# Bounded and Well-Placed Theories in the Lattice of Equational Theories

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*In Memory of Evelyn Nelson*

## 0. Introduction

Among the earliest publications of Evelyn Nelson are four papers which appeared in 1971—Nelson [1971a, 1971b] and Burris and Nelson [1971/72, 1971]. These papers contributed to our understanding of the intricate structural diversity of the lattice of equational theories. In particular, these papers revealed that large partition lattices occur as intervals in the lattice of equational theories of semigroups. In the present paper, we focus on some contrasting features of the lattice of equational theories. We identify a filter in the lattice of equational theories, called the filter of bounded theories. This filter is countably infinite and fairly tractable, especially for similarity types which provide only one operation symbol.

Every equational theory can be viewed as a fully invariant congruence relation on the term algebra of the appropriate similarity type. From this viewpoint, certain classes of equational theories emerge from the imposition of various cardinality restrictions on the equivalence classes of terms induced by these theories. At one extreme are the *term-finite* equational theories—those theories each of whose equivalence classes is finite. At another extreme, one should put those theories which induce only finitely many equivalence classes of terms. However, it is easy to see that, for each similarity type, there is only one such theory: the trivial theory based on $x \approx y$. (Officially, we take equations to be ordered pairs of terms, but we frequently use the suggestive notation $s \approx t$ in place of the official $\langle s, t\rangle$.) A slightly more elaborate notion proves more suitable. Observe that for any equational theory $T$, the image of any $T$-equivalence class of terms with respect to any automorphism of the term algebra is again a $T$-equivalence class. So the automorphism group of the term algebra partitions the $T$-equivalence classes into orbits. We say that an equational theory $T$ is *bounded* iff the automorphism group of the term algebra partitions the $T$-equivalence classes into finitely many orbits.

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The automorphisms of the term algebra have a particularly simple structure. Each one is determined by a permutation of the variables. We will say that terms $s$ and $t$ are *literally similar* iff $t$ is the image of $s$ with respect to some automorphism of the term algebra. For example, $(xy)z$ is literally similar to $(yx)u$ but to neither $(xy)x$ nor $x(yz)$. Evidently, an equational theory $T$ is bounded iff there is a finite set $W$ of terms such that every term is $T$-equivalent to some term which is literally similar to a member of $W$. For similarity types with only finitely many operation symbols, an equational theory $T$ is bounded iff there is a natural number $b_T$ such that every term is equivalent, with respect to $T$, with some term whose length is no greater than $b_T$.

The equational theory of left-zero semigroups and the equational theory of right-zero semigroups are two equational theories with the most trivial structure of all. They are based on the equations $xy \approx x$ and $xy \approx y$, respectively. These theories can be described as follows: An equation $s \approx t$ is valid in left-zero semigroups iff the leftmost variable in the terms $s$ and $t$ coincide; and $s \approx t$ is valid in right-zero semigroups iff the rightmost variables in $s$ and $t$ coincide. So each of these theories is bounded—we can take $W = \{x\}$ in both cases.

These two equational theories are not interesting. However, one can ask if there is an interesting equational theory that would lie, in some sense, just in the middle between them. In this way, many more bounded theories can be discovered. One can try to define an equational theory $E$ of groupoids in the following way: $s \approx t \in E$ iff the centermost variables of $s$ and $t$ coincide. The trouble is that it is not clear how to define the notion of the centermost variable in a term. Each term—of the type of groupoids—which is not a variable, has a uniquely determined left part and a uniquely determined right part. So, given a term $s$, we can try to find its center by taking first its left part, then the right part of the left part, then the left part of the right part of the left part, etc. We can continue in this zigzag way until we arrive at a variable. The variable is then the centermost variable, or the center of the term. There are, however, two ways of starting: we can start by taking either the left part or the right part in the first step. Thus we obtain, in fact, two centers: the left center and the right center. In our equational theory $E$ both must be taken into account. Unfortunately, the set of all groupoid equations $s \approx t$ such that the corresponding centers of $s$ and $t$ coincide fails to be an equational theory. (The equation $xy \approx (w(zx))y$ can be easily derived from $xy \approx (zx)y$. In this last equation, $x$ is the "left center" of both sides and $y$ is the "right center" of both sides, but the "left centers" of the terms in the first equation disagree.) One further condition must be imposed. Namely, that the numbers of steps in going from $s$ and from $t$ to their left centers are either both even or both odd (and the same, of course, applies to the right centers).

If we define $E$ in this way, we get an equational theory. It is based on the set

$$\{(xx)(xx) \approx x, (xy)z \approx (yy)z, x(yz) \approx x(yy)\}$$

of equations. This equational theory and its corresponding variety are interesting; they have been studied by T. Evans [1967]. In fact, Evans was able to prove that the members of this variety are, up to isomorphism, exactly those algebras of the form $\langle S \times S, * \rangle$ where

$$\langle a, b \rangle * \langle c, d \rangle = \langle b, c \rangle \text{ for all } a, b, c, d \in S.$$

It is not difficult to prove that every term is equivalent, with respect to $E$, to some term literally similar to a member of $\{x, xx, x(xx), (xx)x, xy, x(yy), (xx)y, (xx)(yy)\}$. Hence, $E$ is also a bounded equational theory.

The aim of this paper is to introduce a collection of equational theories which can be defined in a manner resembling what we have just done for the equational theory $E$ above. We shall call such equational theories *well-placed* theories. Roughly speaking, a well-placed theory is specified by a collection of directions for how to traverse terms (like the zigzag paths described above) and a parity rule; then two terms $s$ and $t$ are set equal by the theory iff following any of the given directions through either of the terms terminates with the same parity at the same variable. The precise definition is supplied in §2 below.

It turns out that every well-placed theory is bounded. In the special but important case when the similarity type is supplied with only one operation symbol, we shall prove that our collection of well-placed theories is representative, in a sense, for the collection of all bounded theories: an equational theory is bounded iff it extends a well-placed theory. Among those equational theories with just one operation symbol and which have absorption laws—that is, equations of the form $t \approx x$, where $x$ is a variable and $t$ is a term which is not a variable—the bounded theories coincide with the well-placed theories. This allows us to give a complete description of the lattice of absorptive bounded theories (with just one operation symbol). This lattice is distributive and contains all the maximal bounded theories, with a single exception.

For the foundations of equational logic, we refer the reader to the survey paper Taylor [1979] and to the preliminary sections of Ježek [1981] and McNulty [1981]. For more information on varieties and the general theory of algebras, see Burris and Sankappanavar [1981] or McKenzie, McNulty, and Taylor [1987].

## 1. Bounded Equational Theories

The aim of this section is to collect some simple facts concerning bounded equational theories.

THEOREM 1.1. *Let $\rho$ be a similarity type.*

(1) *The set of bounded equational theories of type $\rho$ is a filter in the lattice of all equational theories of type $\rho$.*

(2) *Every bounded equational theory of type $\rho$ has only finitely many extensions of type $\rho$.*
(3) *Every bounded equational theory is the theory of a finite algebra.*
(4) *If $\rho$ is finite, then every bounded equational theory of type $\rho$ is finitely based.*

*Proof.* (1) It is evident that any extension of a bounded equational theory is itself bounded. So it remains to show that $T$ is bounded whenever $T = T_0 \cap T_1$ and both $T_0$ and $T_1$ are bounded. Let $W_0$ and $W_1$ be finite sets of terms such that every term is equivalent, with respect to $T_0$, to a term literally similar to one in $W_0$ and, with respect to $T_1$, to a term literally similar to one in $W_1$. For each pair $\langle u, v\rangle$ of terms, where $u \in W_0$ and $v \in W_1$ select, if possible, one term $t$ such that $t$ is equivalent, with respect to $T_0$, to a term literally similar to $u$ and, with respect to $T_1$, to a term literally similar to $v$. Denote by $W$ the set of all terms selected in this way. Then $W$ is finite and it is not difficult to prove that every term is equivalent, with respect to $T$, to a term literally similar to one in $W$. Hence $T$ is bounded.

(2) Let $T$ be a bounded equational theory and let $W$ be a finite set of terms such that every term is equivalent, with respect to $T$, to a term literally similar to one in $W$. Denote by $x_0, x_1, \dots, x_{n-1}$ all the variables which occur in terms belonging to $W$. Further, let $y_0, y_1, \dots, y_{n-1}$ be pairwise distinct variables different from all the variables $x_0, x_1, \dots, x_{n-1}$. Let $W'$ be the set of all terms literally similar to terms in $W$ and containing no variables other than $x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}$. It is easy to see that every equation is equivalent, with respect to $T$, to an equation both sides of which belong to $W'$. Consequently, every extension of $T$ is based on a subset of $W' \times W'$.

(3) Let $T$ be a bounded equational theory. We first argue that any finitely generated model of $T$ is finite—that is, the variety of all models of $T$ is locally finite. Suppose $\mathbf{A}$ is a model of $T$ which is generated by $\{a_0, \dots, a_{n-1}\}$. Because $T$ is bounded there is a finite set $U$ of terms such that $A = \{t^{\mathbf{A}}(a_0, \dots, a_{n-1}) : t \in U\}$. Thus $A$ is finite. Now $T$ has only finitely many proper extensions, by item (2). For each of these extensions, select a finitely generated—hence finite—model of $T$ which is not a model of the extension. $T$ is the equational theory of the direct product of these finitely many finite algebras.

(4) Let $T$ be a bounded equational theory and let $b$ and $r$ be natural numbers such that $r$ is an upper bound on the ranks of the operation symbols of $\rho$ and every term is equivalent, with respect to $T$, to a term of length no larger than $b$. Let $\Sigma$ be the set of all equations $s \approx t \in T$ such that both $s$ and $t$ have length no larger than $br$ and contain no variables other than $x_0, \dots, x_{br-1}$. Then $\Sigma$ is a finite set of equations and we contend that it is a base for $T$.

We need only prove that for every term $s$ of length greater than $br$ there is a shorter term $t$ such that $\Sigma \vdash s \approx t$. Evidently, $s$ has a subterm $u$ such that its length $\lambda(u)$ satisfies $b < \lambda(u) \leq br$. Then there is a term $v$ such that $\lambda(v) \leq b$

and $u \approx v \in T$. Hence, $\Sigma \vdash u \approx v$ and so $\Sigma \vdash s \approx t$ where $t$ is obtained from $s$ by replacing one occurrence of $u$ by $v$. $t$ is shorter than $s$, as desired. □

Even among equational theories with finite similarity types, the properties attributed to all bounded equational theories by items (2), (3), and (4) of Theorem 1.1 do not characterize the bounded equational theories. Let $\mathbf{A}$ be a finite algebra which belongs to a congruence distributive variety and let $T$ be the equational theory of $\mathbf{A}$. According to Baker's Finite Basis Theorem, cf. Baker [1977], $T$ is finitely based, as required by item (4). Plainly, $T$ fulfills item (3). That $T$ also fulfills (2), i.e., that $T$ has only finitely many extensions, can be found in Jónsson [1967]. But it is easy to invent a finite $\mathbf{A}$, belonging to a congruence distributive variety, whose equational theory is not bounded. Any finite lattice $\mathbf{A} = \langle A, \vee, \wedge \rangle$ with more than one element will do. Indeed, it is easy to verify that no two distinct terms from the following list

$$x_0,\ x_0 \vee x_1,\ x_0 \vee x_1 \vee x_2, \ldots$$

can lie in the same equivalence class of terms, with respect to the equational theory of $\mathbf{A}$ and that, moreover, no two elements selected from two distinct such equivalence classes can be literally similar.

Another way to approach the definition of $T$ being a bounded equational theory is to associate with each $T$-equivalence class $V$ of terms a set $V' = \{s \approx t : s, t \in V\}$ of equations—thus each $T$-equivalence class of terms is viewed as asserting the truth of a certain collection of equations—and insisting that, up to logical equivalence, only finitely many such assertions are associated with $T$. There is one delicate point: Call a term $p$ an *isoterm* for $T$ provided $\{p\}$ is a $T$-equivalence class. Then for any two terms $p$ and $q$ which are isoterms for $T$, the sets $\{p \approx p\}$ and $\{q \approx q\}$ are logically equivalent—for the trivial reason that both $p \approx p$ and $q \approx q$ are logically valid—but $p$ and $q$ may have no particular structural connection with each other. For example, when $T$ is the theory based on the empty set (so $p \approx q \in T$ iff $p = q$) then $T$ fails to be bounded in an extreme way and yet, up to logical equivalence, only one assertion is associated with the $T$-equivalence classes of terms.

THEOREM 1.2. *The following statements are equivalent, for any equational theory $T$:*

(1) *$T$ is a bounded equational theory.*
(2) *Up to literal similarity, there are only finitely many isoterms of $T$ and, up to logical equivalence, $T$ partitions the set of all terms which are not isoterms of $T$ into only finitely many equivalence classes.*

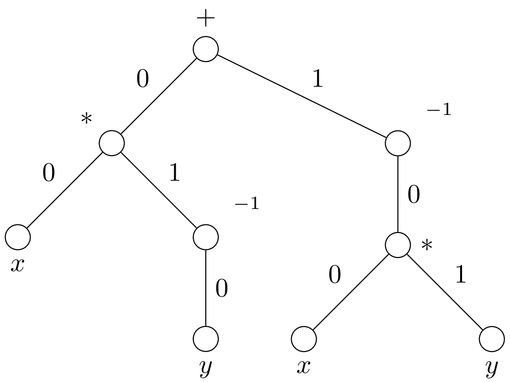
*Proof.* Let $V$ and $Z$ be $T$-equivalence classes of terms, neither of which contains an isoterm of $T$. We will argue that there is an automorphism $\phi$ of the term algebra which carries $V$ to $Z$ iff $V' = \{p \approx q : p, q \in V\}$ and $Z' = \{p \approx q : p, q \in Z\}$ are logically equivalent. Evidently, the existence of such an automorphism guarantees the logical equivalence of the two sets of equations.

Consider the converse. For any terms $p$ and $q$, we write $p \lhd q$ to mean that some substitution instance of $p$ is a subterm of $q$, i.e., there is an endomorphism $\phi$ of the term algebra such that $\phi(p)$ is a subterm of $q$. Note that $\lhd$ is a quasi-order on the set of terms and that $p$ and $q$ are literally similar iff $p \lhd q$ and $q \lhd p$. We suppose that $V'$ and $Z'$ are logically equivalent. Pick $v \in V$ which is $\lhd$-minimal in $V$ and let $u$ be any other element of $V$. Since $Z' \vdash v \approx u$ there must be an equation $p \approx q \in Z'$ with $p \neq q$ such that $p \lhd v$. Similarly, since $V' \vdash p \approx q$ there must be a term $s \in V$ such that $s \lhd p$. By the minimality of $v$, it follows that $s, p$, and $v$ are all literally similar. Let $\phi$ be an automorphism of the term algebra which maps $v$ to $p$. Hence, $\phi$ maps the $T$-equivalence class $V$ to the $T$-equivalence class $Z$. $\square$

## 2. Well-Placed Equational Theories

Fix a similarity type $\rho$. The rank of an operation symbol $Q$ of type $\rho$ will be denoted by $\rho Q$. Operation symbols of rank 0 are called *constant symbols* or just *constants*. We adopt $x_0, x_1, x_2, \ldots$ as our list of variables. Formally, terms are certain finite sequences made up of operation symbols and variables. For any term $t$ (of type $\rho$), $var(t)$ is the set of all variables occurring in $t$ and $\lambda(t)$ is the length of $t$.

It is useful to depict terms as labelled trees. In this view, the leaves are labelled with variables or constants and the internal nodes of the tree are labelled with operation symbols of positive rank and the edges are labelled "left-to-right" with natural numbers. For example, the term $(x * y^{-1}) + (x * y)^{-1}$ is depicted below.




Under this scheme, the subterms of a term correlate with the subtrees and the location of a particular occurrence of a subterm within a given term can be described by specifying the path from the root of the tree to the root of the subtree. Thus the occurrence of the term $x * y$ in $(x * y^{-1}) + (x * y)^{-1}$ can be specified by the sequence $(+, 1)(^{-1}, 0)$. This process of traversing (the tree depicting) a term to arrive at a subterm can be made precise as follows.

By an *elementary place* we mean an ordered pair $\langle Q, i\rangle$ where $Q$ is an operation symbol and $i$ is a natural number with $i < \rho Q$. Finite sequences of elementary places are referred to as *places.* Note that the empty sequence qualifies as a place. The length of a place $e$ will be denoted by $\lambda(e)$. Simple infinite sequences of elementary places are called *directions.* Any two places can be concatenated to form a new place. The set of all places is a monoid with respect to concatenation; this monoid can be identified with the monoid of words over the alphabet consisting of the elementary places, which are considered to be letters. So we write the place $e$ as $a_0a_1\dots a_{n-1}$, where $a_i$ for $i < n$ is an elementary place, instead of $\langle a_0, a_1, \dots, a_{n-1}\rangle$. We denote the concatenation of two places $e$ and $f$ by $ef$. A place $e$ can also be concatenated with a direction $h$ to give the direction $eh$. Thus the union of the set of all places with the set of all directions is a partial monoid with respect to concatenation.

Let $e$ be a place and $h$ be either a place or a direction. We say that $e$ is an *initial segment* of $h$ and that $h$ is an *extension* of $e$ provided $h = eh'$ for some $h'$. This $h'$ is uniquely determined and will be denoted by $h - e$. Two places are said to be *incomparable* if neither is an extension of the other. $e^n$ is defined for each natural number $n$ so that $e^0$ is the empty place and $e^{k+1} = e^k e$ for all natural numbers $k$. For each nonempty place $e$, the (unique) direction $h$ which extends $e^n$, for each natural number $n$, is denoted by $e^\omega$. A direction $h$ is said to be *eventually periodic* iff $h = ef^\omega$ for some places $e$ and $f$, where $f$ is nonempty; $h$ is called *periodic* if, in addition, $e$ is empty.

For terms $t$ and places $e$ we want $t[e]$ to denote the subterm of $t$ occurring at the place $e$. $t[e]$ is defined recursively as follows:

$$t[e] = \begin{cases} t, & \text{if } e \text{ is empty} \\ t_i[f], & \text{if } t = Qt_0\dots t_{n-1} \text{ and } e = \langle Q, i\rangle f \\ \emptyset, & \text{otherwise} \end{cases}$$

So, $t[e]$ is always either a subterm of $t$ or it is the empty set. If $t[e] = u$ is a term, then $e$ is called an *occurrence* of $u$ in $t$. For a given term $t$, the set of occurrences of subterms in $t$ is always finite; it maximal elements (with respect to the ordering by extension) are just the occurrences of variables and constants in $t$.

Let $t$ and $s$ be terms and let $e$ be an occurrence of the subterm $u$ in $t$. Then there is a unique term $r$ such that $r[e] = s$ and $r[f] = t[f]$ for every place $f$ which is incomparable with $e$. This term $r$ is called the term obtained from $t$ by *replacing* $u$ by $s$ at the occurrence $e$. We denote $r$ by $t(e : u \leftarrow s)$.

Let $t$ be a term and $h$ be a direction. We say that $t$ is *traversible* in the direction $h$ provided $t[e]$ is a variable for some initial segment $e$ of $h$. This $e$, if it exists, is unique and we denote it by $\tau_t(h)$; we also denote $t[e]$ as $t[h]$.

By a *coherent triple* we shall mean a triple $\langle J, m, d\rangle$ where $J$ is a finite set of directions, $m$ is a function, called the *threshold function*, from $J$ into the set of natural numbers, and $d$ is a function, called the *period function*, from $J$ into the set of positive integers such that

(1) Every final segment of $h$ is a member of $J$, for all $h \in J$.
(2) If $h \in J$, then $h = ef^\omega$ where $m(h) = \lambda(e)$ and $d(h) = \lambda(f)$.
(3) If $h = ah' \in J$, where $a$ is an elementary place, then $m(h) \leq m(h') + 1$ and $d(h')$ is a multiple of $d(h)$.

With each coherent triple $\langle J, m, d\rangle$ we associate a set $\Theta(J, m, d)$ of equations as follows. $s \approx t \in \Theta(J, m, d)$ iff both of the following conditions hold:

(1) For every direction $h \in J$, $s$ is traversible in the direction $h$ iff $t$ is traversible in the direction $h$.
(2) If $h \in J$ and $s$ and $t$ are traversible in the direction $h$, then $s[h] = t[h]$ and either $\tau_s(h) = \tau_t(h)$ or else $\lambda(\tau_s(h)) \equiv \lambda(\tau_t(h)) \mod d(h)$ and both $m(h) \leq \lambda(\tau_s(h))$ and $m(h) \leq \lambda(\tau_t(h))$.

Given any two natural numbers $m$ and $d$ with $d$ positive, define $\equiv_{m,d}$ to be the equivalence relation on the set of natural numbers such that

$$i \equiv_{m,d} j \text{ iff either } i = j \text{ or } i \equiv j \mod d \text{ and } m \leq i, j.$$

Then the last condition in the definition of $\Theta(J, m, d)$ becomes

$$s[h] = t[h] \text{ and } \lambda(\tau_s(h)) \equiv_{m(h),d(h)} \lambda(\tau_t(h))$$

for all $h \in J$ such that $s$ and $t$ are traversible in the direction $h$. The first condition in the definition of $\Theta(J, m, d)$ is always satisfied, if there is only one operation symbol and it is of positive rank, since in that case every term is traversible in every direction.

THEOREM 2.1. *$\Theta(J, m, d)$ is an equational theory, for every coherent triple $\langle J, m, d\rangle$.*

*Proof.* Let $\Theta$ denote the binary relation $\Theta(J, m, d)$ on the set of all terms. We must prove that $\Theta$ is a fully invariant congruence relation on the term algebra. Evidently, $\Theta$ is an equivalence relation of the set of terms.

To see that $\Theta$ is a congruence relation, let $Q$ be any operation symbol and let $n$ be the rank of $Q$. Let $s_0, t_0, s_1, t_1, \ldots, s_{n-1}, t_{n-1}$ be any terms such that $\langle s_i, t_i\rangle \in \Theta$ for all $i < n$. Put $s = Qs_0s_1 \ldots s_{n-1}$ and $t = Qt_0t_1 \ldots t_{n-1}$. Let $h \in J$. So $h = (G, k)h'$, where $(G, k)$ is an elementary place. If $Q \neq G$, then neither $s$ nor $t$ is traversible in the direction $h$. So, let $G = Q$. Now $h' \in J$ and $\langle s_k, t_k\rangle \in \Theta$. Thus, $s_k$ is traversible in the direction $h'$ iff $t_k$ is, and in the positive case, $s_k[h'] = t_k[h']$ and $\lambda(\tau_{s_k}(h')) \equiv_{m(h'),d(h')} \lambda(\tau_{t_k}(h'))$. Since $\tau_s(h) = (Q, k)\tau_{s_k}(h')$ and $\tau_t(h) = (Q, k)\tau_{t_k}(h')$, we easily obtain $\langle s, t\rangle \in \Theta$ by comparing the definitions. Therefore, $\Theta$ is a congruence on the algebra of terms.

To see that $\Theta$ is fully invariant, let $\langle s, t\rangle \in \Theta$ and let $\phi$ be an endomorphism of the term algebra. It remains to prove that $\langle \phi(s), \phi(t)\rangle \in \Theta$. So let $h \in J$. If $s$ and $t$ are not traversible in the direction $h$, then neither is $\phi(s)$ nor $\phi(t)$. So suppose that $s$ and $t$ are traversible in the direction $h$. Then $s[h] = t[h] = x$, where $x$ is a variable. Evidently, $\phi(s)$ is traversible in the direction $h$ iff $\phi(x)$ is traversible in the direction $h - \tau_s(h)$. Likewise, $\phi(t)$ is traversible in the direction $h$ iff $\phi(x)$ is traversible in the direction $h - \tau_t(h)$. Since $\langle J, m, d\rangle$ is coherent, it follows that $h - \tau_s(h) = h - \tau_t(h)$. Thus $\phi(s)$ is traversible in the direction $h$ iff $\phi(t)$ is, and in the positive case $\phi(s)[h] = \phi(t)[h]$. What remains to check, is very easy. □

An equational theory $T$ is said to be *well-placed* iff $T = \Theta(J, m, d)$, for some coherent triple $\langle J, m, d\rangle$.

In the event that the similarity type supplies only one operation symbol, the notion of a well-placed theory can be formulated in a simpler way. Suppose $Q$ is the only operation symbol and let $n$ be its rank. Then the notion of elementary place can be simplified by suppressing the $Q$—which is always the same. Thus elementary places can be taken to be just the natural numbers less than $n$. Hence places become words on the alphabet $\{0, 1, \ldots, n-1\}$ and directions become simple infinite sequences of natural numbers less than $n$. In this context, every term can be traversed in every direction, allowing us to eliminate the first condition in the definition of $\Theta(J, m, d)$.

Let us see why the theory $E$ described in the introduction is well-placed. Let $J = \{01010101\ldots, 101010\ldots\}$. Thus there are just two directions which belong to $J$. Define $m(h) = 0$ and $d(h) = 2$ for all $h \in J$. It is evident that $\langle J, m, d\rangle$ is a coherent triple. Now $s \approx t \in \Theta(J, m, d)$ iff both $s[h] = t[h]$ for both $h \in J$ and $\tau_s(h) \equiv \tau_t(h) \pmod 2$ for both $h \in J$. The conditions pertaining to traversibility and to the threshold function $m$ are trivially true in this setting.

It can happen that distinct coherent triples determine the same well-placed theory. Here is an example.

EXAMPLE 2.2. Let $J = \{h_0, h_1, h_2\}$ where

$$h_0 = 010101\ldots$$
$$h_1 = 001010\ldots$$
$$h_2 = 101010\ldots$$

Let $d(h_i) = 2$ for $i = 0, 1, 2$, $m(h_0) = m(h_1) = 2$, $m(h_2) = 1$, and $m'(h_0) = m'(h_2) = 1$ while $m'(h_1) = 2$. Then $\langle J, m, d\rangle$ and $\langle J, m', d\rangle$ are distinct coherent triples such that $\Theta(J, m, d) = \Theta(J, m', d)$.

The two coherent triples in this example differ only in their threshold functions. We will call a triple $\langle J, m, d\rangle$ *tight* provided this triple is coherent and if $h, h' \in J$ are directions with a common initial segment of length $m(h)$, then $m(h) = m(h')$. The triple $\langle J, m, d\rangle$ is the example is a tight triple. The set $Ti$ of tight triples carries a natural partial order $\leq$ defined as follows:

$$\langle J, m, d\rangle \leq \langle J', m', d'\rangle$$
$$\text{iff}$$
$$J \subseteq J' \text{ and } m(h) \leq m'(h) \text{ and } d(h) \text{ divides } d'(h) \text{ for all } h \in J$$

It is routine to check that $\leq$ is, in fact, a lattice ordering of $Ti$ and, moreover, that this lattice ordering is *meet-complete* in the sense that if $I$ is any nonempty set and $\langle J_i, m_i, d_i\rangle$ is a tight triple for each $i \in I$, then

$$\bigwedge_{i\in I}\langle J_i, m_i, d_i\rangle = \left\langle \bigcap_{i\in I} J_i, \min_{i\in I} m_i, \gcd_{i\in I} d_i \right\rangle$$

So meets of arbitrary nonempty subsets of $Ti$ always exist. On the other hand, $Ti$ has no largest member, entailing that the lattice order is not complete.

Now notice that because $\Theta(J, m, d)$ is a well-placed theory for every tight triple $\langle J, m, d\rangle$, we can regard $\Theta$ as a function from $Ti$ into the set of well-placed theories. Actually, much more is true.

THEOREM 2.3. *The map $\Theta$ is an anti-isomorphism from the lattice ordered set of tight triples onto the set of well-placed theories lattice-ordered by set inclusion.*

*Proof.* The two lemmas below establish the theorem.

LEMMA 2.4. *For every well-placed theory $T$, there is a tight triple $\langle J, m', d\rangle$ with $T = \Theta(J, m', d)$.*

Suppose $T = \Theta(J, m, d)$, where $\langle J, m, d\rangle$ is a coherent triple. Let $m'$ be the function with domain $J$ such that for each $h \in J$

$$m'(h) = \min_{s\approx t\in T}\{k : \lambda(\tau_s(h)) = k < \lambda(\tau_t(h)), \text{ with } s \text{ traversible in direction } h\}$$

The set from which the minimum is to be extracted is nonempty. For example, let $k$ be the length of the shortest initial segment of $h$ which is not an initial segment of any other direction belonging to the finite set $J$. This $k$ belongs to the specified set. It is routine to verify that $\langle J, m', d\rangle$ is tight and that $T = \Theta(J, m', d)$. □

LEMMA 2.5. *Let $\langle J, m, d\rangle$ and $\langle J', m', d'\rangle$ be two tight triples. Then $\langle J, m, d\rangle \leq \langle J', m', d'\rangle$ iff $\Theta(J', m', d') \subseteq \Theta(J, m, d)$.*

The direct implication is obvious. For the converse, suppose $T' \subseteq T$, where $T = \Theta(J, m, d)$ and $T' = \Theta(J', m', d')$.

First we argue that $J \subseteq J'$. Suppose not. Pick $h \in J$ with $h \notin J'$. Let $e$ be the shortest initial segment of $h$ which cannot be extended to any of the finitely many directions in $J'$. Take two different variables $x$ and $y$ and any terms $s$ and $t$ such that $s[e] = x$, $t[e] = y$, and $s[f] = t[f]$, for all places $f$

incomparable to $e$. Then it is evident that $s \approx t \in T'$, but $s \approx t \notin T$. This contradiction proves $J \subseteq J'$.

For every $h' \in J'$, pick a natural number $c(h')$, divisible by $d'(h')$, which is so large that the initial segment of $h'$ of length $m'(h') + c(h')$ extends to no other direction in $J'$.

Now we argue that $m(h) \leq m'(h)$, for all $h \in J$. So let $h \in J$ and set $h = ef^{\omega}$, where $\lambda(e) = m'(h)$ and $\lambda(f) = d'(h)$. Let $s$ be a term such that $s[e] = x$, where $x$ is a variable. Let $t$ be a term with the following properties:

(1) $t[h'] = x$ and $\lambda(\tau_t(h')) = m'(h') + c(h')$, whenever $h' \in J'$ and $e$ is an initial segment of $h'$.

(2) $t[h'] = s[h']$, whenever $h' \in J'$ and $e$ is not an initial segment of $h'$.

It follows that $s \approx t \in T'$, since $\langle J', m', d' \rangle$ is tight. Consequently, $s \approx t \in T$. But then $m(h) \leq m'(h)$, as desired.

Finally, we argue that $d(h)$ divides $d'(h)$, for all $h \in J$. So let $h \in J$. Evidently, there is an equation $p \approx q \in T'$ such that both $p$ and $q$ are traversible in the direction $h$, $\lambda(\tau_p(h)) = m'(h)+c(h)$, and $\lambda(\tau_q(h)) = m'(h)+c(h)+d'(h)$. But this equation must also belong to $T$. Consequently, $d(h)$ divides $d'(h)$, as desired. □

THEOREM 2.6. *Every well-placed equational theory is bounded.*

*Proof.* Let $\langle J, m, d \rangle$ be a tight triple. Let $F$ be the set of all operation symbols which occur in the elementary places on the directions belonging to $J$. $F$ is a finite set. Let $k = \max_{h \in J} m(h)$ and $p = \mathrm{lcm}_{h \in J}\, d(h)$. Let $J'$ be the set of all directions $h'$ made up using only operation symbols from $F$ such that $h' = ef^{\omega}$, where $\lambda(e) = k$ and $\lambda(f) = p$. For each $h' \in J'$ define $m'(h') = k$ and $d'(h') = p$. Then $\langle J', m', d' \rangle$ is an tight triple and $\langle J, m, d' \rangle \leq \langle J', m', d' \rangle$.

Since $\Theta(J', m', d') \subseteq \Theta(J, m, d)$, to conclude that this latter theory is bounded, it suffices to show that $\Theta(J', m', d')$ is bounded. But this is easy. In such a theory, every term is equivalent to a term $t$ such that every occurrence of a subterm in $t$ is of length smaller than $k + 2p$ and $t$ contains no operation symbols other than those in $F$ (supplemented by an arbitrary operation symbol not in $F$, in case $F$ does not exhaust all available operation symbols). Up to literal similarity, there are only finitely many such terms □

Not every bounded equational theory is well-placed. For example, there are many bounded theories of commutative groupoids—but it is easy to see that the only well-placed theory extending the theory of commutative groupoids is trivial. Moreover, the bounded theories constitute a filter in the lattice of equational theories, but the well-placed theories do not constitute a sublattice—even though they are lattice ordered by set-inclusion.

EXAMPLE 2.7. Fix a similarity type with one operation symbol, that one being binary. The five tight triples specified below form a sublattice isomorphic to $\mathbf{N}_5$ in the lattice of tight triples; however, the corresponding well-placed equational theories do not constitute a sublattice of the lattice of equational theories.

| | | |
|---|---|---|
| $h_0 = 0000\ldots$ | $h_1 = 0111\ldots$ | $h_0 = 1111$ |
| $J_0 = \{h_0, h_1, h_2\}$, | $m_0(h_0) = m_0(h_1) = m_0(h_2) = 1$, | $d_0(h_0) = d_0(h_1) = d_0(h_2) = 1$ |
| $J_1 = \{h_0\}$ | $m_1(h_0) = 1$, | $d_1(h_0) = 1$ |
| $J_2 = \{h_0\}$ | $m_2(h_0) = 0$. | $d_2(h_0) = 1$ |
| $J_3 = \{h_1, h_2\}$ | $m_3(h_1) = m_3(h_2) = 1$, | $d_3(h_1) = d_3(h_2) = 1$ |
| $J_4 = \emptyset$ | | |

Demonstrating the details of this example presents no difficulty. It is helpful to keep in mind that the join of two tight triples must again be a tight triple. If the corresponding well-placed theories are labelled $T_0, T_1, \ldots, T_4$, then $T_4$ is the largest equational theory—the one based on $x \approx y$. But $x \approx y$ is not derivable from $T_1 \cup T_3$. This latter set of equations is contained in the theory based on $xy \approx zw$.

## 3. Bounded and Well-Placed Equational Theories with Only One Operation Symbol

This section is devoted to proving the following theorem.

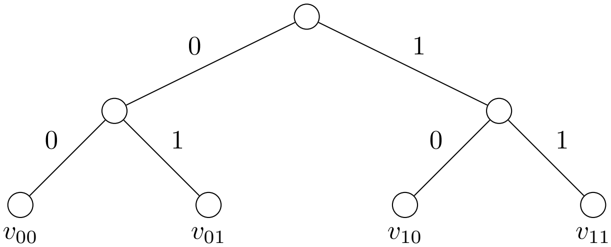
THEOREM 3.1. *In a similarity type which provides only one operation symbol and no constant symbols, an equational theory is bounded iff it is an extension of some well-placed theory.*

This theorem is proven via a sequence of lemmas. Of course, if $T$ is an equational theory which extends a well-placed theory, we know by Theorem 1.1 and Theorem 2.6 that $T$ is bounded. *So from this point through the remainder of this section, we take $T$ to be a fixed but arbitrary bounded equational theory.* Our whole ambition will be to find a well-placed subtheory of $T$.

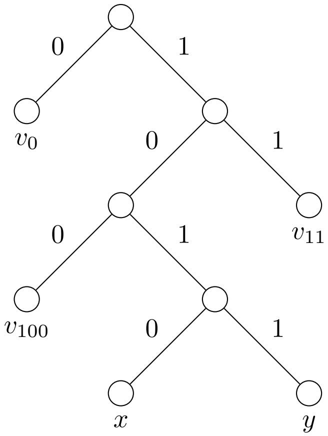
Since there is only one operation symbol, elementary places can differ only in their second coordinates. To simplify notation, we dispense with the operation symbol and take as elementary places the numbers $0, 1, 2, \ldots, n-1$, where $n$ is the rank of the operation symbol. Thus places become sequences or words on $\{0, 1, 2, \ldots, n-1\}$. In illustrating the concepts introduced below, we usually take $n = 2$. Fix a one-to-one map $v$ from the set of all places into the set of all variables such that there are infinitely many variables not in the range of the map $v$. For any place $e$, we write the value of $v$ at $e$ as $v_e$. Fix two distinct variable $x$ and $y$ which are not in the range of $v$.

We say the term $t$ *depends on* the variable $z$ iff $t \approx \sigma(t) \notin T$ for some automorphism $\sigma$ of the term algebra which moves the variable $z$, but fixes every other variable which occurs in $t$. (Actually, we should say that $t$ depends on $z$ with respect to $T$, but $T$ is fixed throughout this section.) Let $k_0$ be the smallest natural number such that every term depends on at most $k_0$ variables. Such a number must exist since $T$ is bounded.

For each natural number $k$, let $w_k$ be the unique term such that $w_k[e] = v_e$, for every place $e$ of length $k$. Thus the tree depicting the term $w_k$ is the full $n$-ary tree of height $k$ with each "leaf" assigned a distinct label. For example, if $n = 2$, then $w_2 = (v_{00}v_{01})(v_{10}v_{11})$ and the tree depicting $w_2$ is the following:




Two places $e$ and $f$ are said to be *complementary* iff $e = gi$ and $f = gj$ for some place $g$ and some distinct elementary places $i$ and $j$. Now let $e$ be any place and $t$ be any term. By $r_e(t)$ we denote the unique term $r$ such that $r[e] = t$ and $r[f] = v_f$ for every place $f$ which is complementary to a nonempty initial segment of $e$. Thus $r_e(t)$ is the most generic term in which $e$ is an occurrence of $t$—in the sense that every other such term is a substitution instance of $r_e(t)$. For example, if $n = 2$, then $r_{101}(xy) = v_0((v_{100}(xy))v_{11})$, which is depicted below.




Define $Z = \{e : r_e(x) \approx r_e(y) \notin T\}$. Thus $e \in Z$ iff $r_e(x)$ depends on $x$. Let $J$ be the set of all directions $h$ such that every finite initial segment of $h$ belongs to $Z$. We will devise a tight triple $\langle J, m, d\rangle$, where $m$ and $d$ are certain functions, such that $\Theta(J, m, d) \subseteq T$. The main obstacle is to prove that $J$ is finite.

LEMMA 3.2. *Every nonempty final segment of $h$ is a member of $J$, for every $h \in J$.*

*Proof.* Let $e$ and $f$ be any places and let $\sigma$ be any automorphism of the term algebra fixing $x$, such that $\sigma(x_g) = x_{eg}$ for every place $g$ which is complementary to some nonempty initial segment of $f$. Then $r_{ef}(x) = r_e(\sigma(r_f(x)))$. Using this fact, it is clear that every initial segment and every final segment of a member of $Z$ belongs again to $Z$. This means that $h \in J$ iff every finite segment, initial or not, of $h$ belongs to $Z$. Now the lemma is immediate. □

Define $Z'$ to be the set of all places $e$ such that $w_{\lambda(e)}$ depends of $v_e$ and let $J'$ be the set of all directions $h$ such that every finite initial segment of $h$ belongs to $Z'$. It is easy to see that if $e \in Z'$, then every initial segment of $e$ and every final segment of $e$ is also a member of $Z'$. Hence, if $h \in J'$, then every final segment of $h$ also belongs to $J'$. But observe that there are at most $k_0$ members of $Z'$ of any fixed length. This entails that $J'$ is finite. It is easy to verify that $Z' \subseteq Z$ and, hence, $J' \subseteq J$. In view of the fact that $J'$ is finite, we fix a natural number $k_1$ such that any place in $Z'$ of length at least $k_1$ extends to exactly one direction in $J'$.

LEMMA 3.3. *Let $e, f, g$ be three places such that $efg \in Z$, $f$ has length greater than $k_1$, and, for any nonempty proper initial segment $f'$ of $f$ and any place $f''$ complementary to $f'$, the term $r_{efg}(x)$ does not depend on $v_{ef''}$. Then $fg \in Z'$ and, for any nonempty initial segment $d$ of $fg$ and any place $d'$ complementary to $d$, the term $r_{efg}(x)$ does not depend on $v_{ed'}$.*

*Proof.* For the sake of contradiction, suppose this lemma fails. Select a failure with $g$ as short as possible. Let $d$ be any place complementary to $f$.

Put $t_0 = r_{efg}(x)$, $t_1 = r_{efg}(y)$, and let $t_2$ be the term obtained from $t_0$ by replacing the variable $v_{ed}$ by $y$. For $i = 0, 1, 2$, let $s_i$ denote the term obtained from $t_i$ by replacing each variable $v_{ef''}$ by the term $w_{\lambda(f)-\lambda(f'')}$, for any nonempty proper initial segment $f'$ of $f$ and any place $f''$ complementary to $f'$. Since $r_{efg}(x)$ does not depend on $v_{ef''}$, we conclude that

$$s_i \approx t_i \in T \qquad \text{for } i = 0, 1, 2.$$

For $i = 0, 1, 2$, let $\sigma_i$ denote an endomorphism of the term algebra such that $\sigma_0(v_f) = \sigma_2(v_f) = t_0[ef]$, $\sigma_0(v_d) = \sigma_1(v_d)$, and such that each $\sigma_i$ fixes every other variable. Then

$$s_i[e] = \sigma_i(w_{\lambda(f)}) \qquad \text{for } i = 0, 1, 2.$$

Now since $efg \in Z$, we know that $t_0 \approx t_1 \notin T$. Hence, $s_0 \approx s_1 \notin T$. But this means that $s_0[e] \approx s_1[e] \notin T$. Consequently, $\sigma_0(w_{\lambda(f)}) \approx \sigma_1(w_{\lambda(f)}) \notin T$. Hence $f \in Z'$. Since $\lambda(f) > k_1$, we conclude that $d \notin Z'$. Because $\sigma_0(v_f) = \sigma_2(v_f)$, it follows that $s_0[e] \approx s_2[e] \in T$. But then $s_0 \approx s_3 \in T$ and consequently $t_0 \approx t_3 \in T$. Hence $r_{efg}(x)$ does not depend on $v_{ed}$.

Since $f \in Z'$ and $r_{efg}(x)$ does not depend on $v_{ed}$ for any place $d$ complementary to $f$, we see that $g$ cannot be empty. Let $fg = f_1 g_1$ where $\lambda(f_1) = \lambda(f)+1$ and $\lambda(g_1) = \lambda(g) - 1$. But then $e, f_1, g_1$ is a failure of the lemma and $g_1$ is shorter than $g$. This is contrary to our original assumption, so the lemma is proved. □

LEMMA 3.4. *$J$ is finite.*

*Proof.* Let $k_2$ be so large that there is an interval in $\{0, 1, , 2 \ldots, k_2 - 1\}$ with more than $k_1$ elements which is disjoint from $M$, whenever $M$ is a subset of $\{0, 1, 2, \ldots, k_2 - 1\}$ with the cardinality of $M$ no greater than $k_0$.

Now consider any $h \in J$. Let $h_0$ be the initial segment of $h$ with length at least $k_2$. Denote by $M$ the set of all numbers $i \in \{0, 1, 2, \ldots, k_2 - 1\}$ such that, for some place $d$ complementary to the initial segment of $h_0$ of length $i$, the term $r_{h_0}(x)$ depends on $v_d$. Thus $h_0 = efg$ where $f$ has the properties described in Lemma 3.3. Hence, $fg \in Z'$. But this means that $h' \in J'$, where $h = h_0 h'$. Thus every member $h$ of $J$ can be written as the concatenation $h_0 h'$ of one the the finitely many members of $Z$ with length $k_2$ followed by one of the finitely many members if $J'$. So $J$ is finite. □

We can now define the tight triple $\langle J, m, d\rangle$. Let $k_3$ be a number so large that for each of the finitely many directions $h$ in $J$, if $g$ is an initial segment of $h$ of length at least $k_3$, then $g$ extends to a unique direction in $J$ and $r_g(x)$ depends only on variables which belong to $P$, where $P = \{z :$ $z$ occurs in $r_e(x)$ for some $e$ with $\lambda(e) = k_3\}$. For each $h \in J$, there are two different initial segments $e$ and $ef$ of $h$ with $\lambda(e) \geq k_3$ such that $r_e(x) \approx r_{ef}(x) \in T$. This follows because $T$ is bounded. Define $d(h)$ to be the smallest positive integer $d$ such that $r_e(x) \approx r_{ef}(x) \in T$, for some initial segment $ef$ of $h$ with $\lambda(f) = d$ and $k_3 \leq \lambda(e)$. For the moment, fix $e, f$, and $h$. Since $r_{ef}(x)$ depends only on the variables in $P$, it follows that $r_e(x) \approx r_{ef}(x) \approx r_{eff}(x) \approx \cdots \in T$. Consequently, $ef^\omega \in J$ and so by the uniqueness of extensions, $h = ef^\omega$. Similarly, if $e'f'$ is another initial segment of $h$ with $k_3 \leq \lambda(e')$ and $1 \leq \lambda(f')$, then $h = e'f'^\omega$. Again, reasoning with $P$, we can conclude that $d(h) = \lambda(f)$ divides $\lambda(f')$, due to the minimality of $d(h)$. It is also clear that $r_{e'}(x) \approx r_{e'f'}(x) \in T$, whenever $e'f'$ is an initial segment of $h$ such that $\lambda(e) \leq \lambda(e')$ and $d(h)$ divides $\lambda(f')$.

Now pick a natural number $k$ so that $r_e(x) \approx r_{ef}(x) \in T$ whenever $ef$ is an initial segment of any direction $h \in J$ such that $k \leq \lambda(e)$ and $d(h)$ divides $\lambda(f)$. Define

$$m(h) = k \qquad \text{for all } h \in J$$

$\langle J, m, d\rangle$ is evidently a tight triple. The proof of Theorem 3.1 is concluded by establishing the next lemma.

LEMMA 3.5. $\Theta(J, m, d) \subseteq T$.

*Proof.* Let $s \approx t \in \Theta(J, m, d)$. Denote by $Q(s,t)$ the set of all those places $e$ such that either $e$ is an occurrence of a variable in exactly one of the terms $s$, $t$ or it is an occurrence in both of them but $s[e] \neq t[e]$. We shall prove $s \approx t \in T$ by induction on the cardinality of $Q(s,t)$. If $Q(s,t)$ is empty, then $s$ and $t$ are the same term. Let $Q(s,t)$ be nonempty. It is enough to consider the case when there exists an occurrence $e$ of a variable $z$ in $s$ such that $e$ is an occurrence of a subterm in $t$ but $s[e] \neq t[e]$. Since $s \approx t \in \Theta(J, m, d)$, it is evident that $k \leq \lambda(e)$. Put $s' = s(e : z \leftarrow s[e])$. Then $Q(s', t)$ is a proper subset of $Q(s,t)$ and that $s' \approx t \in \Theta(J, m, d)$. Hence, $s' \approx t \in T$ by the induction hypothesis. In the event that $e \notin Z$, we obtain $s \approx s' \in T$ immediately—and thus $s \approx t \in T$. So suppose $e \in Z$. Let $h$ be the unique direction in $J$ extending $e$. Evidently, $z = s[e] = s[h] = t[h]$. Let $ef = \tau_t(h)$. Thus $d(h)$ divides $\lambda(f)$. Thus $r_e(x) \approx r_{ef}(x) \in T$. But then a substitution gives $s \approx s' \in T$ and hence $s \approx t \in T$, as desired. □

The conclusion of Theorem 3.1 does not hold for similarity types with more that one operation symbol. For example, the theory $T$ based on $\{F(x,y) \approx G(x,y), F(x,y) \approx x\}$ is evidently bounded, but it is easy to see that $T$ cannot extend any well-placed theory.

## 4. Absorptive Bounded Theories with Only One Operation Symbol

Recall that an equational theory $T$ is called *absorptive* iff there is a term $t$ which is not a variable such that $x \approx t \in T$.

THEOREM 4.1. *In a similarity type which provides only one operation symbol and no constant symbols, an absorptive equational theory is bounded iff it is well-placed.*

*Proof.* We use the notation developed in §3. In particular, $T$ will be a fixed bounded theory—this time absorptive—and $n$ will be the rank of the sole operation symbol $F$. $Z$ and $J$ retain their meanings. We need only argue that $T$ is well-placed.

Because of the assumption that $T$ is absorptive, we can deduce the following additional facts about $Z$ and $J$:

LEMMA 4.2. *Every place from $Z$ can be extended to a direction belonging to $J$.*

*Proof.* It is sufficient to prove that every place $e \in Z$ can be extended to a longer place belonging to $Z$. Suppose, on the contrary, that $ei \notin Z$ for all $i < n$. Then

$$r_{ei}(x) \approx r_{ei}(y) \in T \quad \text{for each } i < n.$$

Let $s_0, \ldots, s_{n-1}$ be terms so that $x \approx F(s_0, \ldots, s_{n-1}) \in T$, using the fact that $T$ is absorptive. For each $i < n$, let $t_i$ be the term obtained from $s_i$ by substituting $y$ for $x$. Therefore,

$$\begin{aligned} r_{ei}(s_i) \approx r_{ei}(t_i) \in T \quad \text{for all } i < n \\ r_e(x) \approx r_e(F(s_0, \ldots, s_{n-1})) \in T \text{ and,} \\ r_e(y) \approx r_e(F(t_0, \ldots, t_{n-1})) \in T. \end{aligned}$$

But this implies that $r_e(x) \approx r_e(y) \in T$, contrary to $e \in Z$. □

LEMMA 4.3. *Every direction in $J$ is periodic.*

*Proof.* Evidently, there is a positive integer $p$ such that whenever $h \in J$, then $h = ef^\omega$ where $e$ and $f$ are both places of length $p$. Let $t$ be a term such that $x \approx t \in T$, $var(t) = \{x\}$, and whenever $g$ is an occurrence of $x$ in $t$, then the length of $g$ is at least $p$. Fix $h \in J$ and suppose $h = ef^\omega$ where $\lambda(e) = \lambda(f) = p$. Let $\sigma$ be a substitution such that $\sigma(x) = r_{ef}(x)$. Hence, $\sigma(t) \approx r_{ef}(x) \in T$. Thus, $\sigma(t)$ depends on $x$. But then there is an occurrence $g$ of $x$ in $\sigma(t)$ such that $r_g(x)$ depends on $x$. Consequently, $g \in Z$. Now $g = e'ef$, where $e'$ is an occurrence of $x$ in $t$. Note that $p \leq \lambda(e')$. Also, since $g \in Z$ we have from Lemma 4.2 that $g$ extends to a direction $h'$ belonging to $J$. But then $h' - e' = f'^\omega$, where $f'$ has length $p$. Consequently, $e = f' = f$ and $h$ is periodic. □

For each $h \in J$, denote by $d(h)$ the smallest positive integer $d$ such that $r_e(x) \approx r_{eg}(x) \in T$, for some initial segment $eg$ of $h$ with $\lambda(g) = d$. Also, denote by $c(h)$ the least number $c$ such that

(1) The initial segment of $h$ of length $c$ differs from the initial segment of $h'$ of length $c$, for every direction $h' \in J$ different from $h$.
(2) $r_f(x)$ depends only on variables occurring in $r_e(x)$, whenever $e$ and $f$ are initial segments of $h$ of length at least $c$.

As observed in the last section, if $e$ and $f$ are initial segments of $h$, each of length at least $c(h)$, such that $\lambda(e) \equiv \lambda(f) \pmod{d(h)}$, then $r_e(x) \approx r_f(x) \in T$.

Let $\Theta = \Theta(J, 0, d)$. Evidently, $\langle J, 0, d\rangle$ is a tight triple and so $\Theta$ is a well-placed theory.

The proof of Theorem 4.1 is concluded with the next two lemmas.

LEMMA 4.4. $T \subseteq \Theta$.

*Proof.* Let $s \approx t \in T$. Let $h \in J$ and put $e = \tau_s(h)$, $f = \tau_t(h)$, $z = s[e]$, and $w = t[e]$. We make the harmless assumption that $z$ and $w$ are not in the range of the map $v$. To conclude that $s \approx t \in \Theta$, we must prove that $z = w$ and that $\lambda(e) \equiv \lambda(f) \pmod{d(h)}$.

We can assume that $e$ is an initial segment of $f$. Let $g$ be an initial segment of $h$ such that $h = g^\omega$, $f$ is an initial segment of $g$, and $c(h) \leq \lambda(g)$. Note that both $ge$ and $gf$ are initial segments of $h$. Since $r_{ge}(z)$ depends only on variables occurring in $r_g(z)$ and since $z = s[e]$, we know that

$$r_{ge}(z) \approx r_g(s) \in T.$$

In an analogous way, we know that

$$r_{gf}(w) \approx r_g(t) \in T$$

But since $s \approx t \in T$, it follows that $r_{ge}(z) \approx r_{gf}(w) \in T$. Because $ge \in Z$, we know that $r_{ge}(z)$ depends on $z$. Thus $r_{gf}(w)$ depends on $z$, forcing $z = w$. So $r_{ge}(x) \approx r_{gf}(x) \in T$. Therefore, $\lambda(ge) \equiv \lambda(gf) \pmod{d(h)}$ and so $\lambda(e) \equiv \lambda(f)$ $\pmod{d(h)}$, as desired. □

LEMMA 4.5. $\Theta \subseteq T$.

*Proof.* Let us call an equation $s \approx t$ long provided any occurrence of any variable in either $s$ or $t$ has length which exceeds $c(h)$ for every $h \in J$. Because $T$ is absorptive, for any equation $s \approx t$, there is a long equation $s' \approx t'$ such that $s \approx s' \in T$ and $t \approx t' \in T$. Since we already know $T \subseteq \Theta$, it follows that $s \approx t \in \Theta$ iff $s' \approx t' \in \Theta$. So the verify this lemma, it suffices to prove that every long equation belonging to $\Theta$ also belongs to $T$. But every long equation belonging to $\Theta$ belongs to $\Theta(J, m, d)$ and this last theory was already shown to be a subtheory of $T$ in Lemma 3.5. □

□

We will call a triple $\langle J, 0, d\rangle$ *absorptive* provided that the triple is coherent and $d(h) = d(h')$ whenever $h \in J$ and $h'$ is a final segment of $h$. It is not difficult to prove that $\Theta(J, 0, d)$ is an absorptive equational theory, if $\langle J, 0, d\rangle$ is an absorptive triple. It is also not difficult to prove that the absorptive triples constitute a distributive sublattice of the lattice of all tight triples. It is also clear that the join, in the lattice of equational theories, of any set of bounded absorptive theories must be itself a bounded absorptive theory. The situation with meets is more complicated. From Theorem 1.1, we know that the meet of two bounded theories is itself bounded, but for arbitrary similarity types, we cannot hope that the meet of two bounded absorptive theories will be absorptive. In the restricted similarity types of this section, we obtain the following result.

COROLLARY 4.6. *In any similarity type which provides only one operation symbol and no constant symbols, the set of all bounded absorptive equational theories is a distributive filter in the lattice of equational theories; the lattice of absorptive triples is dually isomorphic to the lattice of absorptive bounded theories via the dual isomorphism* $\Theta$. □

Even in the similarity type providing just the operation symbols $F$ and $G$, both unary, the meet of two bounded absorptive equational theories can fail to be absorptive. For example, let $T_0$ be the theory based on $\{x \approx F(x), G(x) \approx G(y)\}$ and let $T_1$ be the theory based on $\{x \approx G(x), F(x) \approx F(y)\}$. It is easy to see that these two theories are bounded absorptive theories whose meet is not absorptive.

Fix a similarity type. An equational theory is said to be *equationally complete* provided it is maximal among nontrivial equational theories. With one exception, every equationally complete theory is absorptive. The sole exception is the theory $C$, called the *constant theory*, defined in this way: $s \approx t \in C$ iff either $s = t$ or neither $s$ nor $t$ is a variable. If the similarity type provides only one operation symbol and no constant symbols, then the results of this section lead easily to the following characterization of the equationally complete bounded theories.

Let $e = a_0 a_1 \dots a_{m-1}$ be any place and denote by $\mathrm{cyc}(e)$ the set of all cyclic permutations of the place $e$—that is, all places of the form

$$a_i a_{i+1} \dots a_{m-1} a_0 \dots a_{i-1}.$$

The place $e$ is said to be *irreducible* iff $e$ not a power of any place shorter than $e$. Finally, denote by $P(e)$ the absorptive triple $\langle J, 0, d\rangle$ where $J = \{f^\omega : f \in \mathrm{cyc}(e)\}$ and $d(h) = \lambda(e)$ for all $h \in J$.

COROLLARY 4.7. *In a similarity type which provides only one operation symbol and no constant symbols, the equationally complete bounded theories are the constant theory $C$ and the theories $\Theta(P(e))$ where $e$ is an irreducible place. Moreover, $\Theta(P(e)) = \Theta(P(f))$ iff cyc$(e)$ =cyc$(f)$.* □

In the case of groupoids, $\Theta(P(0))$ is the theory of left-zero semigroups, $\Theta(P(1))$ is the theory of right-zero-semigroups, and $\Theta(P(01))$ is the other equational theory mentioned in the introduction: the theory $E$ which was investigated in Evans [1967].

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This article appeared in Algebra Universalis **26** (1989) 311–331. This online version differs from the published version only in some of the layout matters, like font selection, line breaks and page breaks, and in the sizes of the diagrams. As a result this online version is a bit shorter than the printed version and the page numbers above do not correspond to the published version.