MOD-RETRACTABLE RINGS

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Abstract. A right module M over a ring R is said to be retractable if $\mathrm{Hom}_R(M, N) \neq 0$ for each nonzero submodule N of M. Let MG be a right module over the group ring RG. We proved that MG is a retractable RG-module iff M_R is retractable. A ring R is (finitely) mod-retractable if every (finitely generated) right R-module is retractable. Some comparisons between max rings, semiartinian rings, perfect rings, noetherian rings, nonsingular rings and mod-retractable rings are realized.

1. Introduction

Throughout this paper, we assume that R is an associative ring with unity, M is a unital right R-module and G is a group. A module M is called *retractable* if there exists a non-zero homomorphism into every non-zero submodule $N \subseteq M$, i.e., $Hom_R(M, N) \neq 0$ for every nonzero submodule N of M. This notion was introduced by Khuri [9]. After 1979, retractable modules have been studied extensively by many authors (see for example, [10], [11], [12],[17],[18], [19]). Recently, Ecevit and Koşan introduced the concept of retractability for rings [4]. A ring is said to be *right* (*finitely*) *mod-retractable* if every (finitely generated) right module is retractable.

The notions of the group module of group rings were introduced and studied by Koşan-Lee-Zhou in [8]. Section 2 of this note deals with the transfer of properties of retractable modules between a right R-module and its group RG-module. It is shown that MG is a retractable RG-module if and only if M is a retractable R-module.

In Section 3, we realized some comparisons between max rings, semiartinian rings, perfect rings, noetherian rings, nonsingular rings and mod-retractable rings. We characterize mod-retractable rings as rings whose all torsion theories are hereditary. As a consequence, we prove that a commutative ring is mod-retractable if and only if it is semiartinian. Moreover, we show that a left perfect ring is mod-retractable if and only if it is isomorphic to a ring $\prod_{i \leq k} M_{n_i}(R_i)$ for a finite system of a both left and right perfect local rings R_i, $i \leq k$. This result illustrates the way how to produce new examples of mod-retractable rings proved in [4, Theorem 8] as finite products of matrix rings over mod-retractable rings. Namely, applying this procedure on the class of two-sided perfect local rings we get all examples of left perfect right mod-retractable rings. In case R is a right noetherian ring, then it is shown that R is right mod-retractable if and only if $R \cong \prod_{i \leq k} M_{n_i}(R_i)$ for a system of a local right artinian rings R_i, $i \leq k$.

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In the following the symbols, "$\leq$" will denote a submodule, "$\leq_d$" a module direct summand and "$\trianglelefteq$" an essential submodule. The notions $J(M)$ means Jacobson radical of a module M, $N(R)$ is the prime radical of R and $E(M)$ means an injective envelope of M. The group ring of G over R is denoted by RG. We will refer to [1] and [20] for all undefined notions used in the text.

2. RETRACTABILITY FOR GROUP MODULES

Throughout this section G is a group M is a module over a ring R.

Let MG denote the set all formal linear combinations of the form $\sum_{g\in G} m_g g$ where $m_g \in M$ and $m_g = 0$ almost for every g.

For elements $\sum_{g\in G} m_g g, \sum_{g\in G} n_g g \in MG$ and $\sum_{g\in G} r_g g \in RG$;

$$\sum_{g\in G} m_g g = \sum_{g\in G} n_g g \text{ if and only if } \; m_g = n_g \text{ for all } g \in G$$

$$\sum_{g\in G} m_g g + \sum_{g\in G} n_g g = \sum_{g\in G} (m_g + n_g) g$$

$$(\sum_{g\in G} m_g g)(\sum_{g\in G} n_g g) = \sum_{g\in G} (k_g) g \text{ where } \; k_g = \sum_{hh'=g} m_h r'_h.$$

Under the operation defined above, MG becomes a right R-module over the group ring RG and this RG-module MG is said to be the *group module* of G by M over G (see [8]). Note that M is an R-submodule of MG such that $m = m \cdot 1$, where 1 here denotes the identity element of G. It is well known that the identity element in G is also the identity element of RG.

Lemma 2.1. *If MG is the group module, then $MG \cong_{RG} M \otimes_R RG$.*

Proof. Clearly, there exists an RG-homomorphism $\varphi : M \otimes_R RG \to MG$ satisfying the rule $\varphi(\sum_i m_i \otimes \sum_g r_{gi} g) = \sum_{g,i} (m r_{gi}) g = \sum_g (\sum_i m r_{gi}) g$. Now, it is easy to see that φ is onto MG. Finally, φ is injective since RG is a free left R-module. □

The map $MG \to M$, $\sum m_g g \to \sum m_g$, is an R-homomorphism and is denoted by ε_M. The kernel of ε_M is denoted by $\triangle(M)$. Thus, $\varepsilon_R : RG \to R$ is the usual *augmentation map.*

Lemma 2.2. *Let MG be the group module of G by M over RG. Then for any $x \in MG$ and any $\alpha \in RG$, $\varepsilon_M(x\alpha) = \varepsilon_M(x)\varepsilon(\alpha)$. In particular, ε_M is an R-homomorphism and ε_R is a ring homomorphism.*

Proof. Write $x = \sum_{g\in G} m_g g$ and $\alpha = \sum_{g\in G} r_g g$. Then,

$$\begin{aligned} \varepsilon_M(x\alpha) &= \textstyle\sum_{g\in G} \left(\sum_{hh'=g} m_h r_{h'} \right) \\ &= \left(\sum m_g \right) \left(\sum r_g \right) \\ &= \varepsilon_M(x)\varepsilon(\alpha). \end{aligned}$$

□

Recall that R is called a *right perfect* ring if $J(R)$ is right T-nilpotent and R is semi-local (see [20, Proposition VIII.5.1]). As ε_R is onto R and classes of (finite) mod-retractable rings as well as perfect rings are closed under taking factors, we get the following observations:

Corollary 2.3. *If RG is a (finitely) mod-retractable ring then R is (finitely) mod-retractable.*

Corollary 2.4. *If RG is right perfect then R is right perfect as well.*

Theorem 2.5. *Let G be a finite group. Then RG is right perfect if and only if R is right perfect.*

Proof. By Corollary 2.4, it suffices to prove the reverse implication.

Let R be a right perfect ring and M be an arbitrary nonzero RG-module. As J is right T-nilpotent and M has a natural structure of an R-module, we get that $M(JG) = M(RG)J = MJ \neq M$. This implies that JG is a right T-nilpotent ideal of RG contained in the Jacobson radical of RG. Since $RG/JG \cong (R/J)G$ is artinian R-module, it is a right artinian ring. Hence the Jacobson radical of RG is right T-nilpotent and RG is right perfect by [20, Proposition VIII.5.1]. □

In the following, we suppose that R is a subring of a ring S and $\{e_1, \dots, e_n\}$ is a free R-base of S with $e_iR = Re_i$.

Proposition 2.6. *Let N_R be a submodule of a module M_R. Then* $\mathrm{Hom}_R(M, N) \neq 0$ *if and only if* $\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) \neq 0$.

Proof. First we note that

$$\begin{aligned} \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N \otimes_R S)) \\ &\cong \mathrm{Hom}_R(M, N \otimes_R S) \end{aligned}$$

by [20, Proposition I.9.2]. Since ${}_RS$ is a free module, there exists a natural number n such that ${}_RS \cong R^n$. Now

$$\begin{aligned} \mathrm{Hom}_R(M, N \otimes_R S) &\cong \mathrm{Hom}_R(M, N \otimes_R R^n) \\ &\cong \mathrm{Hom}_R(M, N \otimes_R R)^n \\ &\cong \mathrm{Hom}_R(M, N)^n, \end{aligned}$$

which implies that $\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) \neq 0$ if and only if $\mathrm{Hom}_R(M, N) \neq 0$. □

Recall that $MG \cong_{RG} M \otimes_R RG$ by Lemma 2.1. As the group ring RG is free as an R-module, we get the following easy consequence of Proposition 2.6.

Corollary 2.7. *Let N be a submodule of a right R-module M. Then* $\mathrm{Hom}_R(M, N) \neq 0$ *if and only if* $\mathrm{Hom}_{RG}(MG, NG) \neq 0$.

Lemma 2.8. *Let M be a module over a ring S and N an R-submodule of $M \otimes_R S$. Then every nonzero $\alpha \in \mathrm{Hom}_R(M, N)$ can be extended to a nonzero S-homomorphism of $M \otimes_R S$ into NS.*

Proof. Let $\alpha \in \mathrm{Hom}_R(M, N)$ and $\alpha \neq 0$. Obviously, $\alpha \otimes S$ is an S-homomorphism of $M \otimes_R S$ into $N \otimes_R S$. Moreover, a mapping $\psi : N \otimes_R S \to NS$ defined by the rule $\psi(\sum_i n_i \otimes s_i) = \sum_i n_i s_i$ is an S-homomorphism as well, hence $\phi = \psi\alpha \otimes S$ is the required homomorphism of $M \otimes_R S$ into NS. Finally, we can easily see that $\phi(n \times 1) = \psi(\alpha(n) \otimes 1) = \alpha(n)$, i.e. $\psi \neq 0$. □

Theorem 2.9. *Let all elements e_i commute with all elements of R and M be an R-module. Then $M \otimes_R S$ is a retractable S-module if and only if M is retractable.*

Proof. As ${}_RS$ is a projective module, the functor $- \otimes_R S$ is exact, hence the direct implication follows immediately from Proposition 2.6.

Suppose that M is a retractable R-module and fix a nonzero S-submodule P of $M \otimes_R S$. We have to show that $\mathrm{Hom}_{RG}(M, P) \neq 0$. For each $\mu \in M \otimes_R S$ define

$$\sigma(\mu) = \{F \subset \{1, \dots, n\} \mid \exists m_1, \dots m_n \in M : \mu = \sum_{i \in F} m_i \otimes e_i\}.$$

It is easy to see that $\sigma(\mu) \neq \emptyset$, hence we may put $s(\mu) = \min\{|F| \mid F \in \sigma(\mu)\}$. Now, we can choice a nonzero element $\mu \in P$ with a minimal (nonzero) value of $s(\mu)$. Thus there exist different numbers $i_1, \dots, i_{s(\mu)} \leq n$ and nonzero elements $m_1, \dots, m_{s(\mu)} \in M$ such that $\mu = \sum_{j=1}^{k} m_j \otimes e_{i_j}$. If there are $r \in R$ and $j \leq s(\mu)$ for which $m_j \otimes e_{i_j} r = m_j r \otimes e_{i_j} = 0$, then $s(\mu r) < s(\mu)$, hence $\mu r = 0$ due to the minimality of $s(\mu)$. Thus the annihilators of all $m_1, \dots, m_{s(\mu)}$ coincide and cyclic R-modules μR and $m_i R$ are R-isomorphic. As M is retractable, there exists a nonzero R-homomorphism of M into μR. Now, applying Lemma 2.8 for $N = \mu R$, we obtain a nonzero S-homomorphism of $M \otimes_R S$ to $\mu S \subseteq P$, which finishes the proof. □

As an immediate corollary, we have the following which is one of the main results of this note.

Corollary 2.10. *Let M be an R-module. MG is a retractable RG-module iff M_R is retractable.*

However every tensor product $M \otimes S$ of a retractable R-module M and a ring extension satisfying the hypothesis of Theorem 2.9 has to be retractable over S, the following example shows that such an extension of mod-retractable rings is not necessarily mod-retractable.

Example 2.11. Let F be a field and put $S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then matrices $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commutes with the subring $R = FE_1$ of S and they produce a free base of S a right and left R-module. However every R-module M is retractable, hence every S module $M \otimes S$ is retractable as well by Theorem 2.9, the module $E_2 S$ is not retractable.

3. Mod-retractable rings

The general case and max rings:

In this section, first,we prove a general ring-theoretic criterion of Mod-retractable rings.

Proposition 3.1. *R is a right mod-retractable ring iff for every non-zero module M and every $m \in M$ such that $mR \trianglelefteq M$ there exists a non-zero homomorphism $M \to mR$.*

Proof. It suffices to show the reverse implication. Fix an arbitrary non-zero module M and its non-zero submodule N. Let $n \in N \setminus \{0\}$. Then an identity mapping on nR may be extended to a homomorphism $\nu : M \to E(nR)$. Note that $nR \trianglelefteq \nu(M) \subseteq E(nR)$, hence by the hypothesis there exists a non-zero homomorphism $\nu(M) \to nR \subseteq N$, which finishes the proof. □

We recall that
• a ring is called *right max* provided every non-zero right module contains a maximal submodule,
• an ideal $I \subset R$ is right T-nilpotent, provided for every sequence $a_1, a_2, \dots \in I$ there exist n such that $a_n a_{n-1} \dots a_1 = 0$.

Lemma 3.2. *If R is a right max ring, then $J(R)$ is right T-nilpotent.*

Proof. It is well known (see for example [1, Remark 28.5] or [21, Proposition 1.8]). □

Theorem 3.3. *If R is a right mod-retractable ring, then R is right max.*

Proof. Assume that $0 \neq M$ contains no maximal submodule, fix $0 \neq m \in M$ and an arbitrary maximal submodule N of mR. Then M/N contains no maximal submodule and so there exists no non-zero homomorphism M/N into a simple mR/N, i.e. M/N is not retractable. □

As an immediate consequence of the previous results we obtain

Corollary 3.4. *Jacobson radical of every right mod-retractable ring is right T-nilpotent.*

Recall that a *torsion theory* $\tau = (\mathcal{T}, \mathcal{F})$ is a pair of classes of modules closed under isomorphic images such that $\mathcal{T} \cap \mathcal{F} = 0$, $\mathcal{T}$ is closed under taking factors, $\mathcal{F}$ is closed under submodules and for every module M there exists a submodule $\tau(M)$ for which $\tau(M) \in \mathcal{T}$ and $M/\tau(M) \in \mathcal{F}$. Moreover, a torsion theory is hereditary if $\mathcal{T}$ is closed under submodules.

For a class of right R-modules $\mathcal{C}$, we consider the following *annihilator classes*:

$$^{\circ}\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Hom}_R(M, \mathcal{C}) = 0\}$$

and

$$\mathcal{C}^\circ = \{M \in \text{Mod-}R \mid \text{Hom}_R(\mathcal{C}, M) = 0\}.$$

We notice that the annihilator classes of the form $^\circ\mathcal{C}$ for some $\mathcal{C} \subseteq \text{Mod-}R$ coincide with the torsion classes of modules, and $\mathcal{C}^\circ$ coincide with the torsionfree classes of modules.

Theorem 3.5. *A ring R is mod-retractable if and only if every torsion theory on* Mod-R *is hereditary.*

Proof. Suppose that R is mod-retractable and $\tau = (\mathcal{T}, \mathcal{F})$ is a torsion theory. For $M \in \mathcal{T}$ and $N \le M$, let $\tau(N)$ be the torsion part of N. Then $M/\tau(N) \in \mathcal{T}$, while $N/\tau(N) \in \mathcal{F}$. Then $\text{Hom}(M/\tau(N), N/\tau(N)) = 0$. Since $N/\tau(N)$ is a submodule of $M/\tau(N)$ and $M/\tau(N)$ is retractable, it follows that $N/\tau(N) = 0$. Hence $N \in \mathcal{T}$.

Conversely, suppose that M is an R-module and $0 \ne N \le M$. If $\text{Hom}(M, N) = 0$, then $N \notin {}^\circ(M^\circ)$. This implies that the torsion theory $({}^\circ(M^\circ), M^\circ)$ is not hereditary. □

A chain $(Y_\alpha | \alpha \le \sigma)$ is called a *strictly decreasing continuous chain* of submodules of Y provided that $Y_0 = Y$, $Y_\alpha \supset Y_{\alpha+1}$ for each $\alpha < \sigma$, $Y_\alpha = \bigcup_{\beta<\alpha}$ for each limit ordinal $\alpha \le \sigma$, and $Y_\sigma = 0$.

The following result was proved in [3, Lemma 3].

Lemma 3.6. *Let R be a ring and let X and Y be non-zero R-modules. Then the following are equivalent:*

(1) ${}^\circ X \subseteq {}^\circ Y$;
(2) *there exists a strictly decreasing continuous chain $(Y_\alpha | \alpha \le \sigma)$ of submodules of Y and R-homomorphisms $\varphi_\alpha : Y_\alpha \to X$, $\alpha < \sigma$, such that $Y_{\alpha+1} = Ker(\varphi_\alpha)$ for all $\alpha < \sigma$.*

Theorem 3.7. *The following are equivalent for a ring R:*

(1) *R is mod-retractable*
(2) *If $X \trianglelefteq Y$ then ${}^\circ X = {}^\circ Y$*
(3) *For every module X, ${}^\circ X = {}^\circ E(X)$*

Proof. (1)⇒(2) The inclusion ${}^\circ Y \subseteq {}^\circ X$ is obvious. In order to prove the converse inclusion, we will apply Lemma 3.6. So we construct a strictly decreasing continuous chain $(Y_\sigma | \sigma \le \tau)$ of submodules of Y and a family of non-zero homomorphisms $f_\sigma : Y_\sigma \to X$ for all $\sigma < \tau$.

We put $Y_0 = Y$. Since R is mod-retractable, there is a non-zero homomorphism $f_0 : Y_0 \to X$. Suppose that the submodules Y_ρ and the non-zero homomorphisms $f_\rho : Y_\rho \to X$ are constructed for all $\rho < \sigma$. If $\sigma = \rho + 1$ we denote $Y_\sigma = \text{Ker}(f_\rho)$, and for σ a limit ordinal we put $Y_\sigma = \bigcap_{\rho<\sigma} Y_\rho$. If $Y_\sigma = 0$ then the construction is finished. If $Y_\sigma \ne 0$ then $Y_\sigma \cap X \ne 0$, hence there is a non-zero homomorphism $f_\sigma : Y_\sigma \to X$ such that $f_\sigma(Y_\sigma) \subseteq Y_\sigma \cap X$.

Since for cardinality reasons there is τ with $Y_\tau = 0$, we can apply Lemma 3.6, and the proof is complete.

(2)⇒(3) This is obvious.

(3)⇒(1) Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. If $X \in \mathcal{F}$ then $\mathcal{T} \subseteq {}^\circ X = {}^\circ E(X)$, hence $E(X) \in \mathcal{F}$. Then $\mathcal{F}$ is closed with respect injective envelopes, hence τ is hereditary. □

Commutative rings

To obtain a relation between mod-retractable rings and semiartinian rings for commutative case, we state the main theorem of Ohtake in [15, Theorem 8].

Theorem 3.8. *Let R be a commutative ring. Then the following are equivalent.*
(1) *Every torsion theory in* Mod-R *is of simple type.*
(2) *Every torsion theory in* Mod-R *is hereditary.*
(3) *R is a semiartinian, max ring.*

Theorem 3.9. *Let R be a commutative ring. Then R is mod-retractable if and only if R is semiartinian.*

Proof. Suppose that R is commutative semiartinian. Then $J(R)$ is T-nilpotent by [14, Proposition 3.2] and $R/J(R)$ is von Neumann regular by [14, Theoreme 3.1]. Hence R is mod-retractable by [7, Theorem 3] and Theorem 3.8.

The converse is clear from Theorems 3.5 and 3.8. □

Example 3.10. Let λ be an ordinal and F a field. Put $\kappa = max(card\lambda, \omega)$. Examples of commutative semiartinian regular F-subalgebras of the algebra F^κ of the socle length $\lambda + 1$, which are mod-retractable by Theorem 3.9, is constructed in [5, Theorem 2.6].

Perfect rings

Let M be an R-module. Recall that a submodule N of M is said to be a *superfluous* in M, denoted by $N \ll M$, whenever $L \leq M$ and $M = N + L$ then $M = L$.

Lemma 3.11. *Let M be a non-zero semiartinian module, N its superfluous submodule, and S_i, $i \in I$, simple modules. If $M/N \cong \bigoplus_{i\in I} S_i$ and there exists a simple subfactor of N which is not isomorphic to any S_i, $i \in I$, then there exists a non-retractable factor of M.*

Proof. Let T be a simple submodule of N/X where X is a submodule of N that is not isomorphic to any S_i. Since $N/X \ll M/X$ we may suppose that $X = 0$.

Assume that there exists a non-zero homomorphism $M \to T$. Then there exists a maximal submodule $Y \subset M$ such that $M/Y \cong T$. If $N \not\subseteq Y$, we get $N \neq N + Y = M$ and $Y = M$ because $N \ll M$, a contradiction. Thus $N \subseteq Y$, which implies that $T \cong M/Y$ is a direct summand of $M/N \cong \bigoplus_{i\in I} S_i$. Hence there exists $j \in I$ such that $S_i \cong T$, a contradiction with the hypothesis. We have proved that $Hom(M, T) = 0$. □

Proposition 3.12. *Every local ring which is both right and left perfect is right mod-retractable.*

Proof. Assume that R is a local right and left perfect ring. Let M be a right module over R and N its non-zero submodule. Applying [1, Theorem 28.4], we get that $M/MJ(R) \cong R/J(R)^{(\kappa)}$ is non-zero semisimple since R is right perfect and N is a non-zero semiartinian module (because R is left perfect). Thus there exists a surjective homomorphism $M \to M/MJ(R) \to R/J(R)$ and N contains non-zero socle isomorphic to a direct power of $R/J(R)$, which proved the existence of non-zero homomorphism $M \to N$. □

The following example shows that assumption "right perfect and left perfect" in previous proposition is not superfluous, i.e., there exits local, right perfect, but not left perfect rings, which is neither right nor left mod-retractable.

Example 3.13. Let k be a field and V_k be an infinite dimensional vector space with a countable ordered basis $\{v_n \mid n \in \mathbb{N}\}$, so that every endomorphism of V_k can be described by a column finite $\mathbb{N}\times\mathbb{N}$ matrix with entries in k. Among these matrices, we have the identity matrix I and the unit matrices $e_{n,m}$ $(n, m \in \mathbb{N})$. Consider the k-subalgebra R of $End(V_k)$ generated by I and all the matrices $e_{n,m}$ with $n, m \in \mathbb{N}$ and $n < m$. (This k-subalgebra is just the set of all k-linear combinations of I and finitely many $e_{n,m}$ with $n < m$.) Thus R is the ring of all the $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over the field k that are constant on the diagonal and have only finitely many non-zero entries off the diagonal, all of over the diagonal. We note that all k-linear combinations of finitely many $e_{n,m}$ with $n < m$ are strictly upper triangular matrices with finitely many entries, hence are nilpotent matrices. Hence they form an ideal M of R, and every element in R but not in M is invertible. Thus R is a local ring with maximal ideal M. For every $n > 0$, we have that $e_{0,1}e_{1,2}e_{2,3}...e_{n-1,n} = e_{1,n} \neq 0$, so that M is not left T-nilpotent. It remains to show that M is right T-nilpotent. Take any sequence $a_1, a_2, \dots$ in M. Write a_1, as a linear combination of the $e_{n,m}$ with $n < m$: $a_1 = \sum_{i=1}^{t} \lambda_i e_{n_i,m_i}$. We can suppose $m_1 \leq m_2 \leq \cdots \leq m_t$. It is now easy to verify that $a_{m_t+1}a_{m_t} \dots a_2 a_1 = 0$. Hence M is right T-nilpotent. Note that R contains no simple right ideal, however it is left semiartinian.

Finally, we will show that R is not mod-retractable. First recall that $J(R)$ is not left T-nilpotent, hence R is not left mod-retractable by Corollary 3.4. Suppose that I is an essential right ideal of $J(R)$. It is easy to see that for each i there exists i_j such that $e_{ii_j} \in I$, hence R/I is semiartinian. Now, let $\varphi : E(R_R) \to J(R)$ is a homomorphism. Since $J(R)$ contains no idempotent element, it contains no nonzero injective submodule (cf. Lemma 3.17), hence kernel of φ is essential in $E(R)$. Thus for every $x \in E(R)$ there exists an essential right ideal I of R such that $\varphi(xR) \cong R/I$, which implies that $\varphi(xR)$ is semiartinian submodule of $J(R)$. Since R contains no simple right ideal, $\varphi = 0$, hence $\mathrm{Hom}(E(R), J) = 0$ and R is not right mod-retractable.

In [4, Theorem 8], it is shown for every finite set $\mathcal{I}$ that the ring $\prod_{i\in\mathcal{I}} R_i$ is right mod-retractable if and only if R_i is right mod-retractable for each $i \in \mathcal{I}$. Foe perfect rings we prove more precise structural result.

Theorem 3.14. *Let R be a right and left perfect ring. Then the following conditions are equivalent:*

(1) *R is right mod-retractable;*
(2) *R is right finitely mod-retractable;*
(3) $R \cong \prod_{i\le k} M_{n_i}(R_i)$ *for a system of a local right and left perfect rings R_i, $i \le k$.*

Proof. (1)⇒(2) is clear.

(2)⇒(3) We may suppose without lost of generality that R is an indecomposable ring. Denoted by $\{e_i,\ i \le n\}$ a complete set of orthogonal idempotents of R. Assume that there exists $i, j \le n$ such that $e_iR/e_iJ(R) \not\cong e_jR/e_jJ(R)$ and put $I = \{s \le n|\ e_sR/e_sJ(R) \cong e_iR/e_iJ(R)\}$. Define an idempotent $e = \sum_{j\in I} e_i$ and note that

$$\mathrm{Hom}_R(eR/eJ, (1-e)R/(1-e)J) = 0 = \mathrm{Hom}_R((1-e)R/(1-e)J, eR/eJ).$$

Hence either $\mathrm{Hom}_R(eR, (1-e)R) \neq 0$ and so $(1-e)R$ contains a subfactor isomorphic to e_iR/e_iJ or $\mathrm{Hom}_R((1-e)R, eR) \neq 0$ and so eR contains a subfactor isomorphic to e_jR/e_jJ for a suitable $j \notin I$, otherwise R is indecomposable. Now applying Lemma 3.11, either for $M = (1-e)R$ in the first case or for $M = eR$ in the second case, we see that R is not finitely mod-retractable. Hence $e_iR/e_iJ(R) \cong e_jR/e_jJ(R)$ for all i, j. Now it is well known that $R \cong \mathrm{End}_R(e_1R^n) \cong M_n(e_1Re_1)$ where e_1Re_1 is a local right and left perfect ring.

(3)⇒(1) follows by [4, Corollary 3 and Theorem 8] and Proposition 3.12. □

Now, we can formulate the following easy structural consequence of Theorems 3.3 and 3.14.

Corollary 3.15. *Let R be a left perfect ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\le k} M_{n_i}(R_i)$ for a system of a local rings R_i, $i \le k$, which are both left and right perfect.*

Nonsingular rings

Recall that, a module M_R is said to be *singular* (respectively, *nonsingular*) if $Z(M_R) = M_R$ (respectively, $Z(M_R) = 0$), where $Z(M_R) = \{m \in M : ann_R^r(m) \trianglelefteq R\}$. If $Z(R_R) = 0$ then R is called *a right nonsingular* ring.

Lemma 3.16. *Let R be a right mod-retractable ring and M a non-singular module. Then*

(1) *for every non-zero $m \in M$ there exists a non-zero injective module $E \subseteq mR$,*
(2) $J(M) = 0$.

Proof. (1) Fix a non-zero $m \in M$. Since $E(M)$ is a retractable module and $mR \subseteq E(M)$, there exists a non-zero homomorphism $\varphi : E(M) \to mR$. As M is non-singular, $\mathrm{Ker}\varphi$ is not essential in $E(M)$, hence applying the same technique as in [2, Lemma 3.3] we can find $y \in E(M)$ such that $x = \varphi(y) \neq 0$, $yR \cap \mathrm{Ker}\varphi = 0$ and so $yR \cong \varphi(yR)$. This implies that o $E(yR) \cap \mathrm{Ker}\varphi = 0$ where $E(yR)$ can be expressed as a direct summand of $E(M)$. As $E(yR) \cong \varphi(E(yR))$, the module $\varphi(E(yR))$ is injective.

(2) Let $m \in M$ be a non-zero element. By (1) there exists a non-zero injective submodule E of mR, hence E is direct summand in M, which implies that $J(E) \subseteq$

$J(M)$. Since R is right max by Theorem 3.3, $m =\notin J(E)$, which finishes the proof. □

Corollary 3.17. *If R is a right non-singular right mod-retractable ring, then $J(R) = 0$ and for every non-zero $r \in R$, there exists a non-zero idempotent $e \in rR$ such that eR is an injective module.*

Theorem 3.18. *Every right noetherian right mod-retractable right non-singular ring is semisimple.*

Proof. Let R be a right mod-retractable right non-singular ring. First note that $J(R) = 0$ by Corollary 3.17. Assume that R is not semisimple and define a sequence of right ideals I_n, J_n such that $I_{n+1} \oplus J_{n+1} = J_n$, $I_n \neq 0$ and J_n is not semisimple for each $n \geq 0$.

Take a non-trivial idempotent $e \in R$ which exists by Lemma 3.17. Then either eR or $(1-e)R$ is not semisimple. If eR is not semisimple put $I_0 = (1-e)R$ and $J_0 = eR$ otherwise $I_0 = eR$ and $J_0 = (1-e)R$.

Since J_n is not semisimple, by Corollary 3.17, there exist an idempotent $f \in J_n$ such that $0 \neq fR \neq J_n$ and fR is injective. Thus $J_n = fR \oplus G$ for a suitable submodule G and we put $J_{n+1} = fR$ and $I_{n+1} = G$ if fR is not semisimple and $J_{n+1} = G$ and $I_{n+1} = fR$ otherwise.

Now we can see that $\bigoplus_{n<\omega} I_n$ is infinitely generated right ideal. We have proved that a right mod-retractable right non-singular which is not semisimple is not right noetherian. □

Proposition 3.19. *Let R be a semiartinian mod-retractable ring and I be an ideal. Then*

1. *$J(R/I)$ is T-nilpotent,*
2. *$(R/I)/J(R/I)$ is non-singular,*
3. *every non-zero ideal of R/I contains a non-zero idempotent $e \in R/I$ such that $e(R/I)/eJ(R/I)$ is an injective $(R/I)/J(R/I)$-module.*

Proof. (1) It follows by Corollary 3.4

(2) Without lost of generality, we may assume that $J(R) = 0$ and $x\mathrm{Soc}(R) = 0$ for a non-zero $x \in R$. Note that $xR \cap \mathrm{Soc}(R) \neq 0$ and take $0 \neq y \in xR \cap \mathrm{Soc}(R)$. Then $yRyR = 0$, hence $yR \in J(R)$, a contradiction.

(3) Since $R/J(R)$ is nonsingular by [13, Lemma 7.8], we may apply Corollary 3.17. Thus there exists an idempotent in $R/J(R)$ which can be lifted to an idempotent $e \in R$ such that $eR/eJ(R)$ is injective $R/J(R)$-module. □

Noetherian rings

The following lemma is analogue to [2, Proposition 3.16].

Lemma 3.20. *Let R be a right noetherian ring. If R is right mod-retractable, then it is right artinian and left perfect.*

Proof. Since $R/N(R)$ contains no nilpotent ideal, $R/N(R)$ is right non-singular by [20, Lemma II.2.5]. Note that $J(R) \subseteq N(R)$ in general and $R/N(R)$ is semisimple by Theorem 3.18, which implies that $J(R) = N(R)$. Finally, since $J(R)$ is nilpotent by [20, Lemma XV.1.4] we get that R is right artinian by Hopkins-Levitzki Theorem and R is left perfect by [1, Theorem 28.4]. □

Theorem 3.21. *Let R be a right noetherian ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\le k} M_{n_i}(R_i)$ for a system of local right artinian rings R_i, $i \le k$.*

Proof. ($\Rightarrow$) It follows from Lemma 3.20 and Corollary 3.15.

($\Leftarrow$) Since $R \cong \prod_{i\le k} M_{n_i}(R_i)$ is left and right perfect, the proof is clear from Corollary 3.15 □

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