THE TOP OF THE LATTICE OF CLONES OF QUASIPROJECTIONS

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This paper pertains to the theory of clones — closed sets of operations on a given finite set A. For the basic concepts cf. Á. Szendrei [3]. We shall be concerned with the clones of quasiprojections; quasiprojections are operations f (of an arbitrary arity n) such that $f(a_1,\dots,a_n) \in \{a_1,\dots,a_n\}$ for all $a_1,\dots,a_n \in A$. The aim of this paper is to give a complete description of the lattice of clones contained in the clone of quasiprojections and containing the clone generated by the binary quasiprojections on A. The description is given in Theorem 3. It turns out that the lattice is finite and its elements are in a one–to–one correspondence with (binary) antireflexive symmetric relations on A. (A relation is said to be antireflexive if it containes no ordered pair (a,b) such that $a=b$.)

On the other hand, the lattice of clones of quasiprojections on an at least three–element set is uncountable: this is proved in Theorem 4. (Let us remark that the paper [1] contains an incorrect proof of the same result.)

THEOREM 1. *Let A be a finite nonempty set. A binary relation on A is preserved by all the binary quasiprojections on A iff it is of one of the following three types:*

(1) *a subset of the diagonal $D=\{(a,a); a\in A\}$;*
(2) *a product $U\times V$ with $U,V\subseteq A$;*
(3) *a three–element subset $\{(a,a),(b,b),(a,b)\}$ where $a,b\in A$, $a\neq b$.*

Proof. Firstly, it is clear that every relation of any of these three types is preserved by any binary quasiprojection. Let S be a binary relation on A which is neither of type (1) nor of type (3). Put $U=\{a; \exists b\ (a,b)\in S\}$ and $V=\{b; \exists a\ (a,b)\in S\}$, so that $S\subseteq U\times V$. In order to prove that S is of type (2), take two elements $a\in U$, $b\in V$ and let us show that the pair (u,v) belongs to S.

If there exist elements a',b' such that $(a,a')\in S$, $(b',b)\in S$ and $(a,b')\neq(a',b)$ then we can take a binary quasiprojection f such that $f(a,b')=a$ and $f(a',b)=b$ and we get $(a,b)=(f(a,b'),f(a',b))\in S$. So, it is sufficient to derive a contradiction from the following assumption: whenever $(a,a')\in S$ then $a'=a$ and whenever $(b',b)\in S$ then $b'=b$.

Clearly, we have $(a,a)\in S$ and $(b,b)\in S$. If c,d is any pair such that $(c,d)\in S$ and $c\neq d$ then, taking an appropriate binary quasiprojection f, we get $(a,d)=(f(a,c),f(a,d))\in S$ and consequently $d=a$. Quite similarly, $c=b$. This shows that (b,a) is the only pair in S not belonging to the diagonal. Since S is not of type (3), there exists an element e different from both a and b such that $(e,e)\in S$. Since $(b,a)\in S$ and $(e,e)\in S$, with an appropriate binary quasiprojection f we get $(c,a)=(f(b,c),f(a,c))\in S$. As (b,a) is the only pair in S not belonging to the diagonal, we get the desired contradiction. □

THEOREM 2. *Let A be a finite nonempty set. An n–ary operation f on A belongs to the clone generated by the binary quasiprojections on A iff it is a quasiprojection satisfying the following condition: whenever $f(a_1,\dots,a_n)=a$ where $Card\{a_1,\dots,a_n\}=2$ and whenever $b_1,\dots,b_n$ is an n–tuple such that $a_i\neq b_i$ implies $b_i=a$ for any i then $f(b_1,\dots,b_n)=a$.*

Proof. Denote by Q the clone generated by the binary quasiprojections and by Q' the set of the operations satisfying the above condition. It is easy to verify that Q' is a clone; since all the binary quasiprojections are contained in Q', we get $Q\subseteq Q'$.

The clone Q contains a ternary majority operation $m(x,y,z)$: for example, take an arbitrary linear ordering on A and put $m(x,y,z)=\max(\min(x,y),\min(x,z),\min(y,z))$. Now, it is a well known fact that if a clone C contains a ternary majority operation then an operation belongs to C iff it preserves all the binary invariants of C (cf. [A.S.], Corollary 1.25). So, in order to prove $Q=Q'$, it remains to show that every operation from Q' preserves all the binary relations that are preserved by any binary quasiprojection on A. However, this follows from Theorem 1, as it is clear that all the three types (1), (2) and (3) of relations are preserved by the operations from Q'. □

THEOREM 3. *Let A be a finite nonempty set. Denote by Q the clone of quasiprojections and by Q' the clone generated by the binary quasiprojections on A. The interval $[Q',Q]$ in the lattice of clones on A is antiisomorphic to the lattice of (binary) antireflexive symmetric relations on A. The clone corresponding to a given antireflexive relation r can be described as follows: it consists of the quasiprojections preserving the relation $\{(a,a),(b,b),(a,b)\}$ for any $a,b\in r$; also, it is the clone generated by the binary quasiprojections together with the ternary quasiprojection f defined by*

$$f(x,y,z)=\begin{cases} z & \textit{for } x=y \textit{ and } \{x,z\}\notin r\\ x & \textit{otherwise.}\end{cases}$$

Proof. Every clone in the interval $[Q',Q]$ contains a ternary majority operation, since the clone Q' contains one, and so is uniquely determined by the set of the binary relations that it preserves. We have proved in Theorem 1 that the binary relations preserved by Q' are exactly the relations of types (1), (2) and (3). Now, it is easy to see that the binary relations preserved by Q are exactly the relations of types (1) and (2). From these facts we conclude that for every clone C in the interval $[Q',Q]$ there exists an antireflexive binary relation r on A such that C equals to the clone C_r of the quasiprojections preserving the relation $\{(a,a),(b,b),(a,b)\}$ for any $(a,b)\in r$. However, a quasiprojection preserves $\{(a,a),(b,b),(a,b)\}$ iff it preserves $\{(a,a),(b,b),(b,a)\}$. As a consequence, the clone C_r equals $C_{r'}$ for an antireflexive symmetric relation r'. Clearly, $r\subseteq s$ implies $C_r\supseteq C_s$ and it remains to prove that if r,s are two antireflexive symmetric relations such that $C_r\supseteq C_s$ then $r\subseteq s$. Take a pair $(a,b)\in r$. If $(a,b)\notin s$ then the ternary quasiprojection f defined by $f(a,a,b)=b$ and $f(x,y,z)=x$ for all $(x,y,z)\neq(a,a,b)$ belongs to C_s and consequently to C_r; but this is not possible, as f does not preserve $\{(a,a),(b,b),(a,b)\}$. We get $(a,b)\in s$. Since (a,b) was an arbitrary pair from r, this shows that $r\subseteq s$. The two characterisations of C_r follow easily. □

THEOREM 4. *Let A be a finite set of cardinality at least 3. Then the clone generated by the binary quasiprojections on A has uncountably many subclones.*

Proof. Let us fix three distinct elements $a, b, c \in A$ and take a linear ordering $\leq$ on A such that a, b, c are the top three elements with respect to $\leq$ and $a < b < c$. For every $n \geq 4$ define an n–ary operation f_n on A by

$$f_n(x_1, \dots, x_n) = \begin{cases} b & \text{if } \{x_1, \dots, x_n\} \subseteq \{a, b, c\} \text{ and } |\{i; x_i = a\}| = \\ & \qquad |\{i; x_i = c\}| = 1, \\ \min(x_1, \dots, x_n) & \text{otherwise.} \end{cases}$$

Quite easily, f_n belongs to the clone generated by the binary quasiprojections.

For $n \geq 4$ define an n–ary relation R_n on A by

$$R_n = (\{a, b\}^n - \{(b, \dots, b)\}) \cup \{(a, c, b, \dots, b), (b, a, c, b, \dots, b), \dots, (b, \dots, b, a, c), (c, b, \dots, b, a)\}.$$

It is easy to see that R_n is not preserved by f_n. On the other hand, we are going to prove that if $m \neq n$ (and $m, n \in \{4, 5, 6, \dots\}$) then R_n is preserved by f_m. Let $n \neq m$.

Let $M = (a_{i,j})$ be a matrix of type (n, m) whose every column represents an n–tuple belonging to R_n (so that all the elements of M belong to $\{a, b, c\}$). For $i = 1, \dots, n$ denote by d_i the result of f_m applied to the i–th row. We need to prove $(d_1, \dots, d_n) \in R_n$.

If $d_i = c$ for some i then all the members of the i–th row are equal to c; but then we easily infer that all the columns of M are equal to the (only) n–tuple from R_n having c at its i-th place; consequently, $(d_1, \dots, d_n)$ is equal to an arbitrary column of M; but this arbitrary column belongs to R_n, so that $(d_1, \dots, d_n)$ belongs to R_n, as well.

We can now assume that $d_i \neq c$ for all i. This means that $(d_1, \dots, d_n) \in R_n$ and it is sufficient to prove $(d_1, \dots, d_n) \neq (b, \dots, b)$. Suppose that $(d_1, \dots, d_n) = (b, \dots, b)$.

At least one of the columns must contain the element a, and consequently at least one of the rows must contain a; let it be the k–th row. Since f_m applied to the k–th row gives b, the k–th row contains exactly one occurrence of a, and also exactly one occurrence of c; all the remaining members are equal to b. The $(k-1)$-st row (or the n–th row, if $k = 1$) also contains a, since the column intersecting with the k–th row in the element c belongs to R_n. Again, this means that the $(k-1)$–st row contains exactly one a and exactly one c. We can proceed similarly in this way to obtain the element a in the $(k-2)$–nd row, etc. After n steps we return to the original k–th row, exhausting the whole of the matrix. During the process we have found that each of the rows contains exactly one member equal to a and exactly one member equal to c. Consequently, both a and c occur exactly n times in the matrix M. This implies $m \geq n$; since no column can be equal to $(b, \dots, b)$, we get $m = n$. This is a desired contradiction.

We have proved that f_m preserves R_n iff $m \neq n$. Now it is clear that f_n does not belong to the clone generated by the operations f_i with $i \neq n$. From this it easily follows that the mapping, assigning to any subset S of $\{4, 5, 6, \dots\}$ the clone generated by $\{f_i; i \in S\}$, is an injection. As there are uncountably many subsets of $\{4, 5, 6, \dots\}$, we conclude that there are uncountably many minimal clones contained in the clone generated by binary quasiprojections. □

References

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