ON SEPARATING SETS OF WORDS V

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ABSTRACT. A locally final result concerning transitive closures of special replacement relations in free monoids is proved.

1. Introduction

This article is an immediate continuation of [1], [2], [3] and [4]. References like I.3.3 (II.3.3, III.3.3, IV.3.3, resp.) lead to the corresponding section and result of [1] ([2], [3], [4], resp.) and all definitions and preliminaries are taken from the same source.

2. Technical results (a)

Troughout this note, let $Z \subseteq A^+$ be a strongly separating set of words and let $\psi : Z \to A^*$ be a mapping.

Lemma 2.1. *Let $r, s, t \in A^*$ be reduced words such that neither rt nor ts is reduced. Then:*

(i) *$rt = r_1 z_1 s_1$ and $ts = r_2 z_2 s_2$, where $z_1, z_2 \in Z$ and $r_1, r_2, s_1, s_2 \in A^*$ are reduced.*
(ii) *$r = r_1 r_3$, $s = s_3 s_2$, $z_1 = r_3 r_2$, $z_2 = s_1 s_3$ and $t = r_2 t_1 s_1$, $t_1 \in A^*$, t_1 is reduced.*
(iii) *$r_2, s_1, r_3, s_3 \in A^+$, $|z_1| \geq 2$, $|z_2| \geq 2$ and $|t| \geq 2$.*
(iv) *$rts = r_1 z_1 t_1 z_2 s_2$ and $\mathrm{tr}(rts) = 2$.*
(v) *If $t = \psi(z_0)$ for some $z_0 \in Z$, then the ordered triple (z_1, z_0, z_2) is disturbing (see II.7).*

Proof. See I.6.2 and II.7. □

Corollary 2.2. *Let $r, s, t \in A^*$ be reduced. Then either rt is reduced or ts is reduced, provided that at least one of the following three cases holds:*

(1) *$|t| \leq 1$;*
(2) *rts is meagre;*
(3) $\mathrm{alph}(rts) \subseteq A \cup \{\varepsilon\}$.

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Lemma 2.3. *Assume that, for every $z \in Z$, either $|\psi(z)| \leq 1$ or $\psi(z)$ is reduced. Furthermore, assume that the equivalent conditions of II.7.3 are satisfied (e. g., if $\psi(Z) \subseteq A \cup \{\varepsilon\}$ or $Z \subseteq A$). If $z_1 \in Z$ and $r, s \in A^*$ are reduced, then either $r\psi(z_1)$ or $\psi(z_1)s$ is reduced.*

Proof. Combine 2.1(v) and II.7.3. □

3. Technical results (b)

In this section, let $x, y \in A^*$, $z_1, \dots, z_m \in Z$, $m \geq 1$, $z'_1, \dots, z'_n \in Z$, $n \geq 1$, $z_i = p_i s_i$, $i = 1, 2, \dots, m$, $z'_j = r_j q_j$, $j = 1, 2, \dots, n$, $r = r_1 r_2 \cdots r_n$ and $s = s_m \cdots s_2 s_1$. We will assume that $sx = yr$.

Lemma 3.1. *The following conditions are equivalent:*

(i) $|r| \leq |x|$.
(ii) $|s| \leq |y|$.
(iii) $x = tr$ *and* $y = st$ *for some* $t \in A^*$.

Proof. Obvious. □

In the following six lemmas, assume that $|x| < |r|$ (or, equivalently, $|y| < |s|$).

Lemma 3.2. $r = tx$ *and* $s = yt$ *for some* $t \in A^+$

Proof. Obvious. □

Lemma 3.3. *Assume that* $|s_m| \leq |y|$. *Then:*

(i) $m \geq 2$.
(ii) *There is uniquely determined k such that $1 \leq k < m$ and $|s_m \cdots s_{k+1}| \leq |y| < |s_m \cdots s_k|$.*
(iii) *There is uniquely determined l such that $1 \leq l \leq n$ and $|yr_1 \cdots r_{l-1}| < |s_m \cdots s_k| \leq |yr_1 \cdots r_l|$ (here, $yr_1 \cdots r_{l-1} = y$ for $l = 1$).*
(iv) $ps_{k-1} \cdots s_1 x = qr_l \cdots r_n$, *where* $p = s_m \cdots s_k$ *and* $q = yr_1 \cdots r_{l-1}$ ($p = s$ *and* $px = qr_l \cdots r_n$ *for* $k = 1$; $q = y$ *for* $l = 1$).
(v) $|q| < |p|$ *and* $p = qu$, $u \in A^+$.
(vi) $us_{k-1} \cdots s_1 x = r_l \cdots r_n$ ($ux = r_l \cdots r_n$ *for* $k = 1$).

Proof. We have $|s| = |s_m| + \cdots + |s_1| + |x| = |y| + |r_1| + \cdots + |r_n|$, $|s_m| \leq |y|$ and $|x| < |r_1| + \cdots + |r_n|$. Consequently, $|s_m| + |x| < |y| + |r_1| + \cdots + |r_n|$ and $m \geq 2$. The existence of the uniquely determined number k follows from the inequalities $|s_m| \leq |y|$ and $|y| < |s|$. If $|s_m \cdots s_k| \leq |yr_1|$, we put $l = 1$. If $|yr_1| < |s_m \cdots s_k|$, then the existence of the uniquely determined number l follows easily. The rest follows from the equality $s_m \cdots s_2 s_1 x = yr_1 r_2 \cdots r_n$. □

Lemma 3.4. *Assume that* $|s_m| \leq |y|$ *(see 3.3). Then:*

(i) $z_k = z'_l = s_k = r_l$ *and* $p_k = q_l = \varepsilon$.

(ii) *If* $k \geq 2$ *and* $l < n$, *then* $m \geq 3$, $n \geq 2$, $s_{k-1} \cdots s_1 x = r_{l+1} \cdots r_n$ *and* $s_m \cdots s_{k+1} = yr_1 \cdots r_{l-1}$ *(*$= y$ *for* $l = 1$*)*.

(iii) *If* $k \geq 2$ *and* $l = n$, *then* $m \geq 3$, $s = yr$, $s_{k-1} = \cdots = s_1 = x = \varepsilon$ *and* $s_m \cdots s_{k+1} = yr_1 \cdots r_{n-1}$ *(*$= y$ *for* $n = 1$*)*.

(iv) *If* $k = 1$ *and* $l < n$, *then* $n \geq 2$, $x = r_{l+1} \cdots r_n$, $s = yr_1 \cdots r_l$ *and* $s_m \cdots s_2 = yr_1 \cdots r_{l-1}$ *(*$= y$ *for* $l = 1$*)*.

(v) *If* $k = 1$ *and* $l = n$, *then* $s = yr$, $x = \varepsilon$ *and* $s_m \cdots s_2 = yr_1 \cdots r_{n-1}$ *(*$= y$ *for* $n = 1$*)*.

Proof. If $|r_l| < |u|$ then $|yr_1 \cdots r_l| = |q| + |r_l| < |q| + |u| = |p| = |s_m \cdots s_k|$, a contradiction. Thus $|u| \leq |r_l|$, $r_l = uu_1$, $s_{k-1} \cdots s_1 x = u_1 r_{l+1} \cdots r_n$, $z'_l = r_l q_l = uu_1 q_l$ and $s_m \cdots s_k = p = qu = yr_1 \cdots r_{l-1} u$.

If $|s_k| < |u|$ then $|y| + |u| \leq |q| + |u| = |p| = |s_m \cdots s_{k+1}| + |s_k| < |s_m \cdots s_{k+1}| + |u|$ and $|y| < |s_m \cdots s_{k+1}|$, a contradiction. Thus $|u| \leq |s_k|$, $s_k = u_2 u$, $s_m \cdots s_{k+1} u_2 = yr_1 \cdots r_{l-1}$ and $z_k = p_k s_k = p_k u_2 u$.

We have proved that $z_k = p_k s_k = p_k u_2 u$ and $z'_l = uu_1 q_l$. Since $u \neq \varepsilon$, it follows that $z_k = u = z'_l$, and $p_k = q_l = u_1 = u_2 = \varepsilon$. Then $s_k = z_k = z'_l = r_l = u$. By 3.3 (vi), $us_{k-1} \cdots s_1 x = r_l \cdots r_n$. Consequently, $s_{k-1} \cdots s_1 x = r_{l+1} \cdots r_n$ for $k \geq 2$ and $l < n$; $s_{k-1} = \cdots = s_1 = x = \varepsilon$ for $k \geq 2$, $l = n$; $x = r_{l+1} \cdots r_n$ for $k = 1$, $l < n$; $x = \varepsilon$ for $k = 1$, $l = n$.

If $k \geq 2$ and $l < n$, then $ps_{k-1} \cdots s_1 x = s_m \cdots s_1 x = yr_1 \cdots r_l$ implies $p = yr_1 \cdots r_l$. But $p = s_m \cdots s_k$ and $s_k = r_l$. Thus $s_m \cdots s_{k+1} = yr_1 \cdots r_{l-1}$ in this case. The rest is similar. □

Lemma 3.5. *Assume that* $|y| < |s_m|$. *Then:*

(i) *There is uniquely determined* l *such that* $1 \leq l \leq n$ *and* $|yr_1 \cdots r_{l-1}| < |s_m| \leq |yr_1 \cdots r_l|$ *(here,* $yr_1 \cdots r_{l-1} = y$ *for* $l = 1$*)*.

(ii) $ps_{m-1} \cdots s_1 x = qr_l \cdots r_n$, *where* $p = s_m$ *and* $q = yr_1 \cdots r_{l-1}$ *(*$p = s$ *and* $px = qr_l \cdots r_n$ *for* $m = 1$; $q = y$ *for* $l = 1$*)*.

(iii) $|q| < |p|$ *and* $p = qu$, $u \in A^+$.

(iv) $us_{m-1} \cdots s_1 x = r_l \cdots r_n$ *(*$ux = r_l \cdots r_n$ *for* $m = 1$*)*.

Proof. Similar to that of 3.3. □

Lemma 3.6. *Assume that* $|y| < |s_m|$ *(see 3.5). Then:*

(i) $z_m = z'_l = s_m = r_l$ *and* $p_m = q_l = \varepsilon$.

(ii) *If* $m \geq 2$ *and* $l < n$, *then* $n \geq 2$, $s_{m-1} \cdots s_1 x = r_{l+1} \cdots r_n$ *and* $y = r_1 = \cdots = r_{l-1} = \varepsilon$ *(*$y = \varepsilon$ *for* $l = 1$*)*.

(iii) *If* $m \geq 2$ *and* $l = n$, *then* $s_{m-1} = \cdots = s_1 = x = y = r_1 = \cdots = r_{n-1} = \varepsilon$ *(*$s_{m-1} = \cdots = s_1 = x = y = \varepsilon$ *for* $n = 1$*)*.

(iv) *If* $m = 1$ *and* $l < n$, *then* $n \geq 2$, $x = r_{l+1} \cdots r_n$ *and* $y = r_1 = \cdots = r_{l-1} = \varepsilon$ *(*$y = \varepsilon$ *for* $l = 1$*)*.

(v) *If* $m = 1$ *and* $l = n$, *then* $s = yr$ *and* $x = y = r_1 = \cdots = r_{n-1} = \varepsilon$ *(*$x = y = \varepsilon$ *for* $n = 1$*)*.

Proof. Similar to that of 3.4. □

Lemma 3.7. *There are uniquely determined k and l such that:*

(i) $1 \le k \le m$ *and* $1 \le l \le n$.
(ii) $z_k = z'_l = s_k = r_l$ *and* $p_k = q_l = \varepsilon$.
(iii) $|s_m \cdots s_{k+1}| \le |y| < |s_m \cdots s_k|$ *(*$s_m \cdots s_{k+1} = \varepsilon$ *for* $k = m$*).*
(iv) $|yr_1 \cdots r_{l-1}| < |s_m \cdots s_k| \le |yr_1 \cdots r_l|$ *(*$yr_1 \cdots r_{l-1} = y$ *for* $l = 1$*).*
(v) *If* $1 < k < m$ *and* $1 < l < n$*, then* $m \ge 3$*,* $n \ge 3$*,* $s_{k-1} \cdots s_1 x = r_{l+1} \cdots r_n$ *and* $s_m \cdots s_{k+1} = yr_1 \cdots r_{l-1}$.
(vi) *If* $1 < k < m$ *and* $1 < l = n$*, then* $m \ge 3$*,* $n \ge 2$*,* $s_{k-1} = \cdots = s_1 = x = \varepsilon$ *and* $s_m \cdots s_{k+1} = yr_1 \cdots r_{n-1}$.
(vii) *If* $1 < k < m$ *and* $1 = l < n$*, then* $m \ge 3$*,* $n \ge 2$*,* $s_{k-1} \cdots s_1 x = r_2 \cdots r_n$ *and* $s_m \cdots s_{k+1} = y$.
(viii) *If* $1 < k < m$ *and* $1 = n$ $(= l)$*, then* $m \ge 3$*,* $s_{k-1} = \cdots = s_1 = x = \varepsilon$ *and* $s_m \cdots s_{k+1} = y$.
(ix) *If* $1 < k = m$ *and* $1 < l < n$*, then* $m \ge 2$*,* $n \ge 3$*,* $s_{m-1} \cdots s_1 x = r_{l+1} \cdots r_n$ *and* $y = r_1 = \cdots = r_{l-1} = \varepsilon$.
(x) *If* $1 < k = m$ *and* $1 < l = n$*, then* $m \ge 2$*,* $n \ge 2$*,* $s_{m-1} = \cdots = s_1 = x = y = r_1 = \cdots = r_{n-1} = \varepsilon$.
(xi) *If* $1 < k = m$ *and* $1 = l < n$*, then* $m \ge 2$*,* $n \ge 2$*,* $s_{m-1} \cdots s_1 x = r_2 \cdots r_n$ *and* $y = \varepsilon$.
(xii) *If* $1 < k = m$ *and* $1 = n$ $(= l)$*, then* $m \ge 2$*,* $s_{m-1} = \cdots = s_1 = x = y = \varepsilon$.
(xiii) *If* $1 = k < m$ *and* $1 < l < n$*, then* $m \ge 2$*,* $n \ge 3$*,* $x = r_{l+1} \cdots r_n$ *and* $s_m \cdots s_2 = yr_1 \cdots r_{l-1}$.
(xiv) *If* $1 = k < m$ *and* $1 < l = n$*, then* $m \ge 2$*,* $n \ge 2$*,* $x = \varepsilon$ *and* $s_m \cdots s_2 = yr_1 \cdots r_{n-1}$.
(xv) *If* $1 = k < m$ *and* $1 = l < n$*, then* $m \ge 2$*,* $n \ge 2$*,* $x = r_2 \cdots r_n$ *and* $s_m \cdots s_2 = y$.
(xvi) *If* $1 = k < m$ *and* $1 = n$ $(= l)$*, then* $m \ge 2$*,* $x = \varepsilon$ *and* $s_m \cdots s_2 = y$.
(xvii) *If* $1 = m$ $(= k)$ *and* $1 < l < n$*, then* $n \ge 3$*,* $x = r_{l+1} \cdots r_n$ *and* $y = r_1 = \cdots = r_{l-1} = \varepsilon$.
(xviii) *If* $1 = m$ $(= k)$ *and* $1 < l = n$*, then* $n \ge 2$*,* $x = y = r_1 = \cdots = r_{n-1} = \varepsilon$.
(xix) *If* $1 = m$ $(= k)$ *and* $1 = l < n$*, then* $n \ge 2$*,* $x = r_2 \cdots r_n$ *and* $y = \varepsilon$.
(xx) *If* $1 = m$ $(= k)$ *and* $1 = n$ $(= l)$*, then* $x = y = \varepsilon$.

Proof. Combine 3.4 and 3.6. □

Proposition 3.8. $x = tr$ *and* $y = st$ *for some* $t \in A^*$ *(see 3.1), provided that at least one of the following six conditions holds:*

(1) $m = 1$ *and* $|z_1| \le |y|$*;*
(2) $n = 1$ *and* $|z'_1| \le |x|$*;*
(3) *All the words* $s_1, \ldots, s_m$ *are reduced;*
(4) *All the words* $r_1, \ldots, r_n$ *are reduced;*

(5) $z_i \neq z'_j$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$;
(6) $s_i \neq r_j$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$;

Proof. The result follows easily from 3.7. □

4. Technical results (c)

In this section, let $r, s, t \in A^*$ be reduced words such that $(rs, t) \in \tau$. We have $rs = r_0 z_0 s_0$, $z_0 \in Z$, r_0, s_0 reduced. By I.6.2, $r = r_0 p_0$, $s = q_0 s_0$ and $z_0 = p_0 q_0$, where $p_0, q_0 \in A^+$ are reduced (then $|z_0| \geq 2$).

Since $(rs, t) \in \tau$, there is a ρ-sequence $w_0, w_1, \ldots, w_m$, $m \geq 1$, such that $w_0 = rs$ and $w_m = t$. Clearly, $\mathrm{tr}(w_0) = 1$, $\mathrm{tr}(w_1) \geq 1$, $\ldots$, $\mathrm{tr}(w_{m-1}) \geq 1$ and $\mathrm{tr}(w_m) = 0$. Now, we will assume that $\mathrm{tr}(w_i) = 1$ for $i = 2, \ldots, m-1$ (cf. II.6 and III.4). Consequently, $w_i = r_i z_i s_i$, $z_i \in Z$, r_i, s_i reduced, $i = 0, 1, \ldots, m-1$.

Lemma 4.1.

(i) $rs = r\varepsilon s = w_0 = r_0 z_0 s_0$.
(ii) $r_i \psi(z_i) s_i = w_{i+1} = r_{i+1} z_{i+1} s_{i+1}$ *for every* i, $0 \leq i \leq m-2$.
(iii) $t = w_m = r_{m-1} \psi(z_{m-1}) s_{m-1}$.

Proof. Obvious. □

Lemma 4.2. *Let* $0 \leq i \leq m-2$. *Then just one of the following three cases takes place:*

(1) $r_i \psi(z_i)$ *is reduced,* $\psi(z_i) s_i$ *is not reduced,* $r_{i+1} = r_i p'_{i+1}$, $\psi(z_i) = p'_{i+1} p_{i+1}$, $s_i = q_{i+1} s_{i+1}$, $z_{i+1} = p_{i+1} q_{i+1}$, $r_i \psi(z_i) = r_i p'_{i+1} p_{i+1} = r_{i+1} p_{i+1}$ *and* $\psi(z_i) s_i = p'_{i+1} z_{i+1} s_{i+1}$, $p'_{i+1} \in A^*$ *and* $p_{i+1}, q_{i+1} \in A^+$ *(*p'_{i+1}, p_{i+1}, q_{i+1} *reduced);*
(2) $r_i \psi(z_i)$ *is not reduced,* $\psi(z_i) s_i$ *is reduced,* $r_i = r_{i+1} p_{i+1}$, $\psi(z_i) = q_{i+1} q'_{i+1}$, $s_{i+1} = q'_{i+1} s_i$, $z_{i+1} = p_{i+1} q_{i+1}$, $r_i \psi(z_i) = r_{i+1} z_{i+1} q'_{i+1}$ *and* $\psi(z_i) s_i = q_{i+1} q'_{i+1} s_i = q_{i+1} s_{i+1}$, $q'_{i+1} \in A^*$ *and* $p_{i+1}, q_{i+1} \in A^+$ *(*q'_{i+1}, p_{i+1}, q_{i+1} *reduced);*
(3) *Both* $r_i \psi(z_i)$ *and* $\psi(z_i) s_i$ *are reduced,* $r_i = r_{i+1} p_{i+1}$, $s_i = q_{i+1} s_{i+1}$ *and* $z_{i+1} = p_{i+1} \psi(z_i) q_{i+1}$.

Proof. The word $r_i \psi(z_i) s_i = r_{i+1} z_{i+1} s_{i+1}$ is meagre, and hence it follows from 2.2 that at least one of the words $r_i \psi(z_i)$ and $\psi(z_i) s_i$ is reduced. The rest is easy. □

Lemma 4.3. *Let* $0 \leq i \leq m-2$.

(i) *If 4.2(1) holds and* $|\psi(z_i)| \leq 1$, *then* $\psi(z_i) = p_{i+1} \in A$ *and* $p'_{i+1} = \varepsilon$.
(ii) *If 4.2(2) holds and* $|\psi(z_i)| \leq 1$, *then* $\psi(z_i) = q_{i+1} \in A$ *and* $q'_{i+1} = \varepsilon$.

Proof. Obvious. □

In the remaining part of this section, we will assume that $p'_{i+1} = \varepsilon$ ($q'_{i+1} = \varepsilon$, resp.) whenever $0 \le i \le m-2$ and 4.2(1) (4.2(2), resp.) is true.

If 4.2(1) is satisfied, then $\psi(z_i) = p_{i+1}$, $r_i = r_{i+1}$, $s_i = q_{i+1}s_{i+1}$, $z_{i+1} = \psi(z_i)q_{i+1}$ and we put $g_{i+1} = \varepsilon$ and $h_{i+1} = q_{i+1}$. Then $z_{i+1} = g_{i+1}\psi(z_i)h_{i+1}$, $r_i = r_{i+1}g_{i+1}$ and $s_i = h_{i+1}s_{i+1}$.

If 4.2(2) is satisfied, then $\psi(z_i) = q_{i+1}$, $r_i = r_{i+1}p_{i+1}$, $s_i = s_{i+1}$, $z_{i+1} = p_{i+1}\psi(z_i)$ and we put $g_{i+1} = p_{i+1}$ and $h_{i+1} = \varepsilon$. Again, $z_{i+1} = g_{i+1}\psi(z_i)h_{i+1}$, $r_i = r_{i+1}g_{i+1}$ and $s_i = h_{i+1}s_{i+1}$.

If 4.2(3) is satisfied, then $r_i = r_{i+1}p_{i+1}$, $s_i = q_{i+1}s_{i+1}$ and $z_{i+1} = p_{i+1}\psi(z_i)q_{i+1}$ and we put $g_{i+1} = p_{i+1}$ and $h_{i+1} = q_{i+1}$. As usual, $z_{i+1} = g_{i+1}\psi(z_i)h_{i+1}$, $r_i = r_{i+1}g_{i+1}$ and $s_i = h_{i+1}s_{i+1}$.

Furthermore, we put $g_0 = p_0$ and $h_0 = q_0$, so that $z_0 = g_0h_0 = g_0\varepsilon h_0$. Finally, we put $g_m = r_{m-1}$ and $h_m = s_{m-1}$, so that $t = g_m\psi(z_{m-1})h_m$.

Notice that all the words $g_0, \dots, g_m$ and $h_0, \dots, h_m$ are reduced.

The following three lemmas are easy.

Lemma 4.4.

(i) $z_0 = g_0h_0 = g_0\varepsilon h_0$, $r = r_0g_0$ *and* $s = h_0s_0$.
(ii) *If* $1 \le i \le m-1$, *then* $z_i = g_i\psi(z_{i-1})h_i$, $r_{i-1} = r_ig_i$ *and* $s_{i-1} = h_is_i$.
(iii) $t = g_m\psi(z_{m-1})h_m$.
(iv) *All the words* $g_0, \dots, g_m$ *and* $h_0, \dots, h_m$ *are reduced.*
(v) $r = g_m \cdots g_1g_0$ *and* $s = h_0h_1 \cdots h_m$.

Lemma 4.5. *Put* $r' = g_{m-1} \cdots g_1g_0$, $s' = h_0h_1 \cdots h_{m-1}$, $r'' = g_{m-1} \cdots g_1$, $s'' = h_1 \cdots h_{m-1}$ ($r'' = \varepsilon = s''$ *if* $m = 1$). *Then:*

(i) $r = g_mr'$ *and* $s = s'h_m$.
(ii) $rs = g_mr's'h_m$.
(iii) $r's' = r''z_0s''$.
(iv) $(r's', \psi(z_{m-1})) \in \tau$.
(v) $(rs', g_m\psi(z_{m-1})) \in \tau$.
(vi) $(r's, \psi(z_{m-1}h_m)) \in \tau$.

Lemma 4.6.

(i) *If* $t = r$, *then* $r = g_m\psi(z_{m-1})h_m$ *and* $(g_m\psi(z_{m-1})h_mh_0h_1 \cdots h_{m-1}, g_m\psi(z_{m-1})) = (rs', g_m\psi(z_{m-1})) \in \tau$.
(ii) *If* $t = s$, *then* $s = g_m\psi(z_{m-1})h_m$ *and* $(g_{m-1} \cdots g_1g_0g_m\psi(z_{m-1})h_m, \psi(z_{m-1})h_m) = (r's, \psi(z_{m-1})h_m) \in \tau$.

5. Technical results (d)

In this section, we will assume that $\psi(Z) \subseteq A \cup \{\varepsilon\}$.

Let $r, s, t, p, q \in A^*$ be reduced words such that $(rt, p) \in \tau$ and $(ts, q) \in \tau$. Then, of course, neither rt nor ts is reduced and $r, s, t \in A^+$.

Lemma 5.1. *There are* $m \ge 1$, $z_0, \dots, z_{m-1} \in Z$ *and reduced words* $g_0, \dots, g_m, h_0, \dots, h_m \in A^*$ *such that:*

(i) $z_0 = g_0 h_0$.
(ii) *If* $1 \le i \le m-1$, *then* $z_i = g_i \psi(z_{i-1}) h_i$.
(iii) $p = g_m \psi(z_{m-1}) h_m$.
(iv) $r = g_m \cdots g_1 g_0$.
(v) $t = h_0 h_1 \cdots h_m$.
(vi) $(r h_0 h_1 \cdots h_{m-1}, g_m \psi(z_{m-1})) \in \tau$.

Proof. Use 4.4 and 4.5(v). □

Lemma 5.2. *There are* $m' \ge 1$, $z'_0, \dots, z'_{m'-1} \in Z$ *and reduced words* $g'_0, \dots, g'_{m'}, h'_0, \dots, h'_{m'} \in A^*$ *such that:*

(i) $z'_0 = g'_0 h'_0$.
(ii) *If* $1 \le i \le m'-1$, *then* $z'_i = g'_i \psi(z'_{i-1}) h'_i$.
(iii) $q = g'_{m'} \psi(z'_{m'-1}) h'_{m'}$.
(iv) $s = h'_0 h'_1 \cdots h'_{m'}$.
(v) $t = g'_{m'} \cdots g'_1 g'_0$.
(vi) $(g'_{m'-1} \cdots g'_1 g'_0 s, \psi(z'_{m'-1}) h'_{m'}) \in \tau$.

Proof. Use 4.4 and 4.5(vi). □

Lemma 5.3.

(i) $h_0 h_1 \cdots h_m = t = g'_{m'} \cdots g'_1 g'_0$.
(ii) *There is* $f \in A^*$ *such that* $g'_{m'} = h_0 h_1 \cdots h_{m-1} f$ *and* $h_m = f g'_{m'-1} \cdots g'_1 g'_0$.

Proof.

(i) See 5.1(v) and 5.2(v).
(ii) Combine (i), 3.1 and 3.8.

□

Lemma 5.4. *Put* $t_1 = h_0 h_1 \cdots h_{m-1}$, $t_2 = f$ *and* $t_3 = g'_{m'-1} \cdots g'_1 g'_0$. *Then:*

(i) $t = t_1 t_2 t_3$.
(ii) $(r t_1, g_m \psi(z_{m-1})) \in \tau$.
(iii) $(t_3 s, \psi(z'_{m'-1}) h'_{m'}) \in \tau$.
(iv) $p = g_m \psi(z_{m-1}) t_2 t_3$.
(v) $q = t_1 t_2 \psi(z'_{m'-1}) h'_{m'}$.

Proof. Combine 5.1(iii), 5.2(iii) and 5.3. □

6. Technical results (e)

Assume that $\psi(Z) \subseteq A$ and ψ is strictly length decreasing (equivalently, $Z \cap A = \emptyset$). By III.6.5, for every $w \in A^*$ there exists a uniquely determined reduced word r such that $(w, r) \in \xi$.

Proposition 6.1. *Let* $r, s \in A^*$ *be reduced and let* $p, q \in A^*$ *be such that* $pq \ne \varepsilon$. *Then either* $(rpq, r) \notin \xi$ *or* $(qps, s) \notin \xi$.

Proof. Since $pq \neq \varepsilon$, we have $rpq \neq r$ and $qps \neq s$. Now, proceeding by contradiction, assume that $(rpq, r) \in \tau$, $(qps, s) \in \tau$ and $|rs|$ is minimal. Of course (III.6.4, III.6.5), we can assume that both p and q are reduced. The rest of the proof is divided into five parts:

(i) Let $q = \varepsilon$. Then $p \neq \varepsilon$, $(rp, r) \in \tau$ and $(ps, s) \in \tau$. According to 5.4, $p = p_1p_2p_3$, $(r, u) \in \tau$, $(p_3s, v) \in \tau$, $r = up_2p_3$, $s = p_1p_2v$, u, v reduced. We get $(up_2p_3p_1, u) \in \tau$, $(p_3p_1p_2v, v) \in \tau$ and, if $(p_3p_1, p_4) \in \xi$, where p_4 is reduced, then $(up_2p_4, u) \in \xi$, $(p_4p_2v, v) \in \xi$. If $p_2 = \varepsilon = p_4$, then $p_3p_1 \neq \varepsilon$ (since $p \neq \varepsilon$) and $p_4 \neq \varepsilon$ (since $\varepsilon \notin \psi(Z)$), a contradiction. Thus $p_2p_4 \neq \varepsilon$ and $(up_2p_4, u) \in \tau$, $(p_4p_2v, v) \in \tau$. But $|u| + |v| < |r| + |s|$, a contradiction with the minimality of $|rs|$.

(ii) Let $q = \varepsilon$. This case is analogous to (i).

(iii) Let $p \neq \varepsilon \neq q$ and $r = r'q$, where $(rp, r') \in \xi$ and r' is reduced. Furthermore, let $(qp, t) \in \xi$, where t is reduced. Then $(r'qp, r') = (rp, r') \in \xi$, $(r'qp, r't) \in \xi$ (since $(qp, t) \in \xi$), and hence $(r't, r') \in \xi$. Similarly, $(qps, ts) \in \xi$ (since $(qp, t) \in \xi$), and hence $(ts, s) \in \xi$ (since $(qps, s) \in \tau$). Since $qp \neq \varepsilon$, we have $t \neq \varepsilon$ and $(r't, r') \in \tau$, $(ts, s) \in \tau$. But this is a contradiction since $|r'| + |s| < |r| + |s|$.

(iv) Let $p \neq \varepsilon \neq q$ and $s = qs'$, where $(ps, s') \in \xi$ and s' is reduced. This case is analogous to (iii).

(v) Let $p \neq \varepsilon \neq q$ and $r'q \neq r$, $qs' \neq s$, where r', s' are reduced and such that $(rp, r') \in \xi$ and $(ps, s') \in \xi$. We have $(r'q, r) \in \tau$ and $(qs', s) \in \tau$. According to 5.4, $q = q_1q_2q_3$, $(r'q_1, u) \in \tau$, $(q_3s', v) \in \tau$, $r = uq_2q_3$ and $s = q_1q_2v$, u, v reduced. Now, $(rp, r') \in \xi$ implies $(uq_2q_3pq_1, r'q_1) = (rpq_1, r'q_1) \in \xi$, and hence $(uq_2q_3pq_1, u) \in \tau$. Quite similarly, $(q_3pq_1q_2v, v) \in \tau$. Finally, if $(q_3pq_1, t) \in \xi$, where t is reduced, then $(uq_2t, u) \in \xi$ and $(tq_2v, v) \in \xi$. Of course, $t \neq \varepsilon$, $(uq_2t, u) \in \tau$, $(tq_2v, v) \in \tau$ and $|u| + |v| < |r| + |s|$ (since $q \neq \varepsilon$), a contradiction. □

7. Main result

Assume that $\psi(Z) \subseteq A$ and ψ is strictly length decreasing.

Theorem 7.1. *Let $z_1, z_2 \in Z$ be such that $z_1 \neq z_2$ and $\psi(z_1) = a = \psi(z_2)$ ($a \in A$). Furthermore, let $r, s \in A^*$ and $w \in A^*$. Then either $(w, rz_1s) \notin \xi$ or $(w, rz_2s) \notin \xi$ (of course, $(rz_1s, ras) \in \rho$ and $(rz_2s, ras) \in \rho$).*

Proof. We can assume without loss of generality that both r and s are reduced. If $(w, rz_1s) \in \xi$ and $(w, rz_2s) \in \xi$, then $P(rz_1s, rz_2s) \neq \emptyset$ (see IV.5) and we can assume that $w \in Q(rz_1s, rz_2s)$ (use IV.5.3). According to IV.6.1, either $w = rz_1xz_2s$, $(rz_1x, r) \in \tau$, $(xz_2s, s) \in \tau$, x reduced or $w = rz_2xz_1s$, $(rz_2x, r) \in \tau$, $(xz_1s, s) \in \tau$, x reduced. In both cases, $(rax, r) \in \xi$ and $(xas, s) \in \xi$, a contradiction with 6.1. □

8. Examples

Example 8.1. Let $z_1 = a^2b^2$, $z_2 = a^2bab^2$, $r_1 = \varepsilon$, $r_2 = b^2$, $s_1 = a$, $s_2 = \varepsilon$, $r = a$, $s = bab^2$ and $t = b^2a$. Then all the words r_1, r_2, s_1, s_2, r, s, t are reduced and $rat = a^2b^2a = r_1z_1s_1$ and $tas = b^2a^2bab^2 = r_2z_2s_2$. Furthermore, $(rat, \psi(z_1)a) \in \rho$ and $(tas, b^2\psi(z_2)) \in \rho$.

If $\psi(z_1) = \varepsilon$, then $(rat, a) \in \rho$. If $\psi(z_1) = b^2$, then $(rat, t) \in \rho$. If $\psi(z_2) = a$, then $(tas, t) \in \rho$.

Notice also that $sat = bab^2ab^2a$ and $tar = b^2a^3$ are reduced.

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