ON REPRESENTABLE MAPPINGS OF SEMIGROUPS INTO CARDINALS

Jaroslav Ježek

Let S be a semigroup and f be a mapping of S into the class of nonzero cardinal numbers. The mapping f is said to be representable if there exist a groupoid G and a homomorphism h of G onto S such that $\mathrm{Ker}(h)$ is the least congruence of G for which the corresponding factor is a semigroup and $f(a) = \mathrm{Card}\, h^{-1}(a)$ for all $a \in S$.

The investigation of representable mappings (see the papers [2] and [3]) is closely connected with and originates from the study of the notion of associativity semihypergroup, which was introduced in [4] and further studied e.g. in [1].

The purpose of the present paper is to introduce a new condition (C), necessary for the representability of a mapping f on a semigroup S; our condition, which is a refinement of a similar condition from [3], turns out to be also sufficient on a large class of semigroups. The necessity and the (restricted) sufficiency of (C) will be the two main results of this paper. In their proofs we shall make use of the following rather simple observation of set-theoretical character, which we are not going to prove.

LEMMA. *Let A be a nonempty set and $\mathcal{K}$ be a system of pairwise disjoint nonempty sets. The following two conditions are equivalent:*

(1) *there exists a mapping g of $\bigcup\mathcal{K}$ onto A such that $A \times A$ is the only equivalence on A containing all the relations $g(K) \times g(K) \quad (K \in \mathcal{K})$;*
(2) $\mathrm{Card}\, A + \mathrm{Card}\, \mathcal{K} \leq 1 + \mathrm{Card} \bigcup \mathcal{K}$.

For a semigroup S and an element $a \in S$ we denote by M_a the set of the pairs (b, c) of elements of S such that $bc = a$; denote by E_a the equivalence on M_a generated by the pairs $((uv, w), (u, vw))$ where u, v, w are elements with $uvw = a$; and denote by ν_a the number of the blocks of E_a. (If M_a is empty then $\nu_a = 0$.)

We introduce the following condition for a mapping f of S into the class of nonzero cardinals:

$$\text{(C)} \qquad f(a) + \nu_a \leq 1 + \sum_{(b,c)\in M_a} f(b)f(c) \qquad \text{for any } a \in S.$$

THEOREM 1. *Let S be a semigroup and f be a mapping of S into the class of nonzero cardinal numbers. If f is representable then it satisfies the condition (C).*

Proof. Let G be a groupoid and h be a homomorphism of G onto S with the properties formulated above. For an element $a \in S$ such that $f(a) = 1$ the inequality in (C) is trivially true; we shall therefore consider the case $f(a) \geq 2$ only.

Define a binary relation s on G by $(u, v) \in s$ iff $(u, v) \in \mathrm{Ker}(h)$ and if $u, v \in h^{-1}(a)$ then either $u = v$ or $u, v \in GG$. One can easily prove that s is a congruence of G, $s \subseteq \mathrm{Ker}(h)$ and G/s is a semigroup. Consequently, $s = \mathrm{Ker}(h)$ and we have thus proved that $h^{-1}(a) \subseteq GG$.

Define a binary relation r on G as follows: $(u, v) \in r$ iff $(u, v) \in \mathrm{Ker}(h)$ and if $u, v \in h^{-1}(a)$ then there exists a finite sequence $u_0, \dots, u_k$ $(k \geq 0)$ of elements of $h^{-1}(a)$ such that $u_0 = u$, $u_k = v$ and such that for any $i \in \{1, \dots, k\}$ there exist elements $b, c, d, e \in G$ with $u_{i-1} = bc$, $u_i = de$ and $((h(b), h(c)), (h(d), h(e))) \in E_a$.

It is easy to see that r is an equivalence on G. It is a congruence, since if $(u, v) \in r$ then in the case $uw, vw \in h^{-1}(a)$ we can put $k = 1, u_0 = uw, u_1 = vw, b = u, c = w, d = v$ and $e = w$ to obtain $(uw, vw) \in r$ (we have $(h(b), h(c)) = (h(d), h(e))$). In order to be able to assert that G/r is a semigroup, we have to prove $(uv.w, u.vw) \in r$ for all $u, v, w \in G$. We have, of course, $(uv.w, u.vw) \in \mathrm{Ker}(h)$. Let both $uv.w$ and $u.vw$ belong to $h^{-1}(a)$. We can put $k = 1, u_0 = uv.w, u_1 = u.vw, b = uv, c = w, d = u, e = vw$ to obtain $(uv.w, u.vw) \in r$.

Since $\mathrm{Ker}(h)$ is the least congruence with the property that the corresponding factor is a semigroup, we get $r \supseteq \mathrm{Ker}(h)$ and consequently $r = \mathrm{Ker}(h)$. We have proved that whenever $u, v \in h^{-1}(a)$ then there exists a finite sequence $u_0, \dots, u_k$ $(k \geq 0)$ of elements of $h^{-1}(a)$ such that $u_0 = u, u_k = v$ and such that for any $i \in \{1, \dots, k\}$ there exist elements $b, c, d, e \in G$ with $u_{i-1} = bc, u_i = de$ and $((h(b), h(c)), (h(d), h(e))) \in E_a$.

For any block B of E_a denote by K_B the set of the elements $e \in h^{-1}(a)$ such that $e = bc$ for some b, c with $(h(b), h(c)) \in B$. From what we have proved it follows that the system $\mathcal{K}$ of the sets K_B has the following properties: its union equals $h^{-1}(a)$; and $h^{-1}(a) \times h^{-1}(a)$ is the only equivalence on $h^{-1}(a)$ containing all the relations $K_B \times K_B$. It need not be a system of pairwise disjoint sets, but we can take its disjoint union and the canonical mapping g of this disjoint union onto $h^{-1}(a)$; condition (1) of the Lemma is clearly satisfied and so we get $\mathrm{Card}\, h^{-1}(a) + \nu_a \leq 1 + \sum_{K \in \mathcal{K}} \mathrm{Card}(K)$; but $\mathrm{Card}\, h^{-1}(a) = f(a)$ and $\sum_{K \in \mathcal{K}} \mathrm{Card}(K) \leq \sum_{(b,c) \in M_a} f(b) f(c)$. $\square$

Let S be a semigroup and a be an element of S. If there exists a largest positive integer n with the property that $a = a_1 \dots a_n$ for some n–tuple $a_1, \dots, a_n$ of elements of S then this number n is called the breadth of a; in the opposite case a is said to be of infinite breadth. (Notice that a semigroup without elements of infinite breadth is necessarily infinite.)

THEOREM 2. *Let S be a semigroup (which may but need not contain a zero) in which every nonzero element is of finite breadth. A mapping f of S into the class of nonzero cardinals is representable iff it satisfies the condition (C).*

Proof. Let (C) be satisfied. For every element $a \in S$ take a set A_a of cardinality $f(a)$ and denote by G the disjoint union of the sets $A_a \quad (a \in S)$. Define a mapping h of G onto S by $h(x) = a$ for $x \in A_a$.

Let a be a nonzero element of SS. We get from (C) that Condition (2) of the Lemma is satisfied for the system $\mathcal{K} = \{\bigcup\{A_b \times A_c; (b,c) \in B\}; B \in M_a/E_a\}$. Consequently, there exists a mapping g_a of the union $\bigcup\{A_b \times A_c; (b,c) \in M_a\}$ onto A_a such that $A_a \times A_a$ is the only equivalence on A_a containing, for any block B of E_a, the relation $(\bigcup\{g_a(A_b \times A_c); (b,c) \in B\})^2$. For $x \in A_b$ and $y \in A_c$, where $(b,c) \in M_a$, put $xy = g_a(x,y)$.

So far, we have defined xy for all the pairs $x, y \in G$ such that $x \in A_b$ and $y \in A_c$ where bc is a nonzero element. If S has no zero element, G has become a groupoid. In case when S contains a zero element 0, we need to complete the definition by considering the pairs $x \in A_b, y \in A_c$ with $bc = 0$. Take a fixed element $o \in A_0$ and define xy by $xo = x$ if $x \in A_0$ and $xy = o$ in the remaining cases.

Clearly, h is a homomorphism of G onto S and it remains to prove that if r is a congruence of G containing all the pairs $(xy.z, x.yz)$, with $x, y, z \in G$, then $r \supseteq \mathrm{Ker}(h)$.

We have to prove $r \supseteq A_a \times A_a$ for any element $a \in S$. If S contains a zero 0 then $r \supseteq A_0 \times A_0$ is easy to see: for any element $x \in A_0 - \{o\}$ we have $xo.x = o$ and $x.ox = x$, so that $(o, x) \in r$. (We could also say that the subgroupoid A_a of G is contra-associative, according to [2].)

So, it remains to prove $r \supseteq A_a$ for any nonzero element a of S. This will be done by induction on the breadth of a. If a is of breadth 1 (i.e., if $a \in S - SS$), then $f(a) = 1$ and everything is clear. Let $a \in SS$. By induction we can suppose that $r \supseteq A_b \times A_b$ for any element $b \in S$ of breadth smaller than the breadth of a.

According to the construction of g_a, it is sufficient to prove that if B is a block of E_a and if (b,c) and (d,e) are two elements of B then $(xy, zu) \in r$ for any $x \in A_b, y \in A_c, z \in A_d, u \in A_e$. In other words, to prove that the equivalence E_a is contained in the binary relation E on M_a defined in this way: E is the set of the ordered pairs $(b,c),(d,e)) \in M_a \times M_a$ such that if $x \in A_b, y \in A_c, z \in A_d$ and $u \in A_e$ then $(xy, zu) \in r$.

By the definition of E_a, it is sufficient to prove that E is an equivalence relation containing all the pairs $((bc, d), (b, cd))$ where $b, c, d \in S$ are elements with $bcd = a$. The reflexivity of E can be verified in the following way: if $(b,c) \in M_a$ and $x \in A_b, y \in A_c, z \in A_b, u \in A_c$ then $(x,z) \in r$ and $(y,u) \in r$ (since b, c have smaller breadth than a), so that $(xy, zu) \in r$, which yields $((b,c),(b,c)) \in E$. The symmetry and the transitivity of E are easy to see. Let b, c, d be three elements of S with $bcd = a$. Take four elements $x \in A_{bc}, y \in A_d, z \in A_b, u \in A_{cd}$. Further, take an element $v \in A_c$. Since the elements bc and cd have smaller breadth than a, we have $(zv, x) \in r$ and $(vy, u) \in r$; we have $(zv.y, z.vy) \in r$ by the assumption on r; since r is a congruence, we get $(xy, zu) \in r$, which yields $((bc, d), (b, cd)) \in r$. $\square$

COROLLARY. *Let f be a mapping of the additive semigroup N of positive integers into the class of nonzero cardinals. Then f is representable iff*

$$f(1) = 1 \qquad \text{and} \qquad f(n) \leq \sum_{1 \leq i \leq n-1} f(i)f(n-i) \qquad \text{for all } n \geq 2.$$

This follows from the fact that for all $n \in N - \{1\}$ we have $\nu_n = 1$ (and $\nu_1 = 0$). The result also shows that the upper bound for the numbers $f(n)$ found in [2] — the sequence of the Catalan numbers — is the best possible.

Let us remark that the class of semigroups, for which the condition (C) proved to be both necessary and sufficient, includes all nilpotent semigroups and also all free semigroups and their subsemigroups.

We close the paper with two open problems.

PROBLEM 1. Find a semigroup S and a mapping of S into the class of nonzero cardinals which satisfies the condition (C) but is not representable.

PROBLEM 2. Characterize the class of the semigroups S such that any mapping of S into the class of nonzero cardinals is representable. (Consult [2] for partial results.) Is this class closed for some algebraic constructions?

References

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