COMMUTATIVE SEMIGROUPS WITH ALMOST TRANSITIVE ENDOMORPHISM SEMIRINGS

Jaroslav Ježek, Tomáš Kepka, and Petr Němec

Abstract. In the paper, commutative semigroups with almost transitive endomorphism semirings are investigated.

In many classical situations, endomorphisms and/or automorphisms operate transitively on some algebraic structures. Such considerations appeared e.g. in our investigation of commutative semigroups that are simple over their endomorphism semirings (see [1]). In this note, we present a slight generalization of the transitive action.

Throughout the paper, let $A = A(+)$ be a commutative semigroup and $E = \mathrm{End}(A(+))$ be the full endomorphism semiring of A (clearly, E is a unitary semiring and A is a left E-semimodule). Further, $\mathrm{Aut}(A)$ is the group of automorphisms of $A(+)$, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ is the set of non-negative integers. As usual, $0 = 0_A$ ($o = o_A$, resp.) will denote the neutral (absorbing, resp.) element of A and $0_A \in A$ ($o \in A$, resp.) means that A has the neutral (absorbing, resp.) element. An element $a \in A$ is *idempotent* if $a = a + a$ and $\mathrm{Id}(A)$ denotes the set of all idempotent elements. A is a *semilattice* if $A = \mathrm{Id}(A)$. A subset I of A is an *ideal* if $I \neq \emptyset$ and $A + I \subseteq I$. A subsemigroup B of A is fully invariant if $f(B) \subseteq B$ for every $f \in E$. We shall say that A is *ems-simple* if $|A| \geq 2$ and $|B| = 1$ whenever B is a fully invariant subsemigroup with $B \neq A$ (then $B = \{a\}$ for some $a \in \mathrm{Id}(A)$).

Obviously, for each $a \in A$, $E(a) = \{\, f(a) \mid f \in E \,\}$ is a fully invariant subsemigroup of A and $a \in E(a)$. In particular, if $E(a) = \{a\}$ then $a \in \mathrm{Id}(A)$. We shall say that E operates on A

– *transitively* if for all $a, b \in A$ there is $f \in E$ such that $f(a) = b$ (i.e., $E(a) = A$ for every $a \in A$);
– *almost transitively* if there is $w \in A$ such that for all $a, b \in B_w = A \setminus \{w\}$ there is $f \in E$ such that $f(a) = b$ (i.e., $B_w \subseteq E(a)$ for every $a \in B_w$.

Clearly, if E operates on A transitively then it operates almost transitively and for w can be chosen any element. Further, if $|A| = 2$ then E operates almost transitively on A (indeed, if $w \in A$ then $B_w = \{v\}$ and $v \in E(v)$).

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In the rest of the paper, we shall always assume that E operates almost transitively on A (i.e., $w \in A$ is such that $B_w = A \setminus \{w\} \subseteq E(a)$ for every $a \in B_w$) and $|A| \geq 2$.

1. Basic properties

1.1 Lemma. *If $w \in \mathrm{Id}(A)$ then $E(a) = A$ for every $a \in B_w$.*

Proof. The mapping f defined by $f(x) = w$ for each $x \in A$ is an endomorphism, and hence $w = f(a) \in E(a)$ for every $a \in B_w$. □

1.2 Lemma. *If $a \in \mathrm{Id}(A)$ then $E(a) \subseteq \mathrm{Id}(A)$.*

Proof. Obvious. □

1.3 Lemma. *Just one of the following two cases takes place:*

(1) $E(w) = \{w\}$ *(and then $w \in \mathrm{Id}(A)$).*
(2) $E(w) = A$. □

Proof. If $E(w) \neq \{w\}$ then there is $f \in E$ such that $a = f(w) \neq w$. Then $B_w \subseteq E(a) = E(f(w)) \subseteq E(w)$ and, of course, $w \in E(w)$. □

1.4 Lemma. *If $w \in \mathrm{Id}(A)$ and either $\mathrm{Id}(A) \neq \{w\}$ or $E(w) \neq \{w\}$ then A is a semilattice and E operates transitively on A.*

Proof. Combine 1.1, 1.2 and 1.3. □

1.5 Lemma. *Assume that $w \notin E(a_0)$ for at least one $a_0 \in B_w$. Then:*
(i) *B_w is a fully invariant subsemigroup of A and $w \notin E(a) = B_w$ for every $a \in B_w$.*
(ii) $\mathrm{End}(B)$ *operates transitively on B.*

Proof. (i) $B_w = E(a_0)$ is a fully invariant subsemigroup of A. If $a \in B$ and $f \in E$ are such that $w \in E(a)$ then $a = g(a_0)$ for some $g \in E$ and $w = fg(a_0) \in E(a_0)$, a contradiction.
(ii) For every $f \in E$, the restriction $f|B_w$ is an endomorphism of B_w by (i). □

1.6 Corollary. *Just one of the following two cases takes place:*

(1) $E(a) = A$ *for every $a \in B_w$.*
(2) $w \notin E(a)$ *for every $a \in B_w$.* □

1.7 Remark. Let $T = \{\, (u, v) \in A \times A \mid u \notin E(v) \,\}$. According to 1.5, either $u \neq w$ for all $(u, v) \in T$ or $(w, a) \in T$ for every $a \in B_w$. Similarly, using 1.3, either $v \neq w$ for all $(u, v) \in T$ or $(a, w) \in T$ for every $a \in B_w$.

1.8 Proposition. *If E does not operate transitively on A and $|A| \geq 3$ then w is uniquely determined.*

Proof. Suppose that there are $v, w \in A$ such that $B_w \subseteq E(x)$ for all $x \in B_w$, $B_v \subseteq E(y)$ for all $y \in B_v$ and $v \neq w$. As $|A| > 2$, there is $c \in A$ with $v \neq c \neq w$. With respect to 1.6, if $E(a) \neq A$ for some $a \neq w$ then $w \notin E(c)$ and $w \notin E(v)$, hence $E(v) = \{v\}$ by 1.3, $v \in \mathrm{Id}(A)$ and $E(c) = A$ by 1.1, a contradiction. Thus $E(a) = A$ for all $a \neq w$. Symmetrically, $E(a) = A$ for all $a \neq v$, hence $E(w) = A$ and E operates transitively on A. □

2. Classification with respect to idempotents

2.1 Assume now that $w \notin \mathrm{Id}(A)$ and $\mathrm{Id}(A) \cap B_w \neq \emptyset$. By 1.2, B is a semilattice. Of course, $E(w) = A$ by 1.3, $w \neq v = 2w = 4w = 2v$, $B = \mathrm{Id}(A)$ is a fully invariant subsemigroup of A and $A = B \cup \{w\}$. Since $w \notin \mathrm{Id}(A)$, $f(w) = w$ and $f(v) = v$ for each $f \in \mathrm{Aut}(A)$. Thus automorphisms do not operate almost transitively on A whenever $|A| \geq 3$. If $|A| \leq 3$ then A is isomorphic to one of the following semigroups A_1, A_2, A_3, A_4:

A_1	w	v
w	v	v
v	v	v

A_2	w	v	u
w	v	v	v
v	v	v	v
u	v	v	u

A_3	w	v	u
w	v	v	u
v	v	v	u
u	u	u	u

A_4	w	v	u
w	v	v	w
v	v	v	v
u	w	v	u

2.2 Now, suppose that $w \in \mathrm{Id}(A) = \{w\}$. Then $E(a) = A$ for every $a \in B_w$ by 1.1 and $E(w) = \{w\}$. Of course, A is ems-simple and E does not operate transitively on A. Further, $f(w) = w$ and $f(B) = B$ for every $f \in \mathrm{Aut}(A)$. Nevertheless, it may happen that $\mathrm{Aut}(A)$ operates transitively on B (i.e., for all $a, b \in B$ there is $f \in \mathrm{Aut}(A)$ such that $f(a) = b$).

2.3 Now, let us suppose that $w \in \mathrm{Id}(A)$ and $B \cap \mathrm{Id}(A) \neq \emptyset$. Then A is a semilattice, $A = B \cup \{w\}$ and E operates transitively on A.

2.4 Finally, suppose that $\mathrm{Id}(A) = \emptyset$. Then A is infinite. Moreover, $E(w) = A$ and $B \subseteq E(a)$ for every $a \in B$. If $E(a) = A$ (i.e., $w \in E(a)$) for at least one $a \in B$ then E operates transitively on A. On the other hand, if $E(a) = B$ for every $a \in B$ then B is a fully invariant subsemigroup of A and $\mathrm{End}(B)$ operates transitively on B.

2.5 Suppose that A is not ems-simple. Then just one of the following two cases takes place:

(1) $\mathrm{Id}(A) = B$, $A = B \cup \{w\}$, $2w \neq w$ and B is a fully invariant subsemigroup of A (and a semilattice).

(2) $\mathrm{Id}(A) = \emptyset$, $A = B \cup \{w\}$, $2w \neq w$, B is a fully invariant subsemigroup of A and $\mathrm{End}(B)$ operates transitively on B.

References

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Department of Algebra, MFF UK, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: jezek@karlin.mff.cuni.cz

Department of Algebra, MFF UK, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: kepka@karlin.mff.cuni.cz

Department of Mathematics, ČZU, Kamýcká 129, 165 21 Praha 6 – Suchdol, Czech Republic
E-mail address: nemec@tf.czu.cz