is a finite constant semigroup;
(3) A/r is a cyclic p-group for some prime p and then A is a finite p-elementary group.
Notice that A is finite anyway.

3. Emc-simple cancellative commutative semigroups

3.1 Proposition. *Let A be a cancellative commutative semigroup. The following conditions are equivalent:*

(i) id_A *and $A \times A$ are the only fully invariant cancellative congruences of A.*

(ii) *For all* $a, b, c, d \in A$ *such that* $a \neq b$ *there are* $f, g \in E$ *such that* $f(a) + g(b) + c = g(a) + f(b) + d$.

Proof. If $a \neq b$ then $\varepsilon_{a,b} \neq \mathrm{id}_A$ (see 1.1(x)). $\square$

3.2 Remark. Notice that the equivalent conditions of 3.1 imply the following three conditions:
(1) For all $a, b, c \in A$, $a \neq b$, there are $f, g \in E$ such that $f(a)+g(b)+c = g(a)+f(b)$.
(2) For all $a, c, d \in A$, $a \neq 0_A$ there are $f, g \in E$ such that $f(a) + c = g(a) + d$.
(3) For all $a, b \in A$, $a \neq 0_A$, there are $f, g \in E$ with $f(a) + b = g(a)$.

Further, if A satisfies the conditions of 3.1 and $0_A \notin A$ then $\delta_n = A \times A$ for all $n \in \mathbb{N}$, and hence the difference group $G = A - A$ is torsionfree and divisible, i.e., for every $a, b \in A$ and $n \in \mathbb{N}$ there are $c, d \in A$ such that $a + nd = b + nc$.

3.3 Lemma. *Let A be an archimedean commutative semigroup such that A has no proper fully invariant ideal. Then:*
(i) *If r is a fully invariant congruence of A such that $(w, 2w) \in r$ for some $w \in A$ then A/r is a group.*
(ii) *If $w \in A$ is such that $w \neq 2w$ and r, s are fully invariant congruences of A such that $r \subsetneqq s$ and r is maximal with respect to $(w, 2w) \notin r$ then A/s is a group.*
(iii) *If $w \in A$ is such that $2w \neq 4w$, r is a fully invariant congruence maximal with respect to $(2w, 4w) \notin r$, and A/s is not a group whenever s is a fully invariant congruence of A such that $r \subsetneqq s \neq A \times A$ then $B = A/r$ is emc-simple of type* 2.6(2) *or* 2.6(3).

Proof. (i) If $v \in A$ is such that $(v, 2v) \in r$ then $(w, v) \in r$. Indeed, since A is archimedean, we have $w+x = mv$ and $v+y = nw$ for suitable $x, y \in A$ and $m, n \in \mathbb{N}$. However $(w, nw) \in r$, $(v, mv) \in r$, and hence $(w + x, v) \in r$ and $(v + y, w) \in r$. Consequently, $(2w + x, v + w) \in r$ and $(2v + y, w + v) \in r$. As $(w + x, 2w + x) \in r$ and $(v + y, 2v + y) \in r$, we obtain $(w + x, v + w) \in r$ and $(v + y, w + v) \in r$, hence $(w + x, v + y) \in r$ and $(v, w) \in r$, as desired. If $f \in E$ then $(f(w), 2f(w)) \in r$, and so $(w, f(w)) \in r$. Since $E(w) + A$ is a fully invariant ideal, for every $a \in A$ there are $b \in A$ and $g \in E$ with $g(w) + b = a$. Then $(w + b, a) = (w + b, g(w) + b) \in r$, and hence $(2w + b, a + w) \in r$. Since $(2w + b, w + b) \in r$, we have $(2w + b, a) \in r$ and $(a, a + w) \in r$. Finally, there are $c \in A$ and $m \in \mathbb{N}$ with $a + c = mw$, hence $(a + c, w) \in r$ and we conclude that A/r is a group.
(ii) We have $(w, 2w) \in s$ and (i) applies.
(iii) Let $\pi : A \to B$ be the natural projection and $t \neq \mathrm{id}_B$ be a fully invariant congruence of B. Then $s = \pi^{-1}(t)$ ($(a, b) \in s$ iff $(\pi(a), \pi(b)) \in t$) is a fully invariant congruence of A and $r \subsetneqq s$. By (ii), A/s is a group. Thus $s = A \times A$ and $t = B \times B$. The rest is clear from 2.6. $\square$

3.4 Theorem. *A non-trivial cancellative commutative semigroup A is emc-simple if and only if it satisfies the following three conditions:*

(1) *For all $a, b \in A$ there are $c \in A$ and $m \in \mathbb{N}$ such that $a + c = mb$ (i.e., A is archimedean).*
(2) *For all $a, b \in A$ there are $c \in A$ and $f \in E$ such that $f(a) + c = b$ (i.e., A has no proper fully invariant ideal).*
(3) *For all $a, b, c, d \in A$ such that $a \neq b$ there are $f, g \in E$ with $f(a)+g(b)+c = g(a) + f(b) + d$.*

Proof. First, suppose that the conditions (1),(2) and (3) are satisfied and let r be a fully invariant congruence of A. If $(w, 2w) \in r$ for some $w \in A$ then A/r is a group by 3.3(i), hence r is cancellative and $r = A \times A$ by 3.1. On the other hand, if $(w, 2w) \notin r$ for every $w \in A$ then $(2w, 4w) \notin r$ as well. Fixing $w \in A$, let r_1 be a fully invariant congruence of A maximal with respect to $(2w, 4w) \notin r$ and $r \subseteq r_1$. Combining 3.1 and 3.3(iii), we get $r_1 = \mathrm{id}_A$, and hence $r = \mathrm{id}_A$. The converse implication is clear (cf. 2.7 and 3.1). □

3.5 Remark. Now, let A be a cancellative commutative semigroup without neutral element. Denote by $G = A - A$ the difference group of A and define a relation $\leq_A$ on G by $\leq_A = \{ (u, v) \mid v - u \in A \} \cup \mathrm{id}_A$. Clearly, $\leq_A$ is an order relation on the group G and $A = \{ u \in G \mid 0_G <_A u \}$ is the cone of positive elements. Obviously, if A is archimedean then, for all $a \in A$ and $u \in G$, there is $m \in \mathbb{N}$ with $u <_A ma$. For every $f \in E$, define an endomorphism $\overline{f}$ of G by $\overline{f}(a - b) = f(a) - f(b)$ for all $a, b \in A$. The mapping $f \mapsto \overline{f}$ is an injective unitary homomorphism of the semiring $E = \mathrm{End}(A)$ into the ring $\mathrm{End}(G)$. The image $\overline{E}$ is a subsemiring of $\mathrm{End}(G)$. Obviously, if $\varphi \in \mathrm{End}(G)$ then $\varphi \in \overline{E}$ iff $\varphi(A) \subseteq A$. Put $R = \overline{E} - \overline{E}$ (i.e., $R \subseteq \mathrm{End}(G)$ and R is the difference ring of the semiring $\overline{E}$). It is easy to see that the semigroup A satisfies the equivalent conditions of 3.1 iff G is a simple R-module.

3.6 Remark. Let P be an additively cancellative parasemifield (i.e., a semiring, where the multiplicative semigroup is a group). Clearly, $P(+)$ satisfies the condition 3.4(1) iff for every $a \in P$ there are $b \in P$ and $m \in \mathbb{N}$ such that $1_P + b = ma$ (equivalently, for every $a \in P$ there are $b \in P$ and $m \in \mathbb{N}$ with $a + b = m1_P$). Further, we have $2b^{-1}a \cdot a + 2_P^{-1}b = b$ for all $a, b \in P$, and hence the condition 3.4(2) is always true for $P(+)$.

3.7 Example. Let $r \in \mathbb{R}^+$ be a transcendental number and $F = \mathbb{Q}(r)$. Put $A = \{ \sum a_i^2 \mid a_i \in F \setminus \{0\}, 1 \leq i \leq n \in \mathbb{N} \}$ and $B = F^+$ $(= F \cup \mathbb{R}^+)$. Clearly, $A \subseteq B$, both A abd B are subparasemifields of $\mathbb{R}^+$ and A is just the set of the numbers

$$\frac{f_1^2(r) + \cdots + f_n^2(r)}{g^2(r)}, \; n \in \mathbb{N}, \; f_1, \ldots, f_n, g \in \mathbb{Q}[x] \setminus \{0\}.$$

Now, we see that $r \in B \setminus A$, $A \neq B$ and $A - A = F = B - B$. Both the additive semigroups $A(+)$ and $B(+)$ satisfy the conditions 3.4(2),(3) and $B(+)$ satisfies 3.4(1) (thus $B(+)$ is emc-simple). On the other hand, $A(+)$ does not satisfy 3.4(1) (we have $\frac{m}{r^2} \notin 1 + A$ for every $m \in \mathbb{N}$).

3.8 Example. Let $A = \{ q \in \mathbb{Q}^+ \mid q \geq 1 \}$. Then A is a subsemiring of $\mathbb{Q}^+$ and $A(+)$ satisfies 3.4(1),(3). On the other hand, since $f(q) \neq 1$ for every $q > 1$ and $f \in \mathrm{End}(A(+))$, $I = \{ q \in A \mid q > 1 \}$ is a fully invariant ideal of A, and so $A(+)$ does not satisfy 3.4(2).

3.9 Example. Let $A = \mathbb{Q} \setminus \{0, 1, -1\}$. Then $A(\cdot)$ is a cancellative commutative semigroup without neutral element. Clearly, A satisfies 3.4(1),(2), however A does not satisfy 3.4(3).

3.10 Example. Let F be a subfield of $\mathbb{R}$ and $A = F^+(+)$. Then A^n is emc-imple for every $n \in \mathbb{N}$.

4. Homomorphisms of strongly archimedean commutative semigroups into $\mathbb{R}^+(+)$

In 4.1 – 4.6, we assume that A is archimedean. Let B be a subsemigroup of A and $\varphi : B \to \mathbb{R}^+(+)$ be a homomorphism such that $\varphi(a) \geq \varphi(b)$ whenever $a, b \in B$ and $a \in b + A$. Further, in A_0 we formally put $\varphi(0) = 0$. Let $w \in A$ be arbitrary and put

$$p(w) = \sup \left\{ \frac{\varphi(a_1 - \varphi(a_2)}{m} \,\middle|\, a_1 \in B, a_2 \in B_0, m \in \mathbb{N}, a_3 \in A_0, mw + a_2 = a_1 + a_3 \right\}$$

and

$$q(w) = \inf \left\{ \frac{\varphi(b_1) - \varphi(b_2)}{n} \,\middle|\, b_1 \in B, b_2 \in B_0, n \in \mathbb{N}, b_3 \in A_0, b_1 = nw + b_2 + b_3 \right\}.$$

The very basic idea of the following five easy lemmas goes back to [8] (but see and consult also [1], [2], [3] and [13]).

4.1 Lemma. (i) $0 < p(w) \leq q(w) < +\infty$.
(ii) *If $w \in B$ then $p(w) = q(w) = \varphi(w)$.* $\square$

Now, put $C_w = (B + \mathbb{N}w) \cup B$, i.e., C_w is the subsemigroup of A generated by $B \cup \{w\}$.

4.2 Lemma. *Let $\psi : C_w \to \mathbb{R}^+(+)$ be a homomorphism such that $\psi|B = \varphi$ and $\psi(u) \geq \psi(v)$ whenever $u, v \in C_w$ are such that $u \in v + A$. Then $p(w) \leq \psi(w) \leq q(w)$.* $\square$

4.3 Lemma. (i) *If $r \in \mathbb{R}^+$ is such that $p(w) \leq r$ and $m \in \mathbb{N}$, $a \in B_0$, $b \in B$, $c \in A_0$ are such that $mw + a = b + c$ then $\varphi(a) + mr \geq \varphi(b)$.*
(ii) *If $r \in \mathbb{R}^+$ is such that $r \leq q(w)$ and $n \in \mathbb{N}$, $a \in B_0$, $b \in B$, $c \in A_0$ are such that $b = nw + a + c$ then $nr + \varphi(a) \leq \varphi(b)$.*
(iii) *If $r \in \mathbb{R}^+$ is such that $p(w) \leq r \leq q(w)$ and $k \in \mathbb{N}$, $a \in B_0$, $b \in B$ are such that $kw = a + b$ then $\varphi(b) = kr + \varphi(a)$.* $\square$

4.4 Lemma. *Let A be cancellative and $r \in \mathbb{R}^+$ be such that $p(w) \leq r \leq q(w)$. If $k_1, k_2 \in \mathbb{N}_0$ and $b_1, b_2 \in B_0$ are such that $k_1 w + a_1 = k_2 w + a_2 \in A$ then $k_1 r + \varphi(a_1) = k_2 r + \varphi(a_2)$.* $\square$

4.5 Lemma. *Assume that A is cancellative, $0_A \notin A$ and $r \in \mathbb{R}^+$ is such that $p(w) \leq r \leq q(w)$. Then there is a homomorphism $\psi : C_w \to \mathbb{R}^+(+)$ such that $\psi|B = \varphi$, $\psi(w) = r$ and $\psi(u) \geq \psi(v)$ whenever $u, v \in C_w$ are such that $u \in A + w$.* $\square$

4.6 Proposition. *Let A be cancellative and $0_A \notin A$. For every $a \in A$ there is a homomorphism $\varphi_a : A \to \mathbb{R}^+(+)$ such that $\varphi_a(a) = 1$.*

Proof. Let $\mathcal{B}$ denote the set of all ordered pairs (B, φ), where B is a subsemigroup of A and $\varphi : B \to \mathbb{R}^+(+)$ is a homomorphism such that $\varphi(a) \geq \varphi(b)$ whenever $a, b \in A$, $a \in b + A$. The set $\mathcal{B}$ is ordered by inclusion. By 4.5, if (B, φ) is maximal in $\mathcal{B}$ then $B = A$. Finally, if $a \in A$ then $\mathbb{N}a$ is isomorphic to $\mathbb{N}(+)$, and hence the mapping $na \mapsto n$ is an injective homomorphism of $B = \mathbb{N}a$ into $\mathbb{R}^+(+)$ and $a \mapsto 1$. $\square$

4.7 Theorem. *Let A be emc-simple of type* 2.6(3). *Then A imbeds into a cartesian power of $\mathbb{R}^+(+)$.*

Proof. By 3.4, A is strongly archimedean. The rest is clear from 4.6. $\square$

4.8 Example. Let $G = \mathbb{Z}_2(+) \times \mathbb{Z}(+)$ and $A = \{\, (i, m) \in G \,|\, m \geq 1 \,\}$. Then A is a subsemigroup of G, $G = A - A$ is the difference group of A and A is a strongly archimedean semigroup. Of course, $2(1, 1) = 2(0, 1)$, and hence A does not imbed into any cartesian power of $\mathbb{R}^+(+)$.

4.9 Proposition. *Let A be an archimedean commutative semigroup. The following conditions are equivalent:*

(i) *A has no idempotent.*
(ii) *For every $a \in A$ there is a homomorphism $\varphi_a : A \to \mathbb{R}^+(+)$ with $\varphi_a(a) = 1$.*

Proof. If (i) holds then A/σ_A is strongly archimedean and 4.6 applies. The converse is obvious. $\square$

4.10 Remark. Let A be a commutative semigroup and $\varphi : A \to \mathbb{R}^+(+)$ be a homomorphism. Define a relation $\preceq$ on A by $a \preceq b$ iff either $a = b$ or $\varphi(a) < \varphi(b)$. One checks readily that this relation is a compatible (partial) order relation and $a \prec a + b$ for all $a, b \in A$ (i.e., every element from A is strictly positive in the order). Furthermore, the order is archimedean in the following sense: For all $a, b \in A$ there is $m \in \mathbb{N}$ with $a \prec mb$.

5. Endo-c-simple zeropotent semigroups

5.1 Lemma. *Let $w \in A$ be such that $E(w) = \{w\}$ and $\lambda_w = A \times A$. Then $w = o_A$ is absorbing and $A + A = \{w\}$ (i.e., A is a constant semigroup).*

Proof. $E(w) = \{w\}$ implies $2w = w$. If $f \in E$ and $a \in A$ then $(a, w) \in \lambda_w$, $f(w) + w = 2w = 2w = w$, and hence $f(a) + w = w$. In particular, $a + w = w$ and w is absorbing. Then, of course, $\mathrm{id}_A(w) + b = w + b = w$ for every $b \in A$, and therefore $a + b = \mathrm{id}_A + b = w$ for all $a, b \in A$ $\square$

5.2 Lemma. *Let $o = o_A \in A$ be absorbing and $E(o) = \{o\}$. If r is a fully invariant congruence of A such that $r \nsubseteq \lambda_o$ then the set $I = \{\, a \in A \,|\, (a, o) \in r \,\}$ is a fully invariant ideal of A and $|I| \geq 2$.*

Proof. Since $r \nsubseteq \lambda_o$, we have $(a, b) \in r \setminus \lambda_o$ for some $a, b \in A$. Then, say, $f(a)+x = o \neq v = f(b) + x$ for some $f \in E$ and $x \in A$. Of course, $(o, v) \in r$ and $|I| \geq 2$. The rest is clear. $\square$

5.3 Proposition. *Let A be emi-simple, $o = o_A \in A$, $E(o) = \{o\}$ and $A + A \neq \{o\}$ (i.e., A is not constant). Then the semigroup A/λ_o is emc-simple.*

Proof. By 5.1, $\lambda_o \neq \mathrm{id}_A$. Hence, using 5.2, λ_o is a maximal proper fully invariant congruence of A, and so A/λ_o is emc-simple. $\square$

The following assertion is obvious:

5.4 Lemma. *Let $o = o_A \in A$ be absorbing and $E(o) = \{o\}$. Then:*
(i) *A/λ_o is a semilattice iff $a + b = o$ for all $a, b \in A$ such that $2a + b = o$;*
(ii) *A/λ_o is constant iff $A + A + A = \{o\}$.* $\square$

5.5 Proposition. *Let A be emi-simple, $o = o_A \in A$, $E(o) = \{o\} \neq A + A$ and $2a + b = o \neq a + b$ for some $a, b \in A$. Then the factorsemigroup A/λ_o is emc-simple of type* 2.6(5)*. In particular, A is emc-simple iff $\lambda_o = \mathrm{id}_A$.*

Proof. By 5.3, the factorsemigroup $B = A/\lambda_o$ is emc-simple, B is not a semilattice with respect to 5.4(i) and, of course, B is not cancellative. Finally, $I = A + A$ is a fully invariant ideal, however $I \neq \{o\}$, and hence $I = A$. Then $A = A + A + A$ and B is not constant due to 5.4(ii). Now it remains to use 2.6. $\square$

5.6 Theorem. *Let A be a non-trivial zeropotent semigroup and $o = o_A$. Then A is emc-simple of type* 2.6(5) *(i.e., A is not constant) if and only if the following two conditions are satisfied:*

(1) *$E(a) + A = A$ for every $a \in A \setminus \{o\}$ (i.e., for all $a, b \in A \setminus \{o\}$ there are $f \in E$ and $c \in A$ with $f(a) + c = b$).*
(2) *For all $a, b \in A \setminus \{o\}$, $a \neq b$, there are $g \in E$ and $d \in A$ such that $\{o\} \subsetneqq \{g(a) + d, g(b) + d\}$.*

Proof. First, assume that A is emc-simple and not constant. If $a \in A \setminus \{o\}$ then $E(a)+A$ is a fully invariant ideal, and hence $E(a)+A = A$ (otherwise $E(a)+A = \{o\}$ and A is constant). Of course, $\lambda_o = \mathrm{id}_A$ (otherwise $\lambda_o = A \times A$ and A is constant by 5.1) and (2) immediately follows. Conversely, assume that the conditions (1) and (2) are satisfied. Of course, (1) implies that A is not constant and $I = A$ whenever I is a fully invariant ideal of A with $I \neq \{o\}$. Further, the set $J = \{\, a \in A \mid (a, o) \in \lambda_o \,\}$ is a fully invariant ideal of A, hence $J = \{o\}$ (otherwise $J = A$, $\lambda_o = A \times A$ and A is constant) and (2) implies that $(a, b) \notin \lambda_o$ whenever $a, b \in A \setminus \{o\}$ and $a \neq b$. Thus $\lambda_o = \mathrm{id}_A$, however A is neither idempotent nor cancellative, and so A is emc-simple of type 2.6(5) by 5.3 and 2.6. $\square$

5.7 Remark. Suppose that A is non-trivial, zeropotent and satisfies 5.6(1) (i.e., A is emi-simple with $A + A = A$).
(i) Take $b, c \in A$ such that $a = b + c \neq o_A$ and put $K = \{\, f \in E \mid b \in A + f(a) \,\}$, $L = \{\, g \in E \mid c \in A + g(a) \,\}$. Obviously, both K and L are non-empty subsets of E. If $f_1, f_2 \in K$ then, for some $u, v \in A$, $u + f_1(a) = b = v + f_2(a)$ and $b = u + f_1(b + c) = u + f_1(v) + f_1 f_2(a) + f_1(c)$ and we see that $f_1 f_2 \in K$. Proceeding similarly, we can show that $KK \cup KL \subseteq K$ nad $LL \cup LK \subseteq L$. Furthermore, if $h \in K \cap L$ then, for some $u, v \in A$, $b = u + h(a)$, $c = v + h(a)$ and $o_A \neq a = b+c = u+v+2h(a) = u+v+o_A = o_A$, a contradiction. Thus $K \cap L = \emptyset$. The set $M = K \cup L$ is a subsemigroup of the multiplicative semigroup $E(\cdot)$ and both K and L are right ideals of M. Notice that $\mathrm{id}_A \notin M$.
(ii) Take $z \in A$, $z \neq o_A$, and define a relation t on the set $E(z)$ by $t = \mathrm{id}_{E(z)} \cup \{\, (f(z), g(z)) \mid f, g \in E \text{ and } f(z) \in A + g(z) \,\}$. Clearly, t is reflexive and transitive. Moreover, if $f(z) = a + g(z)$ and $g(z) = b + f(z)$ for some $a, b \in A$ then $f(z) = a+b+f(z) = 2(a+b)+f(z) = o_A+f(z) = o_A$. Symmetrically, $g(z) = o_A$ and we see that t is antisymmetric. Thus t is an order relation on $E(z)$ and o_A is the smallest element in $E(z)$. On the other hand, if $o_A \neq f(z) \in E(z)$ then $f(z) = b+c$ for some $b, c \in A$ and, with respect to 5.6(1), $b = g(z)+u$ and $c = k(z)+v$ for some $g, k \in E$ and $u, v \in A$. Clearly, $(f(z), g(z)) \in t$ and $(f(z), k(z)) \in t$. If $f(z) = g(z) = k(z)$ then $f(z) = b + c = 2f(z) + u + v = o_A$, a contradiction. Thus the set $E(z)$ has no maximal element and the sets $E(z)$, E and A are infinite.

5.8 Example. Let $\mathcal{T}$ denote the set of all infinite countable subsets of an uncountable set S and put $\mathcal{S} = \mathcal{T} \cup \{S\}$. Define an addition on $\mathcal{S}$ by $R + T = R \cup T$

if $R \cap T = \emptyset$ and $R + T = S$ otherwise. Then $\mathcal{S}$ becomes a commutative zeropotent semigroup, $\mathcal{S} + \mathcal{S} = \mathcal{S}$ and $o_{\mathcal{S}} = S$. One checks easily that automorphisms operate transitively on $\mathcal{T}$ and the conditions 5.6(1),(2) are satisfied, i.e., $\mathcal{S}$ is emc-simple.

5.9 Example. Let $\mathcal{I}$ denote the set of all infinite subsets of $\mathbb{N}$ and define an addition on $\mathcal{I}$ by $I + J = I \cup J$ if $I \cap J = \emptyset$ and $I + J = \mathbb{N}$ otherwise. Again, $\mathcal{I}$ becomes a commutative zeropotent semigroup, $\mathcal{I} + \mathcal{I} = \mathcal{I}$ and $o_{\mathcal{I}} = \mathbb{N}$.

Let $I, J \in \mathcal{I}$ be such that $I \neq \mathbb{N} \neq J$ and α be a bijection of I onto J. For $K \in \mathcal{I}$, put $f(K) = \{\, \alpha(m) \mid m \in K \,\} \subseteq J$ whenever $K \subseteq I$ and $f(K) = \mathbb{N}$ otherwise. Then $f(I) = J$ and f is an endomorphism of $\mathcal{I}(+)$. For, let $K, L \in \mathcal{I}$. If $f(K) = \mathbb{N}$ then $f(K) + f(L) = \mathbb{N}$, $K \nsubseteq I$, $K + L \nsubseteq I$ and $f(K + L) = \mathbb{N}$. Symmetrically, if $f(L) = \mathbb{N}$ then $f(K) + f(L) = \mathbb{N} = f(K + L)$. Now, suppose that $f(K) \neq \mathbb{N} \neq f(L)$. Then $K \cup L \subseteq I$. If $K \cap L = \emptyset$ then, α being injective, $f(K) \cap f(L) = \emptyset$ and we have $f(K + L) = f(K \cup L) = f(K) \cup f(L) = f(K) + f(L)$. Finally, if $K \cap L \neq \emptyset$ then $f(K \cap L) \subseteq f(K) \cap f(L)$, hence $f(K) \cap f(L) \neq \emptyset$ and $f(K + L) = \mathbb{N} = f(K) + f(L)$.

Now it is clear that endomorphisms operate transitively on $\mathcal{I} \setminus \{\mathbb{N}\}$ and $\mathcal{I}$ satisfies the conditions 5.6(1),(2). Thus $\mathcal{I}$ is emc-simple. Notice that automorphisms do not operate transitively on $\mathcal{I} \setminus \{\mathbb{N}\}$. Namely, if f is an automorphism of $\mathcal{T}$ and I is a cofinite subset of $\mathbb{N}$ then $f(I)$ is cofinite.

5.10 Example.([7]) Let R be a subsemigroup of a left cancellative semigroup $S = S(\cdot)$ such that $aS \cap bR \neq \emptyset$ for all $a \in S$ and $b \in R$. Furthermore, assume that $uR \cap vR = \emptyset$ for some $u, v \in R$. Now, denote by $\mathcal{R}$ the set of all non-empty subsets A of S such that $AR \subseteq A$ and define an addition on $\mathcal{R}$ by $A + B = A \cup B$ if $A \cap B = \emptyset$ and $A + B = S$ otherwise. Then $R, S \in \mathcal{R}$, $\mathcal{R}$ becomes a commutative zeropotent semigroup, $o_{\mathcal{R}} = S$ and $r = \{\, (A, B) \in \mathcal{R} \times \mathcal{R} \mid \text{for all } C \in \mathcal{R}, A \cap C = \emptyset \text{ iff } B \cap C = \emptyset \,\}$ is a congruence of $\mathcal{R}$. Put $\mathcal{S} = \mathcal{R}/r$ and denote by π the natural projection of $\mathcal{R}$ onto $\mathcal{S}$.

(i) We have $(aS, S) \in r$ for every $a \in S$ (indeed, if $C \in \mathcal{R}$ and $c \in C$ then $\emptyset \neq aS \cap cR \subseteq aS$). Hence, if $r = \mathrm{id}_{\mathcal{R}}$ then S is a group.

(ii) If $A, B \in \mathcal{R}$ and $a \in S$ then $(a(A + B), aA + aB) \in r$. Indeed, if $A \cap B = \emptyset$ then $A + B = A \cup B$ and $a(A + B) = aA \cup aB$. Since S is left cancellative, we have $aA \cap aB = \emptyset$ and $aA + aB = aA \cup aB$. On the other hand, if $A \cap B \neq \emptyset$ then $A + B = S$, $a(A + B) = aS$, $aA \cap aB \neq \emptyset$, $aA + aB = S$ and $(aS, S) \in r$ by (i).

(iii) If $(A, B) \in r$ then $(aA, aB) \in r$ for every $a \in S$. Indeed, if $C \in \mathcal{R}$ is such that $aA \cap C \neq \emptyset$ then $A \cap D \neq \emptyset$, where $D = \{\, d \in S \mid ad \in C \,\} \in \mathcal{R}$, hence $B \cap D \neq \emptyset$ and $aB \cap C \neq \emptyset$.

(iv) Using (ii) and (iii), we get a multiplicative homomorphism $\alpha : S \to \mathrm{End}(\mathcal{R})$, where $\alpha(a)(\pi(A)) = \pi(aA)$ for all $a \in S$, $A \in \mathcal{R}$.

(v) If σ is a fully invariant congruence of $\mathcal{S}$ such that $(\pi(R), \pi(S)) \in \sigma$ then $\sigma = \mathcal{S} \times \mathcal{S}$. Indeed, put $s = \pi^{-1}(\sigma)$. Then s is a congruence of $\mathcal{R}$, $r \subseteq s$ and if $(A, B) \in s$ then $(aA, aB) \in s$ for every $a \in S$. Further, $(R, S) \in s$, $(aR, aS) \in s$ for every $a \in S$ and, since $(aS, S) \in r \subseteq s$, we get $(aR, S) \in s$. In particular, if $A \in \mathcal{R}$ is such that $(aR, A) \in s$ then $(A, S) \in s$. On the other hand, if $a \in S$, $(aR, A) \notin s$ and $B \in \mathcal{R}$ is maximal with respect to $B \subseteq A$ and $B \cap aR = \emptyset$, then $(A, B \cup aR) \in s$, $(B \cup aR, S) = (B + aR, B + S) \in s$ and, again, $(A, S) \in S$. Thus $s = \mathcal{R} \times \mathcal{R}$ and $\sigma = \mathcal{S} \times \mathcal{S}$.

(vi) We have $uR, vR \in \mathcal{R}$, $(uR, vR) \notin r$ and $(R, S) \notin r$. Indeed, R is a subsemigroup, and hence $uR, vR \in \mathcal{R}$. Further, $(uR, vR) \notin r$, since $uR \cap uR \neq \emptyset$ and $vR \cap uR = \emptyset$. Finally, if $(R, S) \in r$ then $(uR, uS) \in r$ and $(vR, vS) \in r$,

however $(uS, S) \in r$ and $(vS, S) \in r$, and hence $(uR, vR) \in r$, a contradiction.
(vii) Now, suppose that τ is a fully invariant congruence of $\mathcal{S}$ maximal with respect to $(\pi(R), \pi(S)) \notin \tau$. Then $\mathcal{T} = \mathcal{S}/\tau$ is emc-simple of type 2.6(5). Indeed, $\mathcal{T}$ is a non-trivial zeropotent semigroup. If ϱ is a fully invariant congruence of $\mathcal{T}$ such that $\varrho \neq \mathrm{id}_{\mathcal{T}}$ then $\sigma = \psi^{-1}(\varrho)$, ψ being the natural projection of $\mathcal{S}$ onto $\mathcal{T}$, is a fully invariant congruence of $\mathcal{S}$ and $\tau \subsetneqq \sigma$. Thus $(\pi(R), \pi(S)) \in \sigma$, $\sigma = \mathcal{S} \times \mathcal{S}$ by (v), and hence $\varrho = \mathcal{T} \times \mathcal{T}$. Finally, it remains to show that $\mathcal{T}$ is not constant. Let I be a right ideal of R maximal with respect to $vR \subseteq I$ and $uR \cap I = \emptyset$ (we know that $uR \cap vR = \emptyset$). Then $uR + I = uR \cup I$, $(uR \cup I, R) \in r$, $\pi(uR) + \pi(I) = \pi(R)$ and $(\pi(uR) + \pi(I), \pi(S)) \notin \tau$.

5.11 Remark. As a particular case of 5.10, we can take for R the free semigroup of words over two letters u, v and for S any group containing R. For instance, S can be the free group over u and v or S can be chosen to be free metabelian over u and v.

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