From recollements of abelian categories to recollements of triangulated categories

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Exact model structures

Definition

Assume that $\mathcal{E}$ is an exact category.

- A pair of subcategories $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{E}$ is a cotorsion pair if

$$\mathcal{X} = {}^{\perp}\mathcal{Y} := \{X \in \mathcal{E} \mid \mathrm{Ext}^1_{\mathcal{E}}(X, Y) = 0 \text{ for each } Y \in \mathcal{Y}\},$$
$$\mathcal{Y} = \mathcal{X}^{\perp} := \{Y \in \mathcal{E} \mid \mathrm{Ext}^1_{\mathcal{E}}(X, Y) = 0 \text{ for each } X \in \mathcal{X}\}.$$

- A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *complete* if for each $M \in \mathcal{E}$ there exist exact sequences in $\mathcal{E}$

$$0 \to Y \to X \xrightarrow{f_M} M \to 0 \text{ and } 0 \to M \xrightarrow{g_M} Y' \to X' \to 0$$

  such that $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

Exact model structures

Definition

Assume that $\mathcal{E}$ is an exact category.

- A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *hereditary* in $\mathcal{E}$ if $\mathcal{X}$ is closed under taking kernels of admissible epimorphisms between objects of $\mathcal{X}$ and if $\mathcal{Y}$ is closed under taking cokernels of admissible monomorphisms between objects of $\mathcal{Y}$.
- $(\mathcal{X}, \mathcal{Y})$ is *perfect* in $\mathcal{E}$ if $\mathcal{X}$ is covering in $\mathcal{E}$ and $\mathcal{Y}$ is enveloping in $\mathcal{E}$.

Exact model structures

Recall that an exact category $\mathcal{E}$ is called *WIC (or weakly idempotent complete)* if every split monomorphism has a cokernel and every split epimorphism has a kernel.

Theorem (Hovey's one-to-one correspondence)

There is a one-to-one correspondence between exact model structures on a WIC exact category $\mathcal{E}$ and complete cotorsion pairs $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ where $\mathcal{W}$ is a thick subcategory of $\mathcal{E}$.

- We will denote the model structure by the triple $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ above and call it a Hovey triple.
- We may say that an exact model structure is *hereditary* and by this we mean that the Hovey triple is hereditary.

Exact model structures

Let W be the class of weak equivalences. The homotopy category of the model category is the localization $\mathcal{C}[W^{-1}]$ and is denoted by $\mathrm{Ho}(\mathcal{M})$.

Fact

If $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a hereditary Hovey triple, then $\mathrm{Ho}(\mathcal{M})$ is a triangulated category and it is triangle equivalent to the stable category $(\mathcal{Q} \cap \mathcal{R})/\omega$, where $\omega := \mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$ is the class of projective-injective objects.

Recollements of abelian categories

Definition of recollements of abelian categories

A recollement, denoted by $(\mathcal{A}, \mathcal{C}, \mathcal{B})$, of abelian categories, is a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{B} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

of abelian categories and additive functors such that

1. (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs;
2. i_*, $j_!$ and j_* are fully faithful;
3. $\mathrm{im} i_* = \ker j^*$.

Recollements of abelian categories

Example (Idempotents)

Let R be a ring and $e^2 = e \in R$ an idempotent. Then we have a recollement between module categories:

$$\begin{tikzcd}[column sep=huge]
\mathrm{Mod}R/ReR \arrow[r, "inc"] & \mathrm{Mod}R \arrow[l, bend right, "R/ReR\otimes_R -"'] \arrow[l, bend left, "\mathrm{Hom}_R(R/ReR,-)"] \arrow[r, "S_e = eR\otimes_R -"] & \mathrm{Mod}eRe, \arrow[l, bend right, "Re\otimes_{eRe} -"'] \arrow[l, bend left, "\mathrm{Hom}_{eRe}(eR,-)"]
\end{tikzcd}$$

where $inc : \mathrm{Mod}R/ReR \to \mathrm{Mod}R$ denotes the inclusion functor induced by the canonical ring homomorphism $R \to R/ReR$.

Recollements of abelian categories

Let $T : \mathcal{B} \to \mathcal{A}$ be a right exact functor between abelian categories. Recall that the *comma category* $(T \downarrow \mathcal{A})$ is defined as follows:

1. The objects are triples $\binom{A}{B}_{\varphi}$, with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\varphi : T(B) \to A$ is a morphism in $\mathcal{A}$;
2. A morphism $\binom{a}{b} : \binom{A}{B}_{\varphi} \to \binom{A'}{B'}_{\varphi'}$ is given by two morphisms $a : A \to A'$ in $\mathcal{A}$ and $b : B \to B'$ in $\mathcal{B}$ such that $\varphi' T(b) = a\varphi$.

For instance, let $\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a triangular matrix ring.

- If we define $T \cong M \otimes_S - : \mathrm{Mod} S \to \mathrm{Mod} R$, then we get that $\mathrm{Mod}\Lambda$ is equivalent to the comma category $(T \downarrow \mathrm{Mod} R)$.
- If we define $T \cong M \otimes_S - : \mathrm{Ch}(S) \to \mathrm{Ch}(R)$, then $\mathrm{Ch}(\Lambda)$ is equivalent to the comma category $(T \downarrow \mathrm{Ch}(R))$.

Recollements of abelian categories

Example (Comma categories)

Let $T : \mathcal{B} \to \mathcal{A}$ be a right exact functor between abelian categories. The comma category $\mathcal{C} = (T \downarrow \mathcal{A})$ can induce the following recollement of abelian categories:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{B}, \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where $i^*((\begin{smallmatrix} A \\ B \end{smallmatrix})_f) = \mathrm{coker} f$, $i^!((\begin{smallmatrix} A \\ B \end{smallmatrix})_f) = A$ and $j^*((\begin{smallmatrix} A \\ B \end{smallmatrix})_f) = B$ for any $(\begin{smallmatrix} A \\ B \end{smallmatrix})_f \in (T \downarrow \mathcal{A})$, $i^*(A) = (\begin{smallmatrix} A \\ 0 \end{smallmatrix})_0$ for any $A \in \mathcal{A}$, and $j^!(B) = \begin{pmatrix} T(B) \\ B \end{pmatrix}_{\mathrm{id}}$ and $j_*(B) = (\begin{smallmatrix} 0 \\ B \end{smallmatrix})_0$ for any $B \in \mathcal{B}$.

Recollements of abelian categories

Proposition

Let $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ be a recollement of abelian categories with enough projective objects.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "i_*"] & & \mathcal{C} \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \arrow[rr, "j^*"] & & \mathcal{B} \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"]
\end{tikzcd}$$

Then $i^!$ is an exact functor if and only if $\mathcal{C} \simeq (i^! j_! \downarrow \mathcal{A})$.

• See Proposition 8.9 in [V. Franjou, T. Pirashvili, *Comparison of abelian categories recollements*, Doc. Math. 9 (2004) 41-56].

Recollements of triangulated categories

Definition of (co)localization sequences

Let $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ be a sequence of exact functors between triangulated categories. We say it is a *localization sequence* if the following diagram satisfies the listed properties:

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "F"] & \mathcal{T} \arrow[l, bend left, "F_\rho"] \arrow[r, "G"] & \mathcal{T}'' \arrow[l, bend left, "G_\rho"] \end{tikzcd}$$

1. The right adjoint F_ρ of F satisfies $F_\rho \circ F = 1_{\mathcal{T}'}$.
2. The right adjoint G_ρ of G satisfies $G \circ G_\rho = 1_{\mathcal{T}''}$.
3. For any object $X \in \mathcal{T}$, we have $GX = 0$ if and only if $X \cong FX'$ for some $X' \in \mathcal{T}'$.

A colocalization sequence is the dual. See [H. Krause, Compos. Math. 141 (5) (2005) 1128-1162].

Recollements of triangulated categories

Definition of recollements of triangulated categories

Let $\mathcal{T}' \stackrel{F}{\to} \mathcal{T} \stackrel{G}{\to} \mathcal{T}''$ be a sequence of exact functors between triangulated categories. We say $\mathcal{T}' \stackrel{F}{\to} \mathcal{T} \stackrel{G}{\to} \mathcal{T}''$ induces a *recollement* if it is both a localization sequence and a colocalization sequence as shown in the picture

$$\begin{tikzcd}
\mathcal{T}' \arrow[r, "F"] & \mathcal{T} \arrow[l, bend right=40, "F_\lambda"'] \arrow[l, bend left=40, "F_\rho"] \arrow[r, "G"] & \mathcal{T}'' \arrow[l, bend right=40, "G_\lambda"'] \arrow[l, bend left=40, "G_\rho"]
\end{tikzcd}$$

• A.A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque 100, Soc. Math. France, Paris, 1982.

Motivation

- It should be noted that recollements of abelian categories appear quite naturally in various settings and are omnipresent in representation theory. However, the existence of recollements of triangulated categories often is difficult to establish and then provides a strong tool.
- One of natural ways to bulid triangulated categories is using the theory of exact model structures, which is inspired by the discovery by Hovey of the bijective correspondence between abelian model structures and cotorsion pairs in abelian categories.
- The main objective of this talk is to build recollements of triangulated categories from recollements of abelian categories by using the theory of exact model structures.

Motivation

- More precisely, for any recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories with enough projective and injective objects

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{B}, \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{1.1}$$

we will study how to glue exact model structures along the recollement (1.1) and demonstrate that this gluing technique is helpful to build recollements of triangulated categories.

The main technical obstacle

- Obstacle: For any recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{B}, \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

  the functor $i^!$ is not exact in general $\Rightarrow \mathcal{C}$ is not equivalent to some comma category $(T \downarrow \mathcal{A})$. [obj$(T \downarrow \mathcal{A})$ = $\{\binom{A}{B}_{\varphi}$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\varphi : T(B) \to A$ in $\mathcal{A}$ $\}$]
- Method: Replace $\mathcal{C}$ (resp. $\mathcal{B}$) with $\mathcal{C}_1 = \{C \in \mathcal{C} \mid i^*j_*j^*(C) = 0\}$ (resp. $\mathcal{B}_1 = \{B \in \mathcal{B} \mid B \cong j^*(C) \text{ for some } C \in \mathcal{C}_1\}$), and demonstrate that this kind of replacement is helpful to construct complete hereditary cotorsion pairs from $\mathcal{A}$ and $\mathcal{B}_1$ to $\mathcal{C}_1$.

  • Fact: $i^!$ is an exact functor if and only if $i^*j_*j^* = 0$.

Cotorsion pairs and recollements of abelian categories

Notation 2.1

In what follows, we assume that $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a recollement of abelian categories.

- $\mathcal{C}_1 = \{C \in \mathcal{C} \mid i^* j_* j^*(C) = 0\}$.
- $\mathcal{B}_1 = \{B \in \mathcal{B} \mid B \cong j^*(C) \text{ for some } C \in \mathcal{C}_1\}$.
- $\mathcal{C}_1$ (resp. $\mathcal{B}_1$) is an additive subcategory of $\mathcal{C}$ (resp. $\mathcal{B}$) closed under extensions and direct summands
- $\mathcal{C}_1$ (resp. $\mathcal{B}_1$) is a WIC exact category with the exact structure induced from $\mathcal{C}$ (resp. $\mathcal{B}$).

Lemma 2.2

The functors $i^! : \mathcal{C}_1 \to \mathcal{A}$ and $j_* : \mathcal{B}_1 \to \mathcal{C}$ are exact.

Cotorsion pairs and recollements of abelian categories

Theorem 2.3

Let $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ be a recollement of abelian categories with enough projective and injective objects. Assume that $(\mathcal{U}', \mathcal{V}')$ and $(\mathcal{U}'', \mathcal{V}'')$ are complete (resp. hereditary) cotorsion pairs in $\mathcal{A}$ and $\mathcal{B}_1$, respectively. Set $\mathcal{U} = \{C \in \mathcal{C} \mid i^*(C) \in \mathcal{U}', j^*(C) \in \mathcal{U}'',\ \varepsilon_M : j_!j^*(M) \to M \text{ is a monomorphism}\} \cap \mathcal{C}_1$ and by $\mathcal{V} = \{C \in \mathcal{C} \mid i^!(C) \in \mathcal{V}', j^*(C) \in \mathcal{V}''\} \cap \mathcal{C}_1$. [$\varepsilon : j_!j^* \to 1_{\mathcal{C}}$ is the counit of $(j_!, j^*)$]

If $\mathcal{C}_1$ has enough projective and injective objects and $j_!$ is $\mathcal{U}''$-exact, then $(\mathcal{U}, \mathcal{V})$ is a complete (resp. hereditary) cotorsion pair in $\mathcal{C}_1$.

Lemma

Let M be an object in $\mathcal{C}_1$. Then $\mathrm{Ext}^1_{\mathcal{C}}(M, i_*(I)) = 0$ for any injective object I in $\mathcal{A}$ if and only if $\varepsilon_M : j_!j^*(M) \to M$ is a monomorphism.

Cotorsion pairs and recollements of abelian categories

Proof of Theorem 2.3

Step 1: One need show that $(\mathcal{U}', \mathcal{V}')$ and $(\mathcal{U}'', \mathcal{V}'')$ are hereditary cotorsion pairs in $\mathcal{A}$ and $\mathcal{B}_1$, respectively if and only if $(\mathcal{U}, \mathcal{V})$ is a hereditary cotorsion pair in $\mathcal{C}_1$. Now assume that $(\mathcal{U}', \mathcal{V}')$ and $(\mathcal{U}'', \mathcal{V}'')$ are complete cotorsion pairs. To prove that the cotorsion pair $(\mathcal{U}, \mathcal{V})$ is complete, by Salce's Lemma, it suffices to show that $\mathcal{V}$ is special preenveloping in $\mathcal{C}_1$.

Lemma (Salce's Lemma)

Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in $\mathcal{E}$. Then the following are equivalent.

1. $(\mathcal{X}, \mathcal{Y})$ is complete.
2. $\mathcal{X}$ is special precovering in $\mathcal{E}$.
3. $\mathcal{Y}$ is special preenveloping in $\mathcal{E}$.

Cotorsion pairs and recollements of abelian categories

Proof of Theorem 2.3

Step 2: Let C be an object in $\mathcal{C}_1$. Then there exists an exact sequence $0 \to j^*(C) \xrightarrow{f_1} V'' \xrightarrow{g_1} U'' \to 0$ in $\mathcal{B}_1$ with $V'' \in \mathcal{V}''$ and $U'' \in \mathcal{U}''$. Since $C \in \mathcal{C}_1$, it follows that $i^*j_*j^*(C) = 0$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & i_*i^!j_!j^*(C) \arrow[r, "\sigma_{j_!j^*(C)}"] \arrow[d, "i_*i^!(\varepsilon_C)"] & j_!j^*(C) \arrow[r] \arrow[d, "\varepsilon_C"] & j_*j^*(C) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & i_*i^!(C) \arrow[r, "\sigma_C"] & C \arrow[r, "\eta_C"] & j_*j^*(C) \arrow[r] & 0.
\end{tikzcd}$$

So we have the following exact sequence in $\mathcal{C}_1$,

$$0 \longrightarrow i_*i^!j_!j^*(C) \xrightarrow{\varphi_1} j_!j^*(C) \oplus i_*i^!(M) \xrightarrow{\psi_1} C \longrightarrow 0,$$

where $\varphi_1 = \begin{pmatrix} \sigma_{j_!j^*(M)} \\ i_*i^!(\varepsilon_M) \end{pmatrix}$ and $\psi_1 = (\varepsilon_C, \sigma_C)$.

Cotorsion pairs and recollements of abelian categories

Proof of Theorem 2.3

Step 3: One can check that $\sigma_{j_!(V'')} : i_* i^! j_!(V'') \to J_!(V'')$ and $\sigma_{j_!(U'')} : i_* i^! j_!(U'') \to J_!(U'')$ are monomorphisms. Then we have the following pushout diagrams

$$\begin{tikzcd}
0 \arrow[r] & i_* i^! j_!(V'') \arrow[r, "\sigma_{j_!(V'')}"] \arrow[d, "i_*(f_2 i^!(\beta_1))"] & j_!(V'') \arrow[d, dotted, "a_1"] \\
0 \arrow[r, dotted] & i_*(V') \arrow[r, dotted, "b_1"] & V,
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & i_* i^! j_!(U'') \arrow[r, "\sigma_{j_!(U'')}"] \arrow[d, "i_*(\gamma_1)"] & j_!(U'') \arrow[d, dotted, "a_2"] \\
0 \arrow[r, dotted] & i_*(X) \arrow[r, dotted, "b_2"] & U.
\end{tikzcd}$$

Cotorsion pairs and recollements of abelian categories

Proof of Theorem 2.3

One can check that there are exact sequences

$$0 \longrightarrow i_* i^! j_!(V'') \xrightarrow{\varphi_2} j_!(V'') \oplus i_*(V') \xrightarrow{\psi_2} V \longrightarrow 0,$$

$$0 \longrightarrow i_* i^! j_!(U'') \xrightarrow{\varphi_3} j_!(U'') \oplus i_*(X) \xrightarrow{\psi_3} U \longrightarrow 0$$

in $\mathcal{C}$, where $\varphi_2 = \begin{pmatrix} \sigma_{j_!(V'')} \\ i_*(f_2) i_* i^!(\beta_1) \end{pmatrix}$, $\varphi_3 = \begin{pmatrix} \sigma_{j_!(U'')} \\ i_*(\gamma_1) \end{pmatrix}$, $\psi_2 = (-a_1, b_1)$ and $\psi_3 = (-a_2, b_2)$.

Cotorsion pairs and recollements of abelian categories

Proof of Theorem 2.3

Step 4: One can check $0 \to C \xrightarrow{\tau_1} V \xrightarrow{\tau_2} U \to 0$ with $V \in \mathcal{V}$ and $U \in \mathcal{U}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d, dotted] & \\
0 \arrow[r] & i_*i^!j_!j^*(C) \arrow[r, "\varphi_1"] \arrow[d, "i_*i^!j_!(f_1)"] & j_!j^*(C) \oplus i_*i^!(C) \arrow[r, "\psi_1"] \arrow[d, "\begin{pmatrix} j^!(f_1) & 0 \\ 0 & i_*(h_1) \end{pmatrix}"] & C \arrow[r] \arrow[d, dotted, "\tau_1"] & 0 \\
0 \arrow[r] & i_*i^!j_!(V'') \arrow[r, "\varphi_2"] \arrow[d, "i_*i^!j_!(g_1)"] & j_!(V'') \oplus i_*(V') \arrow[r, "\psi_2"] \arrow[d, "\begin{pmatrix} j^!(g_1) & 0 \\ 0 & i_*(h_2) \end{pmatrix}"] & V \arrow[r] \arrow[d, dotted, "\tau_2"] & 0 \\
0 \arrow[r] & i_*i^!j_!(U'') \arrow[r, "\varphi_3"] \arrow[d] & j_!(U'') \oplus i_*(X) \arrow[r, "\psi_3"] \arrow[d] & U \arrow[r] \arrow[d, dotted] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

As an application to Enochs Conjectures

- A complete hereditary cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in a WIC exact category is said to satisfy Enochs conjecture if $\mathcal{X}$ is closed under direct limits whenever $(\mathcal{X}, \mathcal{Y})$ is perfect cotoriosn pair.

Corollary 2.4

Let $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ be a recollement of abelian categories with enough projective and injective objects. Assume that $\mathcal{C}_1$ has enough projective and injective objects, $j_!$ is $\mathcal{U}''$-exact and $i^! j_!(\mathcal{U}'' \cap \mathcal{V}'') \subseteq \mathcal{V}'$. If $(\mathcal{U}', \mathcal{V}')$ and $(\mathcal{U}'', \mathcal{V}'')$ are complete hereditary cotorsion pairs satisfying Enochs conjecture in $\mathcal{A}$ and $\mathcal{B}_1$, respectively, then the complete hereditary cotorson pair $(\mathcal{U}, \mathcal{V})$ (constructed in Theorem 2.3) satisfies Enochs conjecture in $\mathcal{C}_1$.

As an application to Enochs Conjectures

Proof of Corollary 2.4

- Step 1: If $(\mathcal{U}, \mathcal{V})$ is a perfect cotorsion pair in $\mathcal{C}_1$, then $(\mathcal{U}', \mathcal{V}')$ is a perfect cotorsion pair in $\mathcal{A}$.
- Step 2: If $(\mathcal{U}, \mathcal{V})$ is a perfect cotorsion pair in $\mathcal{C}_1$ and $i^! j_!(\mathcal{U}'' \cap \mathcal{V}'') \subseteq \mathcal{V}'$, then $(\mathcal{U}'', \mathcal{V}'')$ is a perfect cotorsion pair in $\mathcal{B}_1$.
- Step 3: Now we assume that $(\mathcal{U}, \mathcal{V})$ is a perfect cotorsion pair in $\mathcal{C}_1$. Then $(\mathcal{U}', \mathcal{V}')$ is a perfect cotorsion pair in $\mathcal{A}$ and $(\mathcal{U}'', \mathcal{V}'')$ is a perfect cotorsion pair in $\mathcal{B}_1$. Hence both $\mathcal{U}'$ and $\mathcal{U}''$ are closed under direct limits by hypothesis. So $\mathcal{U}$ is closed under direct limits by noting that i^*, j^* and $j_!$ commute with colimits since all of them have right adjoint functors.

As an application to Enochs Conjectures

Let $\mathcal{A}$ be an abelian category with enough projective objects.

- Recall that an object M in $\mathcal{A}$ is called *Gorenstein projective* if $M = \mathrm{Z}_0(P^\bullet)$ for some exact complex $P^\bullet$ of projective objects which remains exact after applying $\mathrm{Hom}_{\mathcal{A}}(-, P)$ for any projective object P. The complex $P^\bullet$ is called a *complete $\mathcal{A}$-projective resolution*. In what follows, we denote by $\mathcal{GP}(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects.
- Recall that a right exact functor $T : \mathcal{B} \to \mathcal{A}$ between abelian categories with enough projective objects is called *compatible*, if the following two conditions hold: (C1) $T(Q^\bullet)$ is exact for any exact sequence $Q^\bullet$ of projective objects in $\mathcal{B}$. (C2) $\mathrm{Hom}_{\mathcal{A}}(P^\bullet, T(Q))$ is exact for any complete $\mathcal{A}$-projective resolution $P^\bullet$ and any projective object Q in $\mathcal{B}$.

As an application to Enochs Conjectures

Corollary 2.5

Assume that $T : \mathcal{B} \to \mathcal{A}$ is a compatible right exact functor between abelian categories with enough projective and injective objects and $\mathcal{C} = (T \downarrow \mathcal{A})$ is a comma category. Then we have

1. If $(\mathcal{P}(\mathcal{A}), \mathcal{A})$ and $(\mathcal{P}(\mathcal{B}), \mathcal{B})$ satisfy Enochs conjecture, then so is the projective cotorsion pair $(\mathcal{P}(\mathcal{C}), \mathcal{C})$.
2. If complete hereditary cotorsion pairs $(\mathcal{GP}(\mathcal{A}), \mathcal{GP}(\mathcal{A})^{\perp})$ and $(\mathcal{GP}(\mathcal{B}), \mathcal{GP}(\mathcal{B})^{\perp})$ satisfy Enochs conjecture, then so is $(\mathcal{GP}(\mathcal{C}), \mathcal{GP}(\mathcal{C})^{\perp})$.

• Fact: Let R and S be two rings, ${}_RM_S$ an R-S-bimodule, and $\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ the triangular matrix ring. If we define $T \cong M \otimes_S - : \mathrm{Mod}S \to \mathrm{Mod}R$, then we get that $\mathrm{Mod}\Lambda \cong (T \downarrow \mathrm{Mod}R)$. Moreover, if $fd(M_S) < \infty$ and $pd({}_RM) < \infty$, then $T \cong M \otimes_S -$ is compatible.

Building recollements of triangulated categories from abelian categories

Lemma 3.1

Let $\mathcal{C}$ be a WIC exact category and suppose $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ are complete, hereditary cotorsion pairs over $\mathcal{C}$ with (1) $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$, (2) $\mathcal{Q} \cap \widetilde{\mathcal{R}} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$. Then there exists a unique exact model structure $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$, and its class $\mathcal{W}$ of trivial objects is given by

$$\begin{aligned}\mathcal{W} &= \{X \in \mathcal{C} \mid \exists\, 0 \to X \to R \to Q \to 0 \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\} \\ &= \{X \in \mathcal{C} \mid \exists\, 0 \to Q' \to R' \to X \to 0 \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.\end{aligned}$$

• See Theorem 1.1 in [J. Gillespie, How to construct a Hovey triple from two cotorsion pairs, Fund. Math. 230 (2015) 281–289].

Building recollements of triangulated categories from abelian categories

Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$ and $\mathcal{Y}$ a subcategory of $\mathcal{B}$ in the recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories. We set

$$\mathfrak{N}_{\mathcal{Y}}^{\mathcal{X}} := \{C \in \mathcal{C} \mid i^!(C) \in \mathcal{X},\ j^*(C) \in \mathcal{Y}\},$$

$$\mathfrak{M}_{\mathcal{Y}}^{\mathcal{X}} := \{C \in \mathcal{C} \mid i^*(C) \in \mathcal{X},\ j^*(C) \in \mathcal{Y},\ \varepsilon_C : j_! j^*(C) \to C \text{ is monic}\},$$

where $\varepsilon : j_! j^* \to 1_{\mathcal{C}}$ is the counit of the adjoint pair $(j_!, j^*)$ in the recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories.

Building recollements of triangulated categories from abelian categories

Theorem 3.2

Let $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ be a recollement of abelian categories with enough projective and injective objects, and let $\mathcal{M}_{\mathcal{A}} = (\mathcal{U}'_1, \mathcal{W}', \mathcal{V}'_2)$ and $\mathcal{M}_{\mathcal{B}_1} = (\mathcal{U}''_1, \mathcal{W}'', \mathcal{V}''_2)$ be hereditary exact model structures on $\mathcal{A}$ and $\mathcal{B}_1$, respectively. Set $\mathcal{U}_1 = \mathfrak{M}^{\mathcal{U}'_1}_{\mathcal{U}''_1} \cap \mathcal{C}_1$, $\mathcal{V}_1 = \mathfrak{N}^{\mathcal{W}' \cap \mathcal{V}'_2}_{\mathcal{W}'' \cap \mathcal{V}''_2} \cap \mathcal{C}_1$, $\mathcal{U}_2 = \mathfrak{M}^{\mathcal{U}'_1 \cap \mathcal{W}'}_{\mathcal{U}''_1 \cap \mathcal{W}''} \cap \mathcal{C}_1$ and $\mathcal{V}_2 = \mathfrak{N}^{\mathcal{V}'_2}_{\mathcal{V}''_2} \cap \mathcal{C}_1$. Assume that $\mathcal{C}_1$ has enough projective and injective objects, $j_!$ is $\mathcal{U}''_1$-exact and $\mathcal{U}_1 \cap \mathcal{V}_1 = \mathcal{U}_2 \cap \mathcal{V}_2$.

(1) There is a hereditary exact model structure $\mathcal{M}_{\mathcal{C}_1} = (\mathcal{U}_1, \mathcal{W}, \mathcal{V}_2)$ on $\mathcal{C}_1$, where the class $\mathcal{W}$ is given by

$$\begin{aligned}\mathcal{W} &= \{X \in \mathcal{C}_1 \mid 0 \to X \to R \to Q \to 0 \textit{ with } R \in \mathcal{V}_1,\ Q \in \mathcal{U}_2\} \\ &= \{X \in \mathcal{C}_1 \mid 0 \to Q' \to R' \to X \to 0 \textit{ with } R' \in \mathcal{V}_1,\ Q' \in \mathcal{U}_2\}.\end{aligned}$$

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Theorem 3.2

(2) We have the following recollement of triangulated categories

$$\begin{tikzcd}[column sep=huge]
\mathrm{Ho}(\mathcal{M}_{\mathcal{A}}) \arrow[r, "\mathrm{L}\ \mathrm{i}_* \cong \mathrm{R}\ \mathrm{i}_*"] & \mathrm{Ho}(\mathcal{M}_{\mathcal{C}_1}) \arrow[l, bend right, "\mathrm{L}\ \mathrm{i}^*"'] \arrow[l, bend left, "\mathrm{R}\ \mathrm{i}^!"] \arrow[r, "\mathrm{L}\ \mathrm{j}^* \cong \mathrm{R}\ \mathrm{j}^*"] & \mathrm{Ho}(\mathcal{M}_{\mathcal{B}_1}), \arrow[l, bend right, "\mathrm{L}\ \mathrm{j}_!"'] \arrow[l, bend left, "\mathrm{R}\ \mathrm{j}_*"]
\end{tikzcd}$$

where $\mathrm{L}\ \mathrm{i}^*$, $\mathrm{L}\ \mathrm{i}_*$, $\mathrm{L}\ \mathrm{j}_!$, $\mathrm{L}\ \mathrm{j}^*$, $\mathrm{R}\ \mathrm{i}_*$, $\mathrm{R}\ \mathrm{j}^*$, $\mathrm{R}\ \mathrm{i}^!$ and $\mathrm{R}\ \mathrm{j}_*$ are the total derived functors of those in the recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories.

Building recollements of triangulated categories from abelian categories

Proof of Theorem 3.2

- Step 1: $\mathcal{M}_{\mathcal{A}} = (\mathcal{U}_1', \mathcal{W}', \mathcal{V}_2')$ is hereditary exact model structure on $\mathcal{A} \Longrightarrow (\mathcal{U}_1', \mathcal{W}' \cap \mathcal{V}_2')$ and $(\mathcal{U}_1' \cap \mathcal{W}', \mathcal{V}_2')$ are hereditary complete cotorsion pairs on $\mathcal{A}$; $\mathcal{M}_{\mathcal{B}_1} = (\mathcal{U}_1'', \mathcal{W}'', \mathcal{V}_2'')$ is hereditary exact model structure on $\mathcal{B}_1 \Longrightarrow (\mathcal{U}_1'', \mathcal{W}'' \cap \mathcal{V}_2'')$ and $(\mathcal{U}_1'' \cap \mathcal{W}'', \mathcal{V}_2'')$ are hereditary complete cotorsion pairs on $\mathcal{B}_1$. Set $\mathcal{U}_1 = \mathfrak{M}^{\mathcal{U}_1'}_{\mathcal{U}_1''} \cap \mathcal{C}_1$, $\mathcal{V}_1 = \mathfrak{N}^{\mathcal{W}' \cap \mathcal{V}_2'}_{\mathcal{W}'' \cap \mathcal{V}_2''} \cap \mathcal{C}_1$, $\mathcal{U}_2 = \mathfrak{M}^{\mathcal{U}_1' \cap \mathcal{W}'}_{\mathcal{U}_1'' \cap \mathcal{W}''} \cap \mathcal{C}_1$ and $\mathcal{V}_2 = \mathfrak{N}^{\mathcal{V}_2'}_{\mathcal{V}_2''} \cap \mathcal{C}_1$, where $\mathfrak{M}^{\mathcal{U}_1'}_{\mathcal{U}_1''} := \{C \in \mathcal{C} \mid i^*(C) \in \mathcal{U}_1',\ j^*(C) \in \mathcal{U}_1'',\ \varepsilon_C : j_! j^*(C) \to C \text{ is monic}\}$ and $\mathfrak{N}^{\mathcal{W}' \cap \mathcal{V}_2'}_{\mathcal{V}_1''} := \{C \in \mathcal{C} \mid i^!(C) \in \mathcal{W}' \cap \mathcal{V}_2',\ j^*(C) \in \mathcal{V}_1''\}$. By Theorem 2.3, $(\mathcal{U}_1, \mathcal{V}_1)$ and $(\mathcal{U}_2, \mathcal{V}_2)$ are complete hereditary cotorsion pairs on $\mathcal{C}_1$. Since $\mathcal{U}_1 \cap \mathcal{V}_1 = \mathcal{U}_2 \cap \mathcal{V}_2$, by Lemma 3.1, we have a hereditary exact model structure $\mathcal{M}_{\mathcal{C}_1} = (\mathcal{U}_1, \mathcal{W}, \mathcal{V}_2)$.

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Proof of Theorem 3.2

- Step 2: We have the following localization sequence of triangulated categories

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{M}_{\mathcal{A}}) \arrow[rr, "\mathrm{L}\ \mathrm{i}_*"] & & \mathrm{Ho}(\mathcal{M}_{\mathcal{C}_1}) \arrow[ll, bend left, "\mathrm{R}\ \mathrm{i}^!"] \arrow[rr, "\mathrm{L}\ \mathrm{j}^*"] & & \mathrm{Ho}(\mathcal{M}_{\mathcal{B}_1}) \arrow[ll, bend left, "\mathrm{R}\ \mathrm{j}_*"]
\end{tikzcd}$$

where $\mathrm{L}\ \mathrm{i}_*$, $\mathrm{L}\ \mathrm{j}^*$, $\mathrm{R}\ \mathrm{i}^!$ and $\mathrm{R}\ \mathrm{j}_*$ are the total derived functors of those in the recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories.

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Proof of Theorem 3.2

- Step 3: We have the following colocalization sequence of triangulated categories

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}_{\mathcal{A}}) \arrow[r, "\mathrm{R}\ \mathrm{i}_*"'] & \mathrm{Ho}(\mathcal{M}_{\mathcal{C}_1}) \arrow[l, bend right, "\mathrm{L}\ \mathrm{i}^*"'] \arrow[r, "\mathrm{R}\ \mathrm{j}^*"'] & \mathrm{Ho}(\mathcal{M}_{\mathcal{B}_1}) \arrow[l, bend right, "\mathrm{L}\ \mathrm{j}_!"'] \end{tikzcd}$$

where $\mathrm{L}\ \mathrm{i}^*$, $\mathrm{L}\ \mathrm{j}_!$, $\mathrm{R}\ \mathrm{i}_*$ and $\mathrm{R}\ \mathrm{j}^*$ are the total derived functors of those in the recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories.

Building recollements of triangulated categories from abelian categories

Proof of Theorem 3.2

- Step 4: One can check that $\mathrm{L}\,\mathrm{i}_* \cong \mathrm{R}\,\mathrm{i}_*$ and $\mathrm{L}\,\mathrm{j}^* \cong \mathrm{R}\,\mathrm{j}^*$. So we have the following recollement of triangulated categories

$$\begin{tikzcd}[column sep=huge]
\mathrm{Ho}(\mathcal{M}_{\mathcal{A}}) \arrow[r, "\mathrm{L}\,\mathrm{i}_* \cong \mathrm{R}\,\mathrm{i}_*"] & \mathrm{Ho}(\mathcal{M}_{\mathcal{C}_1}) \arrow[l, bend right=50, "\mathrm{L}\,\mathrm{i}^*"'] \arrow[l, bend left=50, "\mathrm{R}\,\mathrm{i}^!"] \arrow[r, "\mathrm{L}\,\mathrm{j}^* \cong \mathrm{R}\,\mathrm{j}^*"] & \mathrm{Ho}(\mathcal{M}_{\mathcal{B}_1}), \arrow[l, bend right=50, "\mathrm{L}\,\mathrm{j}_!"'] \arrow[l, bend left=50, "\mathrm{R}\,\mathrm{j}_*"]
\end{tikzcd}$$

  where $\mathrm{L}\,\mathrm{i}^*$, $\mathrm{L}\,\mathrm{i}_*$, $\mathrm{L}\,\mathrm{j}_!$, $\mathrm{L}\,\mathrm{j}^*$, $\mathrm{R}\,\mathrm{i}_*$, $\mathrm{R}\,\mathrm{j}^*$, $\mathrm{R}\,\mathrm{i}^!$ and $\mathrm{R}\,\mathrm{j}_*$ are the total derived functors of those in the recollment $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of abelian categories.

Building recollements of triangulated categories from abelian categories

- Let R be a ring and $\mathcal{GI}(R)$ the category of Gorenstein injective left R-modules. Then $({}^{\perp}\mathcal{GI}(R), \mathcal{GI}(R))$ is a complete hereditary cotorsion pair. See [J. Šaroch, J. Šťovíček, *Singular compactness and definability for Σ-cotorsion and Gorenstein modules*, Sel Math New Ser, 2020, 26: 23, 40pp].
- Let $\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a triangular matrix ring. Recall that the *monomorphism category* $\mathrm{Mon}(\Lambda)$ induced by bimodule ${}_RM_S$ is the subcategory of Λ-mod consisting of $\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}$ such that $\varphi : M \otimes_S Y \to X$ is a monic R-map. See [B.L. Xiong, Y.H. Zhang, P. Zhang, *Bimodule monomorphism categories and RSS equivalences via cotilting modules*, J. Algebra 503 (2018) 21-55].

Building recollements of triangulated categories from abelian categories

Corollary 3.3

Let $\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a triangular matrix ring. If M_S is flat and S is a left noetherian ring, then we have the following recollement of triangulated categories:

$$\begin{tikzcd}[column sep=huge]
\dfrac{\mathcal{GI}(R)}{\mathrm{Inj}(R)} \arrow[r, "i_*"] & \dfrac{\mathcal{GI}(\mathrm{Mon}(\Lambda))}{\mathrm{Inj}(\mathrm{Mon}(\Lambda))} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \dfrac{\mathcal{GI}(S)}{\mathrm{Inj}(S)} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd},$$

where $\mathrm{Inj}(\mathrm{Mon}(\Lambda)) = \{ \begin{pmatrix} X \\ Y \end{pmatrix}_{\phi} \in \Lambda\text{-}\mathrm{Mod} \mid X \in \mathrm{Inj}(R),\ Y \in \mathrm{Inj}(S) \}$.

In fact, $\mathcal{GI}(R)/\mathrm{Inj}(R) \simeq \mathrm{Ho}(\mathcal{M}_R)$, where $\mathcal{M}_R = (\mathrm{Mod}R, {}^{\perp}\mathcal{GI}(R), \mathcal{GI}(R$ is the Gorenstein injective model structure of R.

Building recollements of triangulated categories from abelian categories

Let R be a ring and let $\mathbf{C}(R)$ denotes the category of chain complexes of R-modules. For a given class $\mathcal{X}$ of R-modules, we have the following classes of chain complexes in $\mathbf{C}(R)$.

- $\widetilde{\mathcal{X}}$ denotes the class of all exact chain complexes X with cycles $Z_nX \in \mathcal{X}$.
- $dw\widetilde{\mathcal{X}}$ denotes the class of all chain complexes X with components $X_n \in \mathcal{X}$.
- $ex\widetilde{\mathcal{X}}$ denotes the class of all exact chain complexes X with components $X_n \in \mathcal{X}$.

Let $\mathbf{K}(\mathrm{Proj}\ R)$ and $\mathbf{K}_{ex}(\mathrm{Proj}\ R)$ denote the chain homotopy category of all complexes of projective modules and the chain homotopy category of all exact complexes of projective modules, respectively.

Building recollements of triangulated categories from abelian categories

Corollary 3.4

Let $\Lambda = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ be a triangular matrix ring. Denote by $\mathcal{M}^{\mathrm{proj}}_{ctr,R} = (dw\widetilde{\mathcal{P}}_R, \mathcal{W}_{ctr,R}, \mathbf{C}(R))$ and $\mathcal{M}^{\mathrm{proj}}_{ctr,R} = (dw\widetilde{\mathcal{P}}_S, \mathcal{W}_{ctr,S}, \mathbf{C}(S))$ the contraderived model structures of $\mathbf{C}(R)$ and $\mathbf{C}(S)$, respectively. Then we have the following recollement of triangulated categories:

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{ctr,R}) \arrow[r] & \mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{ctr,\Lambda}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{ctr,S}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

In fact, $\mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{ctr,R})$ is called the contraderived category which is equivalent to $\mathbf{K}(\mathrm{Proj}\ R)$.

Building recollements of triangulated categories from abelian categories

Corollary 3.5

Let $\Lambda = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ be an upper triangular matrix ring. Denote by $\mathcal{M}^{\mathrm{proj}}_{stb,R} = (ex\widetilde{\mathcal{P}}_R, \mathcal{V}_{prj,R}, \mathbf{C}(R))$ and $\mathcal{M}^{\mathrm{proj}}_{stb,R} = (ex\widetilde{\mathcal{P}}_S, \mathcal{V}_{prj,S}, \mathbf{C}(S))$ the exact Proj model structures of $\mathbf{C}(R)$ and $\mathbf{C}(S)$, respectively. If M has finite flat dimension as a right S-module, then we have the following recollement of triangulated categories:

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{stb,R}) \arrow[r] & \mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{stb,\Lambda}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{stb,S}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

In fact, $\mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}_{stb,R})$ is called the projective stable derived category which is equivalent to $\mathbf{K}_{ex}(\mathrm{Proj}\ R)$.

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Thank you!

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