FREE LATTICES OVER HALFLATTICES

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0. INTRODUCTION. Although the word problem for free lattices is well known to be solvable (cf. Dean [**1**]), the question still remains open to characterize the finite partial lattices P for which the free lattice $F(P)$ over P is finite.

There are partial answers to this question. In Wille [**5**] the problem is solved for the partial lattices P that are both meet– and join–trivial in the sense that whenever the meet xy or the join $x+y$ of two elements x, y is defined in P then the elements are comparable. In [**3**] the problem is solved for join–trivial partial lattices. In the papers [**2**] and [**4**] free lattices over partial lattices from some other special classes are investigated.

In the present paper we shall be concerned with free lattices over halflattices. By a halflattice we mean a partial lattice P such that xy is defined for all pairs $x, y \in P$ and $x+y$ is defined whenever x, y are two elements with a common upper bound in P. It is easy to see that a partial lattice P is a halflattice iff there exists a lattice L containing P as a relative sublattice such that P is an order–ideal in L (i.e., $a \in P$ implies $b \in P$ for all $b \in L$ with $b \leq a$); for a given P we can define L by $L = P \cup \{1\}$ where 1 is the greatest element of L.

We shall not solve in this paper the problem for which halflattices P is the free lattice over P finite. However, we shall prove that $F(L)$ can be finite under a very restrictive condition only. Namely, we prove that if $F(P)$ is finite for a finite halflattice P then the set of the elements of $F(P) - P$ that can be expressed as $x + y$ for some $x, y \in P$ is a chain of at most four elements. And we give an example showing that the number four is possible in this context.

For the terminology and notation see our paper [**3**]; here we shall only briefly recall the construction of the free lattice $F(P)$ over a partial lattice P. The algebra of terms over P is denoted by $T(P)$. For every term t define an ideal $\downarrow t$ and a filter $\uparrow t$ of P by

$\downarrow t = \{a \in P; a \leq t\}$ and $\uparrow t = \{a \in P; a \geq t\}$ for $t \in P$,

$\downarrow t = \downarrow t_1 \vee \downarrow t_2$ and $\uparrow t = \uparrow t_1 \cap \uparrow t_2$ for $t = t_1 + t_2$,

$\downarrow t = \downarrow t_1 \cap \downarrow t_2$ and $\uparrow t = \uparrow t_1 \vee \uparrow t_2$ for $t = t_1 t_2$.

Define a binary relation $\leq$ on $T(P)$ as follows: if $u \in P$ and $v \in T(P)$ then $u \leq v$ iff $u \in \downarrow v$; if $u \in T(P)$ and $v \in P$ then $u \leq v$ iff $v \in \uparrow u$; if $u = u_1 + u_2$ then $u \leq v$ iff $u_1 \leq v$ and $u_2 \leq v$; if $v = v_1 v_2$ then $u \leq v$ iff $u \leq v_1$ and $u \leq v_2$; if $u = u_1 u_2$ and $v = v_1 + v_2$ then $u \leq v$ iff either $u \leq v_1$ or $u \leq v_2$ or $u_1 \leq v$ or $u_2 \leq v$ or $u \leq a \leq v$ for an element $a \in P$. Then $\leq$ is a quasiordering and the relation $\sim$ on $T(P)$ defined by $u \sim v$ iff $u \leq v$ and $v \leq u$ is a congruence. The free lattice over P is isomorphic to $T(P)/\sim$.

1. GENERAL PARTIAL LATTICES. Let P be a partial lattice and a, b, c, d be elements of P such that

(1) $a \parallel c$, $a \parallel d$, $b \parallel c$;

(2) either $b = d$ or else $b < a$ and $d < c$.

Define elements $t_0, t_1, t_2, \ldots$ of P as follows:

$t_0 = a + d$;

$t_i = b + c t_{i-1}$ for i odd;

$t_i = d + a t_{i-1}$ for $i \geq 2$ even.

We have $a + b = t_0 \geq t_1 \geq t_2 \geq \cdots \geq b, d$.

1.1. Lemma. *Let $i \geq 0$ be such that $t_i = t_{i+1}$. Then $t_{i+1} = t_{i+2}$.*

Proof: If $i = 0$ then $t_2 = d + at_1 = d + at_0 = d + a = t_0$. If $i \geq 2$ is even then $t_{i+2} = d + at_{i+1} = d + at_i = d + at_{i-1} = t_i$. If i is odd then $t_{i+2} = b + ct_{i+1} = b + ct_i = b + ct_{i-1} = t_i$. □

1.2. Lemma. *Let $i \geq 0$ be such that $\uparrow t_i = \uparrow t_{i+1}$. Then $\uparrow t_{i+1} = \uparrow t_{i+2}$.*

Proof: Suppose, on the contrary, that there exists an element $x \in P$ with $x \geq t_{i+2}$ and $x \not\geq t_{i+1}$.

Let $i = 0$. We have $x \geq t_2 = d + at_1$, so that $x \geq d$ and $x \geq at_1$. We have $x \in \uparrow a \vee \uparrow t_1 = \uparrow a \vee \uparrow t_0 = \uparrow a$, so that $x \geq a$ and consequently $x \geq a + d = t_0 \geq t_1$, a contradiction.

Let i be odd. We have $x \geq t_{i+2} = b + ct_{i+1}$, so that $x \geq b$ and $x \geq ct_{i+1}$. We have $x \in \uparrow c \vee \uparrow t_{i+1} = \uparrow c \vee \uparrow t_i = \uparrow (ct_i)$. Hence $x \geq ct_i = ct_{i-1}$ and so $x \geq b + ct_{i-1} = t_i \geq t_{i+1}$, a contradiction.

Let $i \geq 2$ be even. We have $x \geq t_{i+2} = d + at_{i+1}$, so that $x \geq d$ and $x \geq at_{i+1}$. We have $x \in \uparrow a \vee \uparrow t_{i+1} = \uparrow a \vee \uparrow t_i = \uparrow (at_i)$. Hence $x \geq at_i = at_{i-1}$ and so $x \geq d + at_{i-1} = t_i \geq t_{i+1}$, a contradiction. □

1.3. Lemma. *Let $i \geq 0$ be such that $\uparrow t_i = \uparrow t_{i+1}$ and $t_{i+1} > t_{i+2}$. Then $t_{i+2} > t_{i+3}$.*

Proof: By 1.1 we have $t_0 > t_1 > \cdots > t_{i+2}$ and by 1.2 we have $\uparrow t_i = \uparrow t_{i+1} = \uparrow t_{i+2} = \ldots$.

Let us prove $a \not\leq t_1$. If $a \leq t_1$ then $t_2 = d + at_1 = d + a = t_0$, a contradiction.

Let us prove $c \not\leq t_2$. If $c \leq t_2$ then $t_2 \geq b + c \geq t_1$, a contradiction.

Suppose $t_{i+2} = t_{i+3}$.

Let i be even. Then we have $at_{i+1} \leq t_{i+3} = b + ct_{i+2}$. There are five cases.

Case 1: $a \leq t_{i+3}$. Then $a \leq t_1$, a contradiction.

Case 2: $t_{i+1} \leq t_{i+3}$. Then $t_{i+1} \leq t_{i+2}$, a contradiction.

Case 3: $at_{i+1} \leq b$. Then $b \in \uparrow a \vee \uparrow t_{i+1} = \uparrow a \vee \uparrow t_i$ and so $b \geq at_i = at_{i-1}$. If $i = 0$ then we get $b \geq a$, a contradiction. If $i > 0$ and $b = d$ then $b \geq b + at_{i-1} = t_i$, so that $t_i = t_{i+1}$, a contradiction. If $i > 0$ and $b < a$ and $d < c$ then $t_{i+1} = b + ct_i \geq at_{i-1} + d = t_i$, a contradiction.

Case 4: $at_{i+1} \leq ct_{i+2}$. Then $at_{i+1} \leq c$, $c \in \uparrow a \vee \uparrow t_{i+1} = \uparrow a \vee \uparrow t_i$, $c \geq at_i$. If $i = 0$, we get $c \geq a$, a contradiction. If $i > 0$ then we get $ct_i \geq at_i = at_{i-1}$, $t_{i+1} = b + ct_i \geq ct_i \geq at_{i-1}$, $t_{i+1} \geq d + at_{i-1} = t_i$, a contradiction.

Case 5: $at_{i+1} \leq x \leq t_{i+3}$ for some $x \in P$. We have $x \in \uparrow a \vee \uparrow t_{i+1} = \uparrow a \vee \uparrow t_i$, so that $x \geq at_i$. If $i = 0$, we get $a \leq x \leq t_3 \leq t_1$, a contradiction. If $i > 0$ then $x \geq at_i = at_{i-1}$, so that $t_{i+3} \geq d + at_{i-1} = t_i$, a contradiction.

Let i be odd. Then we have $ct_{i+1} \leq t_{i+3} = d + at_{i+2}$. There are five cases.

Case 1: $c \leq t_{i+3}$. Then $c \leq t_2$, a contradiction.

Case 2: $t_{i+1} \leq t_{i+3}$. Then $t_{i+1} \leq t_{i+2}$, a contradiction.

Case 3: $ct_{i+1} \leq d$. Then $d \in \uparrow c \vee \uparrow t_{i+1} = \uparrow c \vee \uparrow t_i$ and so $d \geq ct_i = ct_{i-1}$. If $b = d$ then $d \geq b + ct_{i-1} = t_i$, so that $t_i = t_{i+1}$, a contradiction. If $b < a$ and $d < c$ then $t_{i+1} = d + at_i \geq ct_{i-1} + b = t_i$, a contradiction.

Case 4: $ct_{i+1} \leq at_{i+2}$. Then $ct_{i+1} \leq a$, $a \in \uparrow c \vee \uparrow t_{i+1} = \uparrow c \vee \uparrow t_i$, $a \geq ct_i$, $at_i \geq ct_i = ct_{i-1}$, $t_{i+1} = d + at_i \geq at_i \geq ct_{i-1}$, $t_{i+1} \geq b + ct_{i-1} = t_i$, a contradiction.

Case 5: $ct_{i+1} \leq x \leq t_{i+3}$ for some $x \in P$. We have $x \in \uparrow c \vee \uparrow t_{i+1} = \uparrow c \vee \uparrow t_i$, so that $x \geq ct_i = ct_{i-1}$ and $t_{i+3} \geq ct_{i-1}$; hence $t_{i+3} \geq b + ct_{i-1} = t_i$, a contradiction. □

1.4. Lemma. *Let $i \geq 0$ be such that $\uparrow t_i = \uparrow t_{i+1}$ and $t_{i+1} > t_{i+2}$. Then F(P) is infinite.*

Proof: It follows easily from 1.2 and 1.3. □

2. HALFLATTICES: TWO INCOMPARABLE UNDEFINED JOINS.

2.1. Lemma. *Let P be a finite halflattice and a, b, c, d be four elements of P such that the following four conditions are satisfied:*

(1) $a \parallel c$, $a \parallel d$, $b \parallel c$;

(2) *either* $b = d$ *or else* $b < a$ *and* $d < c$*;*
(3) $a + d \notin P$ *and* $b + c \notin P$*;*
(4) $a \not\leq b + c$ *and* $c \not\leq a + d$.

Then $F(P)$ *is infinite.*

Proof: Define the elements t_i as in Section 1, so that $t_0 = a+d$, $t_1 = b+ct_0$ and $t_2 = d+at_1$. If $t_0 \leq t_1$ then $a \leq a + d \leq b + c(a + d) \leq b + c$, a contradiction. We get $t_0 > t_1$. Since $\uparrow t_1 = \uparrow b \cap (\uparrow c \vee \uparrow (a+d)) = \uparrow b \cap (\uparrow c \vee \emptyset) = \uparrow b \cap \uparrow c = \emptyset$, by 1.4 it is sufficient to prove $t_1 > t_2$. Suppose $t_1 \leq t_2$. Then $ct_0 \leq d + at_1$ and there are five possible cases.

Case 1: $c \leq t_2$. Then $c \leq a + d$, a contradiction.

Case 2: $t_0 \leq t_2$. Then $t_0 \leq t_1$, a contradiction.

Case 3: $ct_0 \leq d$. Then $d \in \uparrow c \vee \uparrow t_0 = \uparrow c \vee \emptyset = \uparrow c$, so that $d \geq c$, a contradiction.

Case 4: $ct_0 \leq at_1$. Then $ct_0 \leq a$; as in Case 3, we get $a \geq c$, a contradiction.

Case 5: $ct_0 \leq x \leq t_2$ for some $x \in P$. Then $x \in \uparrow c \vee \uparrow t_0 = \uparrow c$, $c \leq x \leq t_2 \leq a + d$, a contradiction.

We get a contradiction in all cases. □

2.2. Lemma. *Let* P *be a finite halflattice and* $a, b, c \in P$ *be such that* $a + b \notin P$, $b + c \notin P$ *and* $a + b \parallel b + c$. *Then* $F(P)$ *is infinite.*

Proof: It follows from 2.1. □

2.3. Lemma. *Let* P *be a finite halflattice and* $a, b, c, d \in P$ *be such that*

(1) $a + b \notin P$, $c + d \notin P$, $a + b \parallel c + d$*;*
(2) $b < c$*;*
(3) $b + d \notin P$.

Then $F(P)$ *is infinite.*

Proof: If $d < a$ then we can apply 2.1 to the quadruple a, d, c, b. So, we can suppose that the elements a, c, d are pairwise incomparable. If $d \not\leq a + c$ then we can apply 2.2 to the triple a, c, d; so, let $d \leq a + c$. If $d \not\leq a + b$ then we can apply 2.2 to the triple a, b, d; so, let $d \leq a + b$. If $a + d \notin P$ then we can apply 2.2 to the triple a, d, c; so, let $a + d \in P$. Now we can apply 2.1 to the quadruple $c, b, a + d, d$. □

2.4. Lemma. *Let* P *be a finite halflattice and* $a, b, c, d \in P$ *be such that*

(1) $a + b \notin P$, $c + d \notin P$, $a + b \parallel c + d$*;*
(2) $b \leq cd$*;*
(3) *whenever* $x \in P$ *and* $x \leq (a + b)c$ *then* $x \leq b$*;*
(4) *whenever* $x \in P$ *and* $x \leq (a + b)d$ *then* $x \leq b$.

Then $F(P)$ *is infinite.*

Proof: Consider the three pairwise incomparable elements $a, (a+b)c, (a+b)d$ of the relative sublattice $Q = P \cup \{a + b, (a + b)c, (a + b)d\}$ of $F(P)$. Put $t_0 = a + (a + b)c = a + b$, $t_1 = (a + b)d + (a + b)c$, $t_2 = t_1a + (a + b)c$. In Q we have $\uparrow t_0 = \uparrow t_1 = \{a + b\}$, so that by 1.4 it is sufficient to prove $t_0 > t_1 > t_2$.

If $t_0 \leq t_1$ then $a \leq (a+b)d + (a+b)c$, so that in P we have $a \in \downarrow (a+b)d \vee \downarrow (a+b)c = \downarrow b \vee \downarrow b = \downarrow b$; but $a \leq b$ is impossible. We get $t_0 > t_1$.

Suppose $t_1 \leq t_2$. Then $(a + b)d \leq t_1a + (a + b)c$ and we have five possible cases.

Case 1: $(a + b)d \leq t_1a$. Then $b \leq (a + b)d \leq a$, a contradiction.

Case 2: $(a + b)d \leq (a + b)c$. This is impossible.

Case 3: $a + b \leq t_2$. Then $a \leq t_2 \leq t_1$, $t_0 \leq t_1$, a contradiction.

Case 4: $d \leq t_2$. Then $d \leq a + b$, so that $d \leq b$ by (4) and consequently $d \leq c$, a contradiction.

Case 5: $(a+b)d \leq x \leq t_2$ for some $x \in P$. Then $x \in \uparrow (a+b) \vee \uparrow d = \uparrow d$, $d \leq t_2 \leq a+b$, a contradiction. □

2.5. Lemma. *Let* P *be a finite halflattice and* $a, b, c, d \in P$ *be such that*

(1) $a + b \notin P$, $c + d \notin P$, $a + b \parallel c + d$*;*
(2) a, b, c, d *are not pairwise incomparable.*

Then $F(P)$ is infinite.

Proof: We can suppose that a, b, c, d is a maximal quadruple with respect to these two properties. Further, we can suppose that $b < c$. By 2.3 we can assume that $b + d \in P$. Consider the quadruple $a, b, c, b + d$; by the maximality of a, b, c, d we get $b + d = d$ and hence $b \leq cd$. Let $x \in P$ and $x \leq (a + b)c$. Then the element $y = x + b$ belongs to P (since $x, b \leq c$) and $b \leq y \leq (a + b)c$. If $y > b$ then we can take the quadruple a, y, c, d; by the maximality of a, b, c, d we get $y = b$. But then $y \leq b$ and the condition (3) of 2.4 is satisfied. Similarly one can prove that the condition (4) of 2.4 is satisfied. By 2.4 we obtain that $F(P)$ is infinite. □

2.6. Lemma. *Let P be a finite halflattice and $a, b, c, d \in P$ be such that*

(1) $a + b \notin P$, $c + d \notin P$, $a + b \parallel c + d$;
(2) $a \not\leq c + d$, $c \not\leq a + b$;
(3) $b + c \notin P$.

Then $F(P)$ is infinite.

Proof: Consider the three elements $a(c + d)$, $b(c + d)$ and c of the relative sublattice $Q = P \cup \{c + d, a(c + d), b(c + d)\}$ of $F(P)$. Put $t_0 = a(c + d) + b(c + d)$, $t_1 = t_0c + b(c + d)$, $t_2 = t_1a(c + d) + b(c + d) = t_1a + b(c + d)$. In Q we have $\uparrow t_0 = \uparrow t_1 = \{c + d\}$ and so by 1.4 it is sufficient to prove $t_0 > t_1 > t_2$. If $t_0 \leq t_1$ then $a(c + d) \leq t_0c + b(c + d)$; in each of the five possible cases we get easily a contradiction. Similarly, we cannot have $t_1 \leq t_2$. □

2.7. Lemma. *Let P be a finite halflattice and $a, b, c, d \in P$ be such that*

(1) $a + b \notin P$, $c + d \notin P$, $a + b \parallel c + d$.

Then $F(P)$ is infinite.

Proof: Let a, b, c, d be a maximal quadruple with the property (1). By 2.5 we can assume that a, b, c, d are pairwise incomparable. Since $a + b \parallel c + d$, we can suppose that $a \not\leq c + d$ and $c \not\leq a + b$. By 2.6 it is sufficient to consider the case when $b + c \in P$. If $b \leq c + d$ then $a, b, b + c, d$ is a quadruple contradicting the maximality of a, b, c, d; hence $b \not\leq c + d$.

Let there exist an element $x \in P$ such that $x \leq (a + b)(c + d)$, $x \not\leq b$ and $x \not\leq c$. If $x + b \in P$ then the quadruple $a, x + b, c, d$ contradicts the maximality of a, b, c, d. Hence $x + b \notin P$ and similarly $x + c \notin P$. Using $b \not\leq c + d$ and $c \not\leq a + b$ we get $x + b \parallel x + c$; by 2.2, $F(P)$ is infinite. So, we can assume that whenever x is an element of P such that $x \leq (a + b)(c + d)$ then either $x \leq b$ or $x \leq c$.

If $a \leq (a+b)(c+d)+b$ then $a \in {\downarrow}(a+b)(c+d) \vee {\downarrow}b \subseteq ({\downarrow}b \vee {\downarrow}c) \vee {\downarrow}b = {\downarrow}b \vee {\downarrow}c = {\downarrow}(b+c)$, so that $a \leq b + c$ and the elements a, b have a common upper bound $b + c$ in P, a contradiction. We get $a \not\leq (a + b)(c + d) + b$.

Consider the elements a, b and $c + d$ of the relative sublattice $Q = P \cup \{c + d\}$ of $F(P)$. Put $t_0 = a + b$, $t_1 = (a + b)(c + d) + b$ and $t_2 = t_1a + b$. We have $\uparrow t_0 = \uparrow t_1 = \emptyset$ in Q, so that by 1.4 it is sufficient to prove $t_0 > t_1 > t_2$. As we have proved, $a \not\leq t_1$ and so $t_0 \not\leq t_1$. If $t_1 \leq t_2$ then $(a + b)(c + d) \leq t_1a + b$; in each of the five possible cases we get easily a contradiction; hence $t_1 > t_2$. □

3. HALFLATTICES: A CHAIN OF FIVE UNDEFINED JOINS.

For a finite halflattice P we denote by $UJ(P)$ the set of the elements $u \in F(P) - P$ such that $u = x + y$ for some $x, y \in P$.

For $u \in F(P)$ and $a \in P$ denote by $u \odot a$ the greatest element $x \in P$ with the properties $x \leq p$ and $x \leq a$ (its existence is clear).

3.1. Lemma. *Let P be a finite halflattice such that $F(P)$ is finite. Let p, q be two elements of $UJ(P)$ with $p < q$ and let a, b, c be three elements of P with $q = a + b$ and $p = b + c$. Then $b + (p \odot a) = p$.*

Proof: Put $d = p \odot a$. If $c \leq a$ then $b + d = p$ is clear. Consider the opposite case; then a, b, c are pairwise incomparable. Put

$t_0 = p = b + c$,
$t_i = t_{i-1}a + b$ for i odd,
$t_i = t_{i-1}c + b$ for $i \geq 2$ even.

We have $\uparrow t_i = \emptyset$ for all i.

Let us prove that if $t_0 > t_1$ then $t_1 > t_2$. If $t_1 \le t_2$ then $pa \le t_1 c + b$ and there are only five cases possible.

Case 1: $pa \le t_1 c$. Then $pa \le c$ and $c \in \uparrow(pa) = \uparrow a$, a contradiction.

Case 2: $pa \le b$. Then $b \in \uparrow(pa) = \uparrow a$, a contradiction.

Case 3: $p \le t_2$. Then $t_0 \le t_1$, a contradiction.

Case 4: $a \le t_2$. Then $a \le p$, a contradiction.

Case 5: $pa \le x \le t_2$ for some $x \in P$. Then $x \in \uparrow(pa) = \uparrow a$ and $a \le x \le t_2 \le p$, a contradiction.

It follows from 1.4 that $t_0 = t_1$. Hence $c \le pa + b$. From this we get $c \in \downarrow(pa) \vee \downarrow b = \downarrow d \vee \downarrow b$, so that $c \le b + d$; but then $b + d = p$. □

3.2. Lemma. *Let P be a finite halflattice such that $F(P)$ is finite. Let p, q, r be three elements of $UJ(P)$ such that $p < q < r$ and let a, b, c be three elements of P such that $r = a + b$ and $p = b + c$. Then $b + (q \odot a) = q$.*

Proof: Put $d = q \odot a$. By 3.1 we can suppose that $c < a$; then $c \le d$. By 2.7, $UJ(P)$ is a finite chain. Denote by q_0 the predecessor of q in this chain. Since $q \in UJ(P)$, there exists an element $e \in P$ with $e < q$ and $e \not\le q_0$; let us take a maximal element e with these properties. If $b \not\le e$ then $b + e = q$ and $b + d = q$ follows from 3.1. So, let $b < e$. We have $c \not\le e$ (since b, c have no upper bound in P) and $q = c + e$.

Consider the quadruple e, b, a, c. Put

$t_0 = q = e + c$,

$t_i = t_{i-1} a + b$ for i odd,

$t_i = t_{i-1} e + c$ for $i \ge 2$ even.

We have $\uparrow t_i = \emptyset$ for all i.

Let us prove that if $t_0 > t_1$ then $t_1 > t_2$. If $t_1 \le t_2$ then $qa \le t_1 e + c$ and one of the following five cases must take place.

Case 1: $qa \le t_1 e$. Then $qa \le e$ and $e \in \uparrow(qa) = \uparrow a$, a contradiction.

Case 2: $qa \le c$. Then $c \in \uparrow(qa) = \uparrow a$, a contradiction.

Case 3: $q \le t_2$. Then $t_0 \le t_1$, a contradiction.

Case 4: $a \le t_2$. Then $a \le q$, a contradiction.

Case 5: $qa \le x \le t_2$ for some $x \in P$. Then $a \le x \le t_2 \le q$, a contradiction.

By 1.4 we have proved $t_0 = t_1$, so that $e \le qa + b$. We get $e \in \downarrow(qa) \vee \downarrow b = \downarrow d \vee \downarrow b$, $e \le b + d$ and consequently $b + d = q$. □

3.3. Lemma. *Let P be a finite halflattice. If there exist three elements u, v, w of $UJ(P)$ with $u < v < w$ and three elements a, b, c of P with $a < b < c$, $a < w$, $a \not\le v$ and $b \not\le w$ then $F(P)$ is infinite.*

Proof: There are two elements $x, y \in P$ with $u = x + y$. If $av \le u = x + y$ then there are only five cases possible and we get a contradiction in each of them. Hence $av \not\le u$. Put

$t_0 = av$,

$t_i = (t_{i-1} + cu)b$ for i odd,

$t_i = (t_{i-1} + a)v$ for $i \ge 2$ even.

We have $t_i \le bv$ for all i and $t_0 \le t_1 \le t_2 \le \ldots$; further, $\uparrow t_0 = \uparrow a$ and $\uparrow t_i = b$ for $i \ge 1$.

If $t_1 \le t_0$ then $t_1 \le a$, a contradiction. We get $t_0 < t_1$. Now, we can prove $t_i < t_{i+1}$ by induction for all i. If i is even and $t_{i+1} \le t_i$ then $(t_i + cu)b \le t_{i-1} + a$ and we are in one of the following five cases.

Case 1: $t_{i+1} \le t_{i-1}$. Then $t_i \le t_{i-1}$, a contradiction by induction.

Case 2: $t_{i+1} \le a$. Then $a \in \uparrow b$, a contradiction.

Case 3: $t_i + cu \le t_{i-1} + a$. Then $cu \le t_{i-1} + a \le b$, so that $b \in \uparrow(cu) = \uparrow c$, a contradiction.

Case 4: $b \le t_{i-1} + a$. Then $b \le w$, a contradiction.

Case 5: $t_{i+1} \le x \le t_{i-1} + a$ for some $x \in P$. Then $b \le x \le w$, a contradiction.

If $i \ge 3$ is odd and $t_{i+1} \le t_i$ then $(t_i + a)v \le t_{i-1} + cu$ and the five cases are:

Case 1: $t_{i+1} \le t_{i-1}$. Then $t_i \le t_{i-1}$, a contradiction by induction.

Case 2: $t_{i+1} \le cu$. Then $av = t_0 \le cu \le u$, but we have proved $av \not\le u$ above.

Case 3: $t_i + a \le t_{i-1} + cu$. Then $a \le t_{i-1} + cu \le v$, a contradiction.

Case 4: $v \le t_{i-1} + cu$. Then $v \le c$, a contradiction with $v \in UJ(P)$.

Case 5: $t_{i+1} \le x \le t_{i-1} + cu$ for some $x \in P$. Then $b \le x \le w$, a contradiction. □

3.4. Lemma. *Let P be a finite halflattice. If $UJ(P)$ is a chain of at least five elements then $F(P)$ is infinite.*

Proof: Let $u < v < w < r < s$ be the first five elements of $UJ(P)$. We have $u = x + y$ for some $x, y \in P$. Since $s \in UJ(P)$, there exists an element $c \in P$ with $c < s$ and $c \not\le r$; we can assume that c is maximal with these properties. Since c cannot be an upper bound of both x and y, we can assume that $x \not\le c$; then $s = c + x$. Two applications of 3.2 yield the existence of two elements b and a in P such that $b < c$, $r = x + b$, $a < b$, $w = x + a$. The assumptions of 3.3 are evidently satisfied, so that $F(P)$ is infinite. □

4. THE MAIN RESULTS. The following is a consequence of lemmas 2.7 and 3.4:

4.1. THEOREM. *Let P be a finite halflattice. If the free lattice $F(P)$ over P is finite then the set $UJ(P)$ of the elements $u \in F(P) - P$ that are of the form $u = x + y$ for some $x, y \in P$ is an at most four-element chain.* □

4.2. Example. There exist finite halflattices P such that $UJ(P)$ is a chain of exactly four elements. In figures 1 and 2 we present two such examples. In the first of them, P and $F(P)$ are of cardinalities 8 and 29, respectively, and in the second example they are of cardinalities 25 and 58. In both cases full dots represent the elements of P, while blank dots stand for the elements of $F(P) - P$; it is a mechanical task to verify that the pictured lattice is free over the subset consisting of the full dots.

4.3. Example. If P is a finite halflattice such that $UJ(P)$ consists of one element only then $F(P) = P \cup UJ(P)$ is finite. On the other hand, there exist finite halflattices P such that $UJ(P)$ is a two-element chain and $F(P)$ is infinite. For example, the fourteen-element halflattice obtained from the sixteen-element Boolean algebra by omitting the greatest element and one of the coatoms has this property.

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