ONE-ELEMENT EXTENSIONS OF COMMUTATIVE SEMIGROUPS

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Abstract. A classification of one-element extensions of commutative semigroups is presented.

In the investigation of various classes of commutative semigroups, it often happens that $A = B \cup \{w\}$, where B is a subsemigroup of A and $w \notin B$ (see e.g. [1], [2]). In this short note, we present a classification of such one-element extensions.

1. Regular transformations

Throughout the paper, let $A = A(+)$ be a commutative semigroup. Further, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ is the set of non-negative integers. As usual, $0 = 0_A$ ($o = o_A$, resp.) will denote the neutral (absorbing, resp.) element of A and $0_A \in A$ ($o \in A$, resp.) means that A has the neutral (absorbing, resp.) element. An element $a \in A$ is *idempotent* if $a = a + a$ and $\mathrm{Id}(A)$ denotes the set of all idempotent elements. A is a *semilattice* if $A = \mathrm{Id}(A)$. A subset I of A is an *ideal* if $I \neq \emptyset$ and $A + I \subseteq I$. A transformation $f : A \to A$ is said to be *regular* if $f(a + b) = a + f(b)$ for all $a, b \in A$. Regular transformations form a submonoid of the transformation monoid $T(A)$. The following observations are straightforward:
(1) If $a \in \mathrm{Id}(A)$ and f is regular then $f(a) = a + f(a)$.
(2) For each $a \in A$, the translation $\alpha_a : x \mapsto x + a$ is regular. Further, $\alpha_a\alpha_b = \alpha_{a+b} = \alpha_b\alpha_a$ for all $a, b \in A$, $\psi = \{\, (a, \alpha_a) \,|\, a \in A\}$ is a homomorphism of A into $T(A)$ and $\ker\psi = \{\, (a, b) \in A^2 \,|\, \alpha_a = \alpha_b\}$ is a congruence of A.
(3) If $0 \in A$ then $f = \alpha_{f(0)}$ for each regular transformation f of A.
(4) If f is regular and $a \in A$ then $f^2(2a) = 2f(a)$.
(5) If A is a semilattice then $f^2 = f$ for each regular transformation f of A.
(6) If f is regular and φ is an automorphism of A then $\varphi^{-1}f\varphi$ is a regular transformation of A.
(7) If B is an ideal of A then, for each $a \in B$, the restriction $\beta_a = \alpha_a|B$ is a regular transformation of B.
(8) if $o \in A$ then $f(o) = o$ for each regular transformation f of A.

Further, a regular transformation f is called *strongly regular* if $f^2 = \alpha_a$ for some $a = a_f \in A$. Now, we have the following:

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(9) For each $a \in A$, α_a is strongly regular $A_{\alpha_a} = 2a$.
(10) If f is strogly regular and A is uniquely 2-divisible (i.e., for each $a \in A$ there is exactly one $b = a/2 \in A$ with $a = 2b$) then $f = \alpha_{a_f/2}$.

2. Classification of one-element extension

From now on, let $\bar{A}$ be a commutative semigroup such that $\bar{A} = A \cup \{w\}$, $w \notin A$ and A is a subsemigroup of $\bar{A}$. Put $v = 2w$ and

$$B = \{\, a \in A \,|\, a + w \in A \,\}\,,\; C = A \setminus B = \{\, a \in A \,|\, a + w = w \,\}\,.$$

Obviously, either $B = \emptyset$ or B is an ideal of A. Similarly, either $C = \emptyset$ or C is a subsemigroup of A. In the following classification, the only trick is to find an appropriate description. Once a suitable formulation is found, the proofs are already straightforward.

2.1 Lemma. *Let $B = \emptyset$. Then $a + w = w$, $a + v = v$ for all $a \in A$ and $\bar{A} + \bar{A} = (A + A) \cup \{w\}$. Moreover, just one of the following two cases takes place:*

(1) $v = w$ *and* $w = o_{\bar{A}}$.
(2) $v \in A$, $v = o_A$ *and* $\{v, w\}$ *is a 2-element subgroup of* $\bar{A}$. □

2.2 Construction. Let A be a commutative semigroup, $w \notin A$ and $\bar{A} = A \cup \{w\}$. For all $x, y \in A$, put $x * y = x + y$ and $x * w = w * x = w$. Putting $w * w = w$, we obtain a semigroup $\bar{A}(*)$ of type 2.1(1). If $o_A \in A$ and we put $w * w = o_A$, we obtain a semigroup $\bar{A}(*)$ of type 2.1(2).

2.3 Lemma. *Let $C = \emptyset$ and $f(a) = a + w$ for all $a \in A$. Then f is a regular transformation of A and just one of the following two cases takes place:*

(1) $v = w$, $f^2 = f$ *and* $\bar{A} + \bar{A} = (A + A) \cup f(A) \cup \{w\}$.
(2) $v \in A$, f *is strongly regular,* $a_f = v$, $w \notin \bar{A} + \bar{A}$ *and* $\bar{A} + \bar{A} = (A + A) \cup f(A) \cup \{v\}$. □

2.4 Construction. Let A be a commutative semigroup, $w \notin A$, $\bar{A} = A \cup \{w\}$ and f be a regular transformation of A. For all $x, y \in A$, put $x * y = x + y$ and $x * w = w * x = f(x)$. If $f^2 = f$ (e.g., $f = \alpha_a$ for some $a \in \mathrm{Id}(A)$) and we put $w * w = w$, we obtain a semigroup $\bar{A}(*)$ of type 2.3(1). If f is strongly regular (e.g., $f = \alpha_a$ for some $a \in A$) and we put $w * w = a_f$, we obtain a semigroup $\bar{A}(*)$ of type 2.3(2).

2.5 Lemma. *Let $B \neq \emptyset$, $C \neq \emptyset$ and put $f(b) = b + w$ for all $b \in B$. Then $c + v = v$, $f(b + c) = f(b)$, $b + w \in B$ for all $b \in B$ and $c \in C$, f is a regular transformation of B and $v \in \bar{A} + \bar{A}$. Moreover, just one of the following three cases takes place:*

(1) $v = w$, $f^2 = f$ *and* $\bar{A} + \bar{A} = (A + A) \cup f(B) \cup \{w\}$.
(2) $v \in B$, f *is strongly regular,* $a_f = v$ *and* $\bar{A} + \bar{A} = (A + A) \cup f(B) \cup \{w\}$.
(3) $v \in C$, $v = o_C$, f *is strongly regular,* $a_f = v$ *and* $\bar{A} + \bar{A} = (A + A) \cup f(B) \cup \{w\}$ *and* $\{v, w\}$ *is a 2-element subgroup of* A. □

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2.6 Construction. Let A be a commutative semigroup, B be a proper ideal of A such that $C = A \setminus B$ is a subsemigroup, $w \notin A$, $\bar{A} = A \cup \{w\}$ and f be a regular transformation of B such that $f(b + c) = f(b)$ for all $b \in B$, $c \in C$. For all $x, y \in A$, put $x*y = x+y$, $x*w = w*x = f(x)$ whenever $x \in B$ and $x*w = w*x$ otherwise. If $f^2 = f$ and we put $w*w = w$, we obtain a semigroup $\bar{A}(*)$ of type 2.5(1). If f is strongly regular and $c + a_f = a_f$ for all $c \in C$ then, putting $w*w = a_f$, we obtain a semigroup $\bar{A}(*)$ of type 2.5(2). Finally, if the subsemigroup C has the absorbing element and $f^2(b) = b + o_c$ for all $b \in B$ then, putting $w*w = o_c$, we obtain a semigroup $\bar{A}(*)$ of type 2.5(3). As an easy example, we can take $A = \mathbb{N}_0(+)$, $B = \mathbb{N}$, $C = \{0\}$ and $f = id_B$ ($f = \alpha_1$, resp.).

2.7 Remark. (i) Suppose that $\bar{A}$ is a semilattice. Then only the cases 2.1(1), 2.3(1) and 2.5(1) can occur.
(ii) Suppose that $\bar{A}$ is cancellative. If $B = \emptyset$ then $v = o_A$, hence $A = \{v\}$ and $\bar{A}$ is a 2-element group. If $C = \emptyset$ and $v = w$ then $w = 0_{\bar{A}}$. If $c \in C$ then $c+c+w = c+w$, hence $c \in \mathrm{Id}(\bar{A}) = \{0_{\bar{A}}\}$ and the case 2.5(1) cannot occur.
(iii) Suppose that $\bar{A}$ is a nil-semigroup. i.e., $o = o_A \in A$ and for every $x \in \bar{A}$ there is $m \in \mathbb{N}$ with $ma = o$. If $o = w$ then $\bar{A}$ is of type 2.1(1). Now, let $o \in A$. Then $o + w = o \in A$ and $o \in B$. If $c \in C$ then $v = c+v = 2c+v = \cdots = o+v = o$ and $\bar{A}$ is of type 2.5(2). If $C = \emptyset$ then $\bar{A}$ is of type 2.3(2) (indeed, if $w = v = w + w$ then $w = o$, a contradiction).

References

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