

STRONG SUBMODULES OF ALMOST PROJECTIVE MODULES

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ABSTRACT. The structure of almost projective modules can be better understood in case the following Condition (P) holds: ‘The union of each countable pure chain of projective modules is projective.’ We prove this condition, and its generalization to pure-projective modules, for all countable rings using the new notion of a strong submodule of the union.

However, we also show that Condition (P) fails for all Prüfer domains of finite character with uncountable spectrum; in particular, for the polynomial ring $K[x]$ where K is an uncountable field. Moreover, one can even prescribe the Γ -invariant of the union. Our results generalize earlier work of Hill, and complement recent papers by Macías-Díaz, Fuchs, and Rangaswamy.

By a classic theorem of Kaplansky, the structure theory of projective modules over an arbitrary ring reduces to the countably generated ones. In stark contrast, *almost projective modules* (modules possessing a rich supply of small projective submodules) have a very complex structure in general. Perhaps the most successful invariant measuring their complexity is the Γ -invariant. Projective modules are recognized by having trivial Γ -invariant, [2, 3].

There are additional conditions on almost projective modules that guarantee projectivity. In his work on Whitehead groups, Hill discovered a remarkable condition in the particular case of abelian groups: if A is the union of a countable pure chain of (arbitrarily large) projective groups, then A is projective [8]. In the present paper, we call the analogous property for modules over an arbitrary ring *Condition (P)*.

In the past decade, several authors attempted to extend Hill’s result and establish Condition (P) for large classes of rings, notably for commutative domains and noetherian rings, [4, 5]. So far Macías-Díaz [9] obtained the strongest result, namely, Prüfer domains with countable spectrum have Condition (P).

Section 1 of our paper adds more motivation for considering Condition (P) by showing its role in relating various notions of almost projectivity appearing in the literature. In Section 2 we prove Condition (P), and some of its generalizations, for all countable rings using the new notion of a strong submodule.

However, in Section 3 we show that Condition (P) fails completely for all Prüfer domains of finite character with uncountable spectrum (thus, for example, for the polynomial ring $K[x]$ where K is any uncountable field). The term ‘completely’ refers to the fact that there are essentially no restrictions on the Γ -invariant of A .

In what follows, R will denote a ring (that is, an associative ring with 1), and the term module will mean a right R -module.

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1. ALMOST PROJECTIVE MODULES

The following definition is the analogue of [3, IV.1.1] for general rings with ‘free’ replaced by ‘projective’.

Definition 1.1. Let R be a ring and κ a regular uncountable cardinal. A module M is called κ -projective provided there exists a set \mathcal{S} consisting of $< \kappa$ -generated projective submodules of M such that

- (1) each subset of M of cardinality $< \kappa$ is contained in an element of \mathcal{S} , and
- (2) \mathcal{S} is closed under unions of well-ordered chains of length $< \kappa$.

There are other notions relevant for the study of almost projectivity (see [3, IV.1], [14], et al.). We now recall three of them:

Definition 1.2. Let R be a ring and κ a regular uncountable cardinal. A module M is called *weakly κ -projective* provided that each subset of M of cardinality $< \kappa$ is contained in a pure submodule N of M which is $< \kappa$ -generated and projective.

Recall that a module M is *flat* provided that the functor $M \otimes_R -$ is exact, and M is *Mittag-Leffler* if the canonical map

$$M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} (M \otimes_R Q_i)$$

is monic for each family of left R -modules $(Q_i \mid i \in I)$.

Our starting point is the following result from [10] and [7]:

Lemma 1.3. *Let R be a ring and M a module. Then the following conditions are equivalent:*

- (i) M is \aleph_1 -projective.
- (ii) M is weakly \aleph_1 -projective.
- (iii) Each finite subset of M is contained in a projective countably generated and pure submodule of M .
- (iv) M is flat Mittag-Leffler.

Moreover, if κ is a regular uncountable cardinal and M is κ -projective, then M is \aleph_1 -projective.

Proof. The equivalence of (i), (ii) and (iii) is proved in [10] (see also [1]), while (i) and (iv) are equivalent by [7, Theorem 2.9(i)] (see also [11] and [12]). The moreover part is [7, Theorem 2.9(ii)]. \square

The implication (i) \Rightarrow (ii) extends to arbitrary regular uncountable cardinals κ :

Lemma 1.4. *Let R be a ring, M be a module, and κ be an infinite cardinal.*

- (i) *Assume that M is \aleph_1 -projective. Then each subset of M of cardinality $\leq \kappa$ is contained in a $\leq \kappa$ -generated pure submodule of M .*
- (ii) *Assume that κ is regular uncountable and M is κ -projective. Then M is weakly κ -projective.*

Proof. (i) We will prove the claim by induction on κ . The case of $\kappa = \aleph_0$ follows by Lemma 1.3.

Assume $\kappa \geq \aleph_1$ and let $X = \{x_\alpha \mid \alpha < \kappa\}$ be a subset of M of cardinality κ . For each $\alpha < \kappa$, let $X_\alpha = \{x_\beta \mid \beta < \alpha\}$. By induction on α , we define an increasing chain $(P_\alpha \mid \alpha < \kappa)$ of $< \kappa$ -generated pure submodules of M as follows: $P_0 = 0$, $P_{\alpha+1}$ is a $< \kappa$ -generated pure submodule of M containing $X_\alpha \cup P_\alpha$ (which exists by the inductive premise), and $P_\alpha = \bigcup_{\beta < \alpha} P_\beta$ when $\alpha < \kappa$ is a limit ordinal. Then $P = \bigcup_{\alpha < \kappa} P_\alpha$ is a $\leq \kappa$ -generated pure submodule of M containing X .

(ii) Let \mathcal{S} be as in Definition 1.1, and X be a subset of M of cardinality $< \kappa$. By condition (1) of Definition 1.1, X is contained in a $< \kappa$ -generated projective

submodule $P_0 \in \mathcal{S}$. By the moreover part of Lemma 1.3 and by part (i), P_0 is contained in a $< \kappa$ -generated pure submodule Q_0 of M . Proceeding similarly, we obtain a countable chain

$$P_0 \subseteq Q_0 \subseteq P_1 \subseteq Q_1 \subseteq \cdots \subseteq P_n \subseteq Q_n \subseteq \cdots$$

where $P_n \in \mathcal{S}$, so P_n is $< \kappa$ -generated and projective, and Q_n is $< \kappa$ -generated and pure in M , for all $n < \omega$. Let $P = \bigcup_{n < \omega} P_n = \bigcup_{n < \omega} Q_n$. Then $P \in \mathcal{S}$ by condition (2) of Definition 1.1, and P is pure in M . \square

Note that whatever the cardinality of the ring R , Lemma 1.4(i) makes it possible to purify a submodule without increasing the number of generators. So in the particular case when R is a right hereditary ring, κ -projectivity and weak κ -projectivity are equivalent (to the property that each $< \kappa$ -generated submodule is projective). However, the converse of Lemma 1.4(ii) fails in general:

Example 1.5. Let $\kappa > \aleph_1$ be a regular cardinal, K be a field, and let R denote the endomorphism ring of a κ -dimensional K -linear space modulo its maximal ideal. Then there exists a κ -generated right ideal I in R such that I is weakly κ -projective, but not κ -projective (see [14, Theorem 8]).

Another relevant property is the following (where a chain $(P_n \mid n < \omega)$ is a *pure chain* in case P_n is a pure submodule of P_{n+1} for each $n < \omega$):

Definition 1.6. Let R be a ring. Then R satisfies Condition (P) provided that for each pure chain $(P_n \mid n < \omega)$ consisting of projective modules, the module $P = \bigcup_{n < \omega} P_n$ is projective.

Condition (P) yields a characterization of the weak κ -projectivity:

Proposition 1.7. *Let R be a ring satisfying Condition (P). Let M be a module, and κ be a regular infinite cardinal. Then M is weakly κ -projective, if and only if there exists a set \mathcal{S} consisting of $< \kappa$ -generated projective submodules of M such that*

- (1) *each subset of M of cardinality $< \kappa$ is contained in an element of \mathcal{S} , and*
- (2) *\mathcal{S} is closed under unions of countable chains.*

Proof. In view of [7, Corollary 2.3], assumptions (1) and (2) assure \aleph_1 -projectivity of M , so the if-part is proved as in Lemma 1.4. For the only if part, let \mathcal{S} be the set of all $< \kappa$ -generated projective and pure submodules of M . Then (1) holds by the assumption. If $M_0 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$ is a countable chain of elements of \mathcal{S} , then $M_\omega = \bigcup_{n < \omega} M_n$ is projective by Condition (P), so $M_\omega \in \mathcal{S}$. \square

Condition (P) holds for $R = \mathbb{Z}$. This was shown by Hill [8], who proved thus the singular compactness for almost free abelian groups of cardinality \aleph_α where α has cofinality ω .

More generally, Condition (P) is known to hold for all Prüfer domains with countably many maximal ideals [9, Corollary 15], hence for all valuation domains. In Theorem 2.5 below, we will prove it for all countable rings.

However, attempts to prove Condition (P) for arbitrary domains in [5, XVI.1.4] and [4, Theorem 1.3] have gaps; in fact, as we will see in Theorem 3.1, Condition (P) fails even for $R = K[x]$ where K is an uncountable field.

The main goal of the next section will be to prove Condition (P), and hence the equivalence in Proposition 1.7, for all countable rings. Before proceeding to that point, let us note that under additional assumptions on R and M , the equivalence holds even without assuming Condition (P):

Proposition 1.8. *Let κ be an infinite cardinal, R be a ring, which is either a domain or is right $< \kappa$ -noetherian (i.e., every right ideal is $< \kappa$ -generated). Let M be a module of projective dimension ≤ 1 . Then M is weakly κ -projective, if and only if there exists a set \mathcal{S} consisting of $< \kappa$ -generated projective submodules of M such that*

- (1) *each subset of M of cardinality $< \kappa$ is contained in an element of \mathcal{S} , and*
- (2) *\mathcal{S} is closed under unions of countable chains.*

The following lemma will help in finding projective submodules:

Lemma 1.9. *Let M be a module of projective dimension at most 1. Let N be a tight submodule, i.e., M/N has also projective dimension at most 1. If N is contained in a projective submodule of M , then N is projective.*

Proof. Let P be a projective module such that $N \subseteq P \subseteq M$. We can estimate the projective dimensions of various modules built from N , P and M using the long exact sequence for Ext as follows:

- (1) $\text{proj.dim } M/P \leq \max\{\text{proj.dim } M, \text{proj.dim } P + 1\} \leq 2,$
- (2) $\text{proj.dim } P/N \leq \max\{\text{proj.dim } M/N, \text{proj.dim } M/P - 1\} \leq 1,$
- (3) $\text{proj.dim } N \leq \max\{\text{proj.dim } P, \text{proj.dim } P/N - 1\} \leq 0.$

The last line shows that N is projective. □

Proof of Proposition 1.8. As in the proof of Proposition 1.7, the conditions (1) and (2) of Proposition 1.8 imply that M is κ -projective (because neither Condition (P) nor any of our additional assumptions are needed there).

For the other direction, we first note that by the assumptions on R , there is a Hill family consisting of tight submodules of M : in the case when R is a domain, this follows by [5, Proposition VI.5.1] and [6, 4.2.6]; when R is $< \kappa$ -noetherian, we apply [6, 4.1.11 and 4.2.6].

Let \mathcal{S} be the subfamily of the $< \kappa$ -generated members of this family. Conditions (1) and (2) automatically hold. Finally, the assumption of weak κ -projectivity and Lemma 1.9 imply that \mathcal{S} consist of projective modules. □

2. HILL FAMILIES OF STRONG SUBMODULES

We start this section by considering a general version of Condition (P), where the chain $(P_n \mid n < \omega)$ is not necessarily pure, and the modules P_n ($n < \omega$) are direct sums of modules from a given class \mathcal{C} consisting of countably presented modules, or modules of countable rank. The relevant notion here is that of a strong submodule. It is introduced in the following definition where, for a class of modules \mathcal{C} , we denote by $\text{Sum}(\mathcal{C})$ the class of all direct sums of copies of modules from \mathcal{C} .

Definition 2.1. Let R be a ring and \mathcal{C} be a class of modules.

Let $(P_n \mid n < \omega)$ be a countable increasing chain of modules, and $P = \bigcup_{n < \omega} P_n$. Assume that $P_n \in \text{Sum}(\mathcal{C})$ for each $n < \omega$, that is, there exists a decomposition $P_n = \bigoplus_{\alpha < \kappa_n} P_{n,\alpha}$ where $P_{n,\alpha}$ is isomorphic to an element of \mathcal{C} for each $\alpha < \kappa_n$.

We will fix these decompositions, and for each $n < \omega$ and each subset $S \subseteq \kappa_n$, define $P(n, S) = \bigoplus_{\alpha \in S} P_{n,\alpha}$. So in particular, $P_n = P(n, \kappa_n)$.

A submodule N of P is called *strong* provided there exist $(A_n \mid n < \omega)$ such that $A_n \subseteq \kappa_n$ and $N \cap P_n = P(n, A_n)$ for each $n < \omega$. The sequence $(A_n \mid n < \omega)$ is then uniquely determined by N ; it is the *witnessing sequence* for N .

In this section, P will denote the union $\bigcup_{n < \omega} P_n$ where $(P_n \mid n < \omega)$ is a countable increasing chain of modules as in Definition 2.1.

In the case when \mathcal{C} is the class of all countably presented projective modules, Definition 2.1 covers the setting of Condition (P), because by a classic theorem of Kaplansky, each projective module is a direct sum of modules in \mathcal{C} .

Note that 0 and P are strong submodules of P . Also, unions of chains of strong submodules are strong, and so are arbitrary intersections of strong submodules. Indeed, in Theorem 2.9, we will prove that strong submodules are abundant.

If N is strong in P and the chain $(P_n \mid n < \omega)$ is pure, then N is a pure submodule of P because $N = \bigcup_{n < \omega} N \cap P_n$ and $N \cap P_n$ is a direct summand in the pure submodule P_n of P for each $n < \omega$.

For the next lemma, we recall that a ring R is right \aleph_0 -noetherian provided that each right ideal of R is countably generated. For example, all right noetherian rings, and all countable rings, are right \aleph_0 -noetherian. It is easy to see that a ring R is right \aleph_0 -noetherian, if and only if each submodule of a countably generated module is countably generated.

Lemma 2.2. *Assume that R is right \aleph_0 -noetherian and \mathcal{C} consists of countably presented modules or R is a commutative domain and \mathcal{C} consists of torsion-free modules of countable rank, respectively. Let N be a strong submodule of P with witnessing sequence $(A_n \mid n < \omega)$. Let C be a countable subset of P or a subset of P such that $\langle C \rangle$ has countable rank, respectively.*

Then there is a strong submodule N' of P such that $N \cup C \subseteq N'$, and the witnessing sequence $(A'_n \mid n < \omega)$ for N' satisfies $A_n \subseteq A'_n$ and $A'_n \setminus A_n$ is countable for each $n < \omega$.

Proof. We will simultaneously and recursively construct chains $(C_{n,i} : i < \omega)$ of subsets of κ_n .

As a start, for each $n < \omega$, let $A_n \subseteq C_{n,0} \subseteq \kappa_n$ with $C_{n,0} \setminus A_n$ countable and $C \cap P_n \subseteq P(n, C_{n,0})$.

For $i \geq 0$, let $C_{n,i} \subseteq C_{n,i+1} \subseteq \kappa_n$ with $C_{n,i+1} \setminus C_{n,i}$ countable and $P(m, C_{m,i}) \cap P_n \subseteq P(n, C_{n,i+1})$ for all m .

Finally, we define $A'_n = \bigcup_{i < \omega} C_{n,i}$ for each $n < \omega$. Then $A_n \subseteq A'_n \subseteq \kappa_n$, and $A'_n \setminus A_n$ is countable for each $n < \omega$. Let $N' = \bigcup_{n < \omega} P(n, A'_n) = \bigcup_{n,i < \omega} P(n, C_{n,i})$. Note that the $P(n, C_{n,i})$ form an upper directed system of submodules, so their union is a submodule.

Recall that $P(m, C_{m,i}) \cap P_n \subseteq P(n+1, C_{n+1,i})$ for all $m, n, i < \omega$, hence $N' \cap P_n = P(n, A'_n)$. All in all, N' is a strong submodule of P with witnessing sequence $(A'_n \mid n < \omega)$.

Since $C \cap P_n \subseteq P(n, A'_n)$ for each $n < \omega$, we conclude that $N \cup C \subseteq N'$. \square

Lemma 2.2 serves as inductive step for proving

Proposition 2.3. *Assume that either R is right \aleph_0 -noetherian and \mathcal{C} consists of countably presented modules or R is a commutative domain and \mathcal{C} consists of torsion-free modules of countable rank, respectively.*

Then P is the union of a continuous increasing chain $\mathcal{M} = (M_\alpha \mid \alpha < \lambda)$ of strong submodules of P such that for each $\alpha < \lambda$ there is a countably generated or countable rank submodule N_α of P , respectively, with $M_{\alpha+1} = M_\alpha + N_\alpha$.

Proof. Let $\{p_\alpha \mid \alpha < \lambda\}$ be an R -generating subset of P . Since $M_0 = 0$ is strong, and the union of a chain of strong submodules is strong, we are left to perform the non-limit step of the construction. However, applying Lemma 2.2 for $N = M_\alpha$ and $C = \{p_\alpha\}$, we can take $N_\alpha = \sum_{n < \omega} P(n, A'_n \setminus A_n)$ and $M_{\alpha+1} = N'$. \square

We can prove more in the particular case of countable rings. We will say that a class of modules \mathcal{C} has Property (C) provided that for each increasing pure chain of

modules, $(Q_n \mid n < \omega)$, such that $Q_n \in \text{Sum}(\mathcal{C})$ for all $n < \omega$, and each countably presented pure submodule C of $\bigcup_{n < \omega} Q_n$, the module C is \mathcal{C} -filtered. Moreover, \mathcal{C} has Property (C+) if the same assumptions yield the stronger conclusion of $C \in \text{Sum}(\mathcal{C})$.

For example, the class of all countably presented modules and the class of all projective modules have Property (C+), because the union of a pure chain of projective modules is always \aleph_1 -projective (see Lemma 1.3(iii)).

Lemma 2.4. *Let R be a countable ring. Let \mathcal{C} be a class of countably presented modules which has Property (C). Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules such that $P_n \in \text{Sum}(\mathcal{C})$ for all $n < \omega$, and let $P = \bigcup_{n < \omega} P_n$. Then P is \mathcal{C} -filtered.*

Moreover, if \mathcal{C} has Property (C+), then P is the union of a continuous increasing chain $\mathcal{M} = (M_\alpha \mid \alpha < \lambda)$ consisting of strong submodules of P such that $M_{\alpha+1}/M_\alpha \in \text{Sum}(\mathcal{C})$.

Proof. Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules with $P_n \in \text{Sum}(\mathcal{C})$ for all $n < \omega$. Since R is countable, the continuous chain \mathcal{M} from Proposition 2.3 can be taken with the additional property of $M_\alpha + P_n$ being pure in P for all $n < \omega$ and $\alpha < \kappa$. This is arranged by improving Lemma 2.2 for countable R : when for the strong submodule N , all the submodules $N + P_n$ are pure, then N' can be chosen with the $N' + P_n$ also pure.

It follows that for each $\alpha < \kappa$, the factor $Q = P/M_\alpha$ is the union of the pure chain $(Q_n \mid n < \omega)$ where $Q_n = (M_\alpha + P_n)/M_\alpha$. Moreover, $Q_n \cong P_n/(P_n \cap M_\alpha) \in \text{Sum}(\mathcal{C})$ because M_α is strong. Similarly, the countably presented submodule $C = M_{\alpha+1}/M_\alpha$ is pure in Q , so C is \mathcal{C} -filtered by Property (C). Then $P = \bigcup_{\alpha < \kappa} M_\alpha$ is \mathcal{C} -filtered as well.

Moreover, if \mathcal{C} has Property (C+), then $C = M_{\alpha+1}/M_\alpha \in \text{Sum}(\mathcal{C})$. \square

The assumptions of Lemma 2.4 are satisfied for R countable and \mathcal{C} the class of all countably generated projective modules. Since in this case \mathcal{C} -filtered is the same as projective, we get

Theorem 2.5. *Let R be a countable ring. Then R satisfies Condition (P).*

As another consequence, we obtain the general version of Condition (P) for the case when R is countable, \mathcal{C} has Property (C+), and \mathcal{C} consists of finitely presented modules:

Corollary 2.6. *Let R be a countable ring, and \mathcal{C} be a class of finitely presented modules which has Property (C+). Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules such that $P_n \in \text{Sum}(\mathcal{C})$ for all $n < \omega$, and $P = \bigcup_{n < \omega} P_n$. Then $P \in \text{Sum}(\mathcal{C})$.*

Proof. By Lemma 2.4, P is the union of a continuous increasing chain $\mathcal{M} = (M_\alpha \mid \alpha < \lambda)$ consisting of strong submodules of P such that $M_{\alpha+1}/M_\alpha \in \text{Sum}(\mathcal{C})$. In particular, M_α is pure in $M_{\alpha+1}$ for each $n < \omega$. As \mathcal{C} consists of finitely presented modules, $M_{\alpha+1}/M_\alpha$ is pure-projective, and the embedding $M_\alpha \hookrightarrow M_{\alpha+1}$ splits. This proves that $P \in \text{Sum}(\mathcal{C})$. \square

A variation of Corollary 2.6 gives the version of Condition (P) for pure-projective modules over countable rings.

Theorem 2.7. *Let R be a countable ring, $(P_n \mid n < \omega)$ be an increasing pure chain of pure-projective modules, and $P = \bigcup_{n < \omega} P_n$. Then P is pure-projective.*

Proof. By [10, Seconde partie, Corollaire 2.2.2], a countably presented module is pure-projective, if and only if it is Mittag-Leffler, and the latter property is clearly

inherited by pure submodules. As in Lemma 2.4, we infer that P is the union of a continuous chain, \mathcal{M} , consisting of strong submodules of P such that all consecutive factors in \mathcal{M} are pure-projective, hence P is pure-projective as well. \square

Alternatively, we can deduce Theorem 2.5 from Theorem 2.7, because projective = flat + pure-projective.

Of course, the union of a non-pure countable chain of projective modules need not be projective even for countable rings: just consider $R = \mathbb{Z}$ and \mathbb{Q} as the union of the chain of free groups $(1/n! \cdot \mathbb{Z} \mid n < \omega)$.

Moreover, the general version of Condition (P) for pure chains consisting of modules from $\text{Sum}(\mathcal{C})$ may fail even for countable rings and \mathcal{C} having Property (C). That is, even though P is \mathcal{C} -filtered by Lemma 2.4, $P \notin \text{Sum}(\mathcal{C})$ in general:

Example 2.8. Let R be a simple countable von Neumann regular ring which is not artinian — for example, let R be the directed union of the full matrix rings $M_{2^n}(\mathbb{Q})$ ($n < \omega$) with the block diagonal embeddings

$$\mathbb{Q} \subseteq M_2(\mathbb{Q}) \subseteq M_4(\mathbb{Q}) \subseteq \cdots \subseteq M_{2^n}(\mathbb{Q}) \subseteq M_{2^{n+1}}(\mathbb{Q}) \subseteq \cdots .$$

Consider a simple non-projective module S , and let \mathcal{C} be the class of all finitely $\{S\}$ -filtered modules. Then \mathcal{C} is a class of countable modules, and it has Property (C).

Define a chain of finite length modules $(P_n \mid n < \omega)$ so that $P_0 = S$, and P_{n+1} fits in a non-split short exact sequence $0 \rightarrow P_n \subseteq P_{n+1} \rightarrow S \rightarrow 0$ for each $n < \omega$. This is possible by [15, Proposition 3.3]. This chain is pure because R is von Neumann regular, so all R -modules are flat.

Let $P = \bigcup_{n < \omega} P_n$. Then $P_n \in \mathcal{C}$ for all $n < \omega$, and P is \mathcal{C} -filtered, but $P \notin \text{Sum}(\mathcal{C})$. Indeed, $S = P_0$ is an essential submodule of P , so P is uniform, hence indecomposable.

Returning to the general setting and using an idea by Hill, we can extend the chain \mathcal{M} from Lemma 2.3 further, to a large family of strong submodules:

Theorem 2.9. *Assume that R is right \aleph_0 -noetherian and \mathcal{C} consists of countably presented modules or R is a commutative domain and \mathcal{C} consists of torsion-free modules of countable rank, respectively. Let \mathcal{M} be a chain of strong submodules of P with countably generated or countable rank factors, respectively.*

Then there is a family \mathcal{H} of strong submodules of P such that

- (H1) $\mathcal{M} \subseteq \mathcal{H}$.
- (H2) \mathcal{H} is closed under arbitrary sums and intersections; in fact, \mathcal{H} is a complete distributive sublattice of the modular lattice of all submodules of P .
- (H3) Let $N, N' \in \mathcal{H}$ be such that $N \subseteq N'$. Then there exists a continuous increasing chain $(N_\beta \mid \beta \leq \tau)$ consisting of elements of \mathcal{H} such that $\tau \leq \lambda$, $N_0 = N$, $N_\tau = N'$, and for each $\beta < \tau$ there is $\alpha < \kappa$ such that $N_{\beta+1}/N_\beta$ isomorphic to $M_{\alpha+1}/M_\alpha$.
- (H4) Let $N \in \mathcal{H}$ and X be a countable subset of P (a subset of P such that $\langle X \rangle$ has countable rank, respectively). Then there are $N' \in \mathcal{H}$ and a submodule $Y \subseteq P$ such that Y is countably generated (of countable rank, respectively) and $N \cup X \subseteq N' = N + Y$.

Proof. First, for all $\alpha < \lambda$ and $n < \omega$, let $D_{\alpha,n} = A_{\alpha+1,n} \setminus A_{\alpha,n}$ where $(A_{\alpha,n} \mid n < \omega)$ and $(A_{\alpha+1,n} \mid n < \omega)$ are the witnessing sequences for M_α and $M_{\alpha+1}$, respectively. By the construction of the chain \mathcal{M} , all the sets $D_{\alpha,n}$ are countable. Let $\Delta_\alpha = \sum_{n < \omega} P(n, D_{\alpha,n})$. Then Δ_α is countably generated, and $M_{\alpha+1} = M_\alpha + \Delta_\alpha$ for each $\alpha < \lambda$.

As in [6, §4.2], we call a subset S of σ *closed* in case $\Delta_\alpha \cap M_\alpha \subseteq \sum_{\beta < \alpha, \beta \in S} \Delta_\beta$ for each $\alpha \in S$. We define $\mathcal{H} = \{\sum_{\alpha \in S} \Delta_\alpha \mid S \text{ is closed in } \lambda\}$.

Since each ordinal $\sigma \leq \lambda$ is closed, $\mathcal{M} \subseteq \mathcal{H}$, and (H1) holds. Properties (H2) and (H3) are proved in [6, 4.2.6]. If R is \aleph_0 -noetherian, then (H4) is proved in [6, 4.2.6] while in the domain case, (H4) follows by [6, 4.2.8].

It remains to show that all modules in \mathcal{H} are strong. Let S be a closed subset of λ , $N = \sum_{\alpha \in S} \Delta_\alpha$, and $B_n = \bigcup_{\alpha \in S} D_{\alpha, n}$. It suffices to prove that $N \cap P_n = P(n, B_n)$ for each $n < \omega$. The inclusion \supseteq is clear from the definitions above.

Assume there exists $x \in (N \cap P_n) \setminus P(n, B_n)$. Then there is one of the form $x = x_{\alpha_1} + \cdots + x_{\alpha_i}$ where $\alpha_1 < \cdots < \alpha_i$ are elements of S , and $x_{\alpha_k} \in \Delta_{\alpha_k} \setminus P(n, B_n)$ for all $1 \leq k \leq i$. W.l.o.g., we can assume that $\alpha = \alpha_i$ is minimal. Since $x \in M_{\alpha+1} \cap P_n = P(n, A_{\alpha+1, n})$, we also have $x = y_{\beta_1} + \cdots + y_{\beta_j}$ where $\beta_1 < \cdots < \beta_j$ are elements of $A_{\alpha+1, n}$ and $y_{\beta_l} \in P_{n, \beta_l}$ for each $1 \leq l \leq j$. If $\beta_l \in D_{\alpha, n}$ for some $1 \leq l \leq j$, then $0 \neq x_\alpha - y_{\beta_l} \in \Delta_\alpha \setminus P(n, B_n)$. Possibly replacing x by $x - y_{\beta_l}$, we can assume that $\beta_l \in A_{\alpha, n}$ for all $1 \leq l \leq j$. But then $x_\alpha \in \Delta_\alpha \cap M_\alpha \subseteq \sum_{\beta < \alpha, \beta \in S} \Delta_\beta$, in contradiction with the minimality of α .

This proves that N is strong in P . \square

We can now improve the second part of Lemma 2.4:

Corollary 2.10. *Let R be a countable ring. Let \mathcal{C} be a class of countably presented modules which has Property (C+). Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules such that $P_n \in \text{Sum}(\mathcal{C})$ for all $n < \omega$, and let $P = \bigcup_{n < \omega} P_n$.*

Then P is the union of a continuous increasing pure chain $\mathcal{N} = (N_\alpha \mid \alpha < \aleph_1)$ consisting of strong submodules of P such that $N_{\alpha+1}/N_\alpha \in \text{Sum}(\mathcal{C})$ for all $\alpha < \aleph_1$.

Proof. Let \mathcal{M} be the chain constructed in the second part of Lemma 2.4, and consider the corresponding family \mathcal{H} as in Theorem 2.9. By [13], one can select from \mathcal{H} an increasing continuous chain $\mathcal{N} = (N_\alpha \mid \alpha < \aleph_1)$ of length $\leq \aleph_1$, such that $N_{\alpha+1}/N_\alpha$ is isomorphic to a direct sum of some of the successive factors of the original chain \mathcal{M} for all $\alpha < \aleph_1$. By Lemma 2.4, all these factors are in $\text{Sum}(\mathcal{C})$. Since \mathcal{H} consists of strong (and hence pure) submodules of P , so does \mathcal{N} . \square

3. THE FAILURE OF CONDITION (P)

In this section, we will prove that Condition (P) fails for Prüfer domains of finite character with uncountable spectrum, notably for every PID with an uncountable spectrum. We adopt [3, Theorem VII.1.4] to illustrate that failure of Condition (P) has little if any restriction on the Γ -invariant of even large almost-projective modules.

Recall from [5, Chapter III, Lemma 2.7] that in a Prüfer domain of finite character, every maximal ideal contains a finitely generated ideal, which is not contained in any other maximal ideal. Selecting one for every maximal ideal, we obtain a system of pairwise coprime proper invertible ideals. In fact, all we need is such a system of ideals:

Theorem 3.1. *Let R be a commutative domain with uncountably many pairwise coprime invertible proper ideals. Let κ be a regular uncountable cardinal, and E be a non-reflecting stationary subset of κ all of whose elements have cofinality ω . Then there is a κ -projective κ -generated R -module M with Γ -invariant \tilde{E} which is a union of a countable pure chain of projective submodules.*

Before proving Theorem 3.1, we follow the suggestion of the referee and present a simple particular case of the construction.

Example 3.2. Let R be a PID with uncountably many maximal ideals (p_α) for $0 < \alpha < \aleph_1$.

We define our module via generators and relations:

$$(4) \quad P := \langle e_{\alpha,n} : \alpha < \omega_1, n < \omega \mid p_\alpha e_{\alpha,n+1} = e_{\alpha,n} + e_{0,n+1} : \alpha > 0 \rangle.$$

(This is an example for the theorem with $\kappa = \aleph_1$, and $E = \{\alpha < \aleph_1 \mid \text{cf}(\alpha) = \aleph_0\}$.)

We leave it to the reader to verify that for every $0 < \alpha \leq \aleph_1$ and $i < \omega$ the submodule

$$(5) \quad N_{\alpha,i} = \langle e_{\beta,j} : j \leq i, \beta < \alpha \rangle$$

is actually free with a basis formed by the $e_{0,j}$ for $j \leq i$ and the $e_{\beta,i}$ for $0 < \beta < \alpha$. Since $N_{\alpha,i+1}/N_{\alpha,i} \cong \langle R, p_\beta^{-1} : 0 < \beta < \alpha \rangle$ (with $e_{0,i+1}$ corresponding to 1 and $e_{\beta,i+1}$ corresponding to p_β^{-1}) is torsion-free, $N_{\alpha,i}$ is a pure submodule of $N_{\alpha,i+1}$. Hence P is a union of a pure chain $P_i = N_{\aleph_1,i}$ of projective submodules.

On the other hand, P is a union of a continuous chain

$$(6) \quad N_\alpha = \bigcup_{i < \omega} N_{\alpha,i} = \langle e_{\beta,i} : \beta < \alpha, i < \omega \rangle, \quad 0 < \alpha < \aleph_1$$

of (strong) submodules with non-projective factors $N_{\alpha+1}/N_\alpha \cong R[p_\alpha^{-1}]$ (with $e_{\alpha,i}$ corresponding to p_α^{-i}), and hence is not projective.

The proof of Theorem 3.1 is mostly the same as in [3, Theorems VII.1.3–4], so we present only the differences. To include the sequence of submodules in the structure, we work in the category of ω -filtered modules, i.e., modules M together with an increasing sequence $(M(n) : n < \omega)$ of submodules satisfying $\bigcup_{n=0}^{\infty} M(n) = M$. A *filtered submodule* of M is a submodule N together with the filtration $N(n) := M(n) \cap N$. Note that M/N is also a filtered module with the filtration $(M(n)/N(n) \cong (M(n) + N)/N : n < \omega)$.

For the free module $R^{(\lambda \times \omega)}$, we will always use the filtration $(R^{(\lambda \times n)} : n < \omega)$.

For a module N , let $N[n]$ denote the filtered module

$$(7) \quad N[n](m) := \begin{cases} 0, & m < n, \\ N, & m \geq n. \end{cases}$$

For example,

$$R^{(\lambda \times \omega)} = \bigoplus_{n=0}^{\infty} R^{(\lambda)}[n+1]$$

as filtered modules.

Proof of Theorem 3.1. We distinguish the cases $\kappa > \aleph_1$ and $\kappa = \aleph_1$. To avoid repetition, first, we provide the common part of both cases, and then fill out the missing parts separately.

We build a continuous increasing chain of ω -filtered modules $(M_\mu : \mu < \kappa)$ whose filtrations consist of pure and projective submodules. By increasing we mean that M_ν is a filtered submodule of M_μ for $\mu < \nu$.

The union M of the chain will be our κ -projective module with Γ -invariant \tilde{E} .

To ensure that all the $M_\mu(n)$ and $M(n)$ are projective, we shall make the filtrations of the $M_{\mu+1}/M_\mu$ consist of projective modules.

We shall fix an infinite cardinal $\lambda < \kappa$. For $\mu \notin E$ let

$$M_{\mu+1} := M_\mu \oplus P_\mu, \quad P_\mu := R^{(\lambda \times \omega)} = \bigoplus_{n=0}^{\infty} R^{(\lambda)}[n+1]e_{\mu,n}.$$

For the case $\mu \in E$, we select a template as in [3, Corollary VII.1.2], i.e., a non-projective λ -generated module N_μ with an ω -filtration by projective modules.

By adding a projective module, we may assume that the filtration consist of λ -generated free modules, i.e., $N_\mu(n) \cong R^{(\lambda)}$. The filtration induces a short exact sequence $0 \rightarrow K_\mu \rightarrow F_\mu \rightarrow N_\mu \rightarrow 0$ of ω -filtered modules where

$$(8) \quad F_\mu := \bigoplus_{n=0}^{\infty} N_\mu(n)[n]e_n,$$

$$(9) \quad K_\mu := \bigoplus_{n=0}^{\infty} N_\mu(n)[n+1]e_n \cong R^{(\lambda \times \omega)}.$$

The embedding of K_μ into F_μ maps xe_n into $xe_{n+1} - xe_n$, and the homomorphism $F_\mu \rightarrow N_\mu$ maps xe_n into x for all $x \in N_\mu(n)$ and natural number n . In particular, the filtrations of K_μ and F_μ consist of direct summands, hence pure and projective submodules. We see that the modules $F_\mu/(K_\mu(n)) = \bigoplus_{m=n}^{\infty} N_\mu(m)$ are projective for all N .

We define $M_{\mu+1}$ as the pushout of the inclusion $K_\mu \subseteq F_\mu$ by a suitable embedding $K_\mu \rightarrow M_\mu$ identifying K_μ with the direct summand $\bigoplus_{n=0}^{\infty} R^{(\lambda)}[n+1]e_{\mu_n, n}$ of the filtered submodule $\bigoplus_{n=0}^{\infty} P_{\mu_n}$ for an increasing sequence of successor ordinals μ_n with supremum μ . Then $M_{\mu+1}/M_\mu \cong N_\mu$ as filtered modules, therefore $M_{\mu+1}/M_\mu$ is filtered by projective submodules.

The rest of [3, Theorems VII.1.3–4] applies to show that M is a κ -free module of Γ -invariant \tilde{E} . The filtration of M consists of projective submodules by construction.

All that is left is to find λ and the N_μ and to verify that the filtration of M actually consists of *pure* submodules.

When $\kappa > \aleph_1$, we choose $\lambda = \aleph_1$, and let N_μ be an \aleph_1 -generated non-projective module with an ω -filtration by *pure* and projective submodules. (We may choose all the N_μ the same.) Such an N_μ exists by the $\kappa = \aleph_1$ case. Since the filtration of N_μ is by *pure* submodules, it follows that all the $M_\mu(n)$ and $M(n)$ are pure submodules.

When $\kappa = \aleph_1$, we let $\lambda = \aleph_0$. Let $(I_\alpha : \alpha < \aleph_1)$ be a collection of pairwise coprime invertible proper ideals of R . We define the N_μ as submodules of the quotient field of R :

$$(10) \quad N_\mu(n) := I_\mu^{-n},$$

$$(11) \quad N_\mu := I_\mu^{-\infty}.$$

Clearly, N_μ is non-projective and its filtration is by projective submodules.

To show that the filtration of the M_μ are pure, we show that its localization by any maximal ideal Q is pure. When $I_\mu \not\subseteq Q$ and $\mu \in E$, then $N_{\mu, Q} = R_Q[0]$, so the short exact sequence $K_{\mu, Q} \rightarrow F_{\mu, Q} \rightarrow N_{\mu, Q}$ of filtered modules splits, hence $M_{\mu+1, Q} = M_{\mu, Q} \oplus N_{\mu, Q}$ as filtered modules.

There is at most one $\mu \in E$ with $I_\mu \subseteq Q$. Hence by the previous paragraph, if there is such a μ , then $M_{\nu, Q}$ is a direct summand of $M_{\nu+1, Q}$ as filtered modules for all $\nu < \mu$. So $M_{\mu, Q} \cong \bigoplus_{\nu < \mu} M_{\nu+1, Q}/M_{\nu, Q}$ with arbitrary choice of split preimages of the $M_{\nu+1, Q}/M_{\nu, Q}$. Recall that $K_{\mu, Q}$ is a direct summand of a sum of some of these preimages, so it is actually a direct summand of $M_{\mu, Q}$, i.e., $M_{\mu, Q} = K_{\mu, Q} \oplus H_\mu$. It follows that $M_{\mu+1, Q} = F_{\mu, Q} \oplus H_\mu$. These decompositions of filtered modules show that the filtrations of H_μ and $M_{\mu+1, Q}$ consist of pure submodules. \square

We finish by

Problem 3.3. *Characterize the rings R satisfying Condition (P).*

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